J. Appl. Math. & Informatics Vol. **32***(2014), No. 5 - 6, pp. 599 - 607*
http://dx.doi.org/10.14317/jami.2014.599

ON THE DYNAMICS OF $x_{n+1} = \dfrac{a + x_{n-1}x_{n-k}}{x_{n-1} + x_{n-k}}$

A. M. AHMED

Abstract. In this paper, we investigate the behavior of solutions of the difference equation

$$x_{n+1} = \frac{a + x_{n-1}x_{n-k}}{x_{n-1} + x_{n-k}}, \quad n = 0, 1, 2, ...$$

where $k \in \{1, 2\}$, $a \geq 0$, and $x_{-j} > 0,\ j = 0, 1, ..., k$.

AMS Mathematics Subject Classification : 39A10, 39A21, 39A30.
Key words and phrases : Difference equations, Stability, Oscillation.

1. Introduction

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For example,

Abu-Saris et al.[1] investigated the asymptotic stability of the difference equation

$$x_{n+1} = \frac{a + x_n x_{n-k}}{x_n + x_{n-k}}, \quad n = 0, 1, 2, ...$$

For other related results([2-16]).

In this paper, we investigate the behavior of solutions of the difference equation

$$x_{n+1} = \frac{a + x_{n-1}x_{n-k}}{x_{n-1} + x_{n-k}}, \quad n = 0, 1, 2, ... \tag{1.1}$$

Received September 20, 2013. Revised March 12, 2014. Accepted March 22, 2014.

where $k \in \{1,2\}$, $a \geq 0$, and $x_{-j} > 0,\ j = 0,1,...,k.$

We need the following definitions.

Definition 1.1. Let I be an interval of real numbers and let

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0,1,..., \tag{1.2}$$

with $x_{-k}, x_{-k+1}, ..., x_0 \in I$. Let $\overline{x}$ be the equilibrium point of Eq.(1.2). The linearized equation of Eq.(1.2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + ... + c_{k+1} y_{n-k} \tag{1.3}$$

where $c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, ..., \overline{x})$, $c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, ..., \overline{x}), ..., c_{k+1} = \frac{\partial f}{\partial x_{n-k}}(\overline{x}, \overline{x}, ..., \overline{x})$. The characteristic equation of Eq.(1.3) is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} c_i \lambda^{k-i+1} = 0. \tag{1.4}$$

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1},\ x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}$, $x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).

(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

Definition 1.2. A positive semicycle of $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.2) consists of a 'string' of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ and such that either $l = -k$ or $l > -k$ and $x_{l-1} < \overline{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$.

A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.2) consists of a 'string' of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ and such that either $l = -k$ or $l > -k$ and $x_{l-1} \geq \overline{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \overline{x}$.

Definition 1.3. A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.2) is called nonoscillatory if there exists $N \geq -k$ such that either

$$x_n \geq \overline{x} \quad \forall n \geq N \qquad \text{or} \qquad x_n < \overline{x} \quad \forall n \geq N \ ,$$

and it is called oscillatory if it is not nonoscillatory.

We need the following theorems.

Theorem 1.4 ([16]). *(i) If all roots of Eq.(1.4) have absolute value less than one, then the equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable.*

(ii) If at least one of the roots of Eq.(1.4) has absolute value greater than one, then $\overline{x}$ is unstable.

The equilibrium point $\overline{x}$ of Eq.(1.2) is called a saddle point if Eq.(1.4) has roots both inside and outside the unit disk.

Theorem 1.5 ([16]). *Assume that $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \{1, 2, ...\}$. Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \ . \tag{1.5}$$

2. Behavior of solutions of Eq.(1.1) when $k = 1$ and $a = 0$.

In this section we give the closed form of solutions of Eq.(1.1) when $k = 1$ and $a = 0$.

In this case the difference equation (1.1) becomes

$$x_{n+1} = \frac{x_{n-1}^2}{2x_{n-1}} = \frac{1}{2}x_{n-1}, \qquad n = 0, 1, 2, ... \tag{2.1}$$

with positive initial conditions x_{-1} and x_0.

Eq. (2.1) is linear which have the solution

$$x_n = \frac{1}{2}\left(x_0 + \frac{\sqrt{2}}{2}x_{-1}\right)\left(\frac{\sqrt{2}}{2}\right)^n + \frac{1}{2}\left(x_0 - \frac{\sqrt{2}}{2}x_{-1}\right)\left(\frac{-\sqrt{2}}{2}\right)^n, \qquad n = 1, 2, ... \tag{2.2}$$

3. Behavior of solutions of Eq.(1.1) when $k = 2$ and $a = 0$.

In this section we investigate the behavior of solutions of Eq.(1.1) when $k = 2$ and $a = 0$.

In this case the difference equation (1.1) becomes

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_{n-1} + x_{n-2}}, \qquad n = 0, 1, 2, ... \tag{3.1}$$

with positive initial conditions x_{-2}, x_{-1} and x_0.

Eq.(3.1) has a unique equilibrium point $\overline{x} = 0$.

Theorem 3.1. *The equilibrium point* $\overline{x} = 0$ *of Eq.(3.1) is locally asymptotically stable.*

Proof. Since the linearized equation of Eq.(3.1) about the equilibrium point $\overline{x} = 0$ can be written in the following form

$$z_{n+1} = \frac{1}{4} z_{n-1} + \frac{1}{4} z_{n-2},$$

then the proof follows immediately from Theorem B. □

Theorem 3.2. *The equilibrium point* $\overline{x} = 0$ *of Eq.(3.1) is globally asymptotically stable.*

Proof. From Eq.(3.1) it is easy to show that $x_{n+1} < x_{n-1}$ for all $n \geq 0$ and so the even terms converge to a limit (say $L_1 \geq 0$) and the odd terms converge to a limit (say $L_2 \geq 0$). Then

$$L_1 = \frac{L_1 L2}{L_1 + L_2} \quad \text{and} \quad L_2 = \frac{L_1 L2}{L_1 + L_2},$$

which implies that $L_1 = L_2 = 0$, and the proof is complete. □

4. Behavior of solutions of Eq.(1.1) when $k = 1$ and $a > 0$.

In this section we investigate the behavior of solutions of Eq.(1.1) when $k = 1$ and $a > 0$.

In this case the difference equation (1.1) becomes

$$x_{n+1} = \frac{a + x_{n-1}^2}{2x_{n-1}}, \qquad n = 0, 1, 2, ... \tag{4.1}$$

with positive initial conditions x_{-1} and x_0.

The change of variables $x_n = \sqrt{a} y_n$ reduces Eq.(4.1) to the difference equation

$$y_{n+1} = \frac{1 + y_{n-1}^2}{2y_{n-1}}, \qquad n = 0, 1, 2, ... \tag{4.2}$$

Eq.(4.2) has a unique positive equilibrium point $\overline{y} = 1$.

Theorem 4.1. *The equilibrium point* $\overline{y} = 1$ *of Eq.(4.2) is locally asymptotically stable.*

Proof. The linearized equation of Eq.(4.2) about the equilibrium point $\overline{y} = 1$ is

$$z_{n+1} = 0,$$

and so, the characteristic equation of Eq.(4.2) about the equilibrium point $\overline{y} = 1$ is

$$\lambda^2 = 0,$$

which implies that $|\lambda_1| = |\lambda_2| = 0 < 1$. Hence, the proof is complete. □

Theorem 4.2. *The equilibrium point* $\overline{y} = 1$ *of Eq.(4.2) is globally asymptotically stable.*

Proof. Since $1 + y_{n-1}^2 \geq 2y_{n-1}$ for all $n \geq 0$, then we have $y_n \geq 1$ for all $n \geq 1$. Furthermore $y_{n+1} = \dfrac{1 + y_{n-1}^2}{2y_{n-1}} = \dfrac{1}{2y_{n-1}} + \dfrac{y_{n-1}}{2} \leq \dfrac{1}{2} + \dfrac{y_{n-1}}{2} \leq y_{n-1}$ for all $n \geq 2$. So the even terms $\{y_{2n}\}_{n=2}^{\infty}$ converge to a limit (say $L_1 \geq 0$) and the odd terms $\{y_{2n+1}\}_{n=1}^{\infty}$ converge to a limit (say $L_2 \geq 0$). Then

$$L_1 = \frac{1 + L_1^2}{2L_1} \quad \text{and} \quad L_2 = \frac{1 + L_2^2}{2L_2}$$

which implies that $L_1 = L_2 = 1$. Thus, the proof is complete. □

5. Behavior of solutions of Eq.(1.1) when $k = 2$ and $a > 0$.

In this section we investigate the behavior of solutions of Eq.(1.1) when $k = 2$ and $a > 0$.

In this case the difference equation (1.1) becomes

$$x_{n+1} = \frac{a + x_{n-1}x_{n-2}}{x_{n-1} + x_{n-2}}, \qquad n = 0, 1, 2, ... \tag{5.1}$$

with positive initial conditions x_{-2}, x_{-1} and x_0.

The change of variables $x_n = \frac{\sqrt{a}}{z_n}$ reduces Eq.(5.1) to the difference equation

$$z_{n+1} = \frac{z_{n-1} + z_{n-2}}{1 + z_{n-1}z_{n-2}}, \qquad n = 0, 1, 2, ... \tag{5.2}$$

Eq.(5.2) has two equilibrium points $\overline{z_1} = 0$ and $\overline{z_2} = 1$.

Theorem 5.1. *The equilibrium point $\overline{z_1} = 0$ of Eq.(5.2) is unstable equilibrium point.*

Proof. The linearized equation of Eq.(5.2) about the equilibrium point $\overline{z_1} = 0$ is

$$z_{n+1} = z_{n-1} + z_{n-2},$$

and so, the characteristic equation of Eq.(5.2) about the equilibrium point $\overline{z_1} = 0$ is

$$f(\lambda) = \lambda^3 - \lambda - 1 = 0.$$

It is clear that $f(\lambda)$ has a root in the interval $(1, \infty)$, and so, $\overline{z_1} = 0$ is an unstable equilibrium point. This completes the proof. □

Theorem 5.2. *The equilibrium point $\overline{z_2} = 1$ of Eq.(5.2) is locally asymptotically stable.*

Proof. The linearized equation of Eq.(5.2) about the equilibrium point $\overline{z_2} = 1$ is

$$z_{n+1} = 0,$$

and so, the characteristic equation of Eq.(5.2) about the equilibrium point $\overline{z_2} = 1$ is

$$\lambda^3 = 0,$$

which implies that $|\lambda_1| = |\lambda_2| = |\lambda_3| = 0 < 1$, from which the proof is complete. □

Lemma 5.3. *The following identities are true*

$$(i)\quad z_{n+1}-1=\frac{-\left(z_{n-1}-1\right)\left(z_{n-2}-1\right)}{1+z_{n-1}z_{n-2}}\quad for\ \ n\geq 0. \tag{5.3}$$

$$(ii)\quad z_{n+1}-z_{n-1}=\frac{z_{n-2}\left(1-z_{n-1}^{2}\right)}{1+z_{n-1}z_{n-2}}\quad for\ \ n\geq 0. \tag{5.4}$$

$$(iii)\quad z_{n+1}-z_{n-2}=\frac{z_{n-1}\left(1-z_{n-2}^{2}\right)}{1+z_{n-1}z_{n-2}}\quad for\ \ n\geq 0. \tag{5.5}$$

$$(iv)\quad z_{n+1}-z_{n-3}=\frac{\left(z_{n-2}+z_{n-4}\right)\left(1-z_{n-3}^{2}\right)}{1+z_{n-2}z_{n-3}+z_{n-2}z_{n-4}+z_{n-3}z_{n-4}}\quad for\ \ n\geq 2. \tag{5.6}$$

$$(v)\quad z_{n+1}-z_{n-4}=\frac{\left(z_{n-3}+z_{n-4}+z_{n-5}+z_{n-3}z_{n-4}z_{n-5}\right)\left(1-z_{n-4}^{2}\right)}{\left(1+z_{n-3}z_{n-4}\right)\left(1+z_{n-4}z_{n-5}\right)+\left(z_{n-3}+z_{n-4}\right)\left(z_{n-4}+z_{n-5}\right)}\quad for\ \ n\geq 3. \tag{5.7}$$

Proof. (i) $z_{n+1}-1=\dfrac{z_{n-1}+z_{n-2}}{1+z_{n-1}z_{n-2}}-1=\frac{-(z_{n-1}-1)(z_{n-2}-1)}{1+z_{n-1}z_{n-2}}$ for $n\geq 0.$

(ii) $z_{n+1}-z_{n-1}=\dfrac{z_{n-1}+z_{n-2}}{1+z_{n-1}z_{n-2}}-z_{n-1}=\frac{z_{n-2}\left(1-z_{n-1}^{2}\right)}{1+z_{n-1}z_{n-2}}$ for $n\geq 0.$

(iii) $z_{n+1}-z_{n-2}=\dfrac{z_{n-1}+z_{n-2}}{1+z_{n-1}z_{n-2}}-z_{n-2}=\frac{z_{n-1}\left(1-z_{n-2}^{2}\right)}{1+z_{n-1}z_{n-2}}$ for $n\geq 0.$

(iv) $z_{n+1}-z_{n-3}$

$$=\frac{z_{n-1}+z_{n-2}}{1+z_{n-1}z_{n-2}}-z_{n-3}=\frac{\left(\dfrac{z_{n-3}+z_{n-4}}{1+z_{n-3}z_{n-4}}\right)+z_{n-2}}{1+\left(\dfrac{z_{n-3}+z_{n-4}}{1+z_{n-3}z_{n-4}}\right)z_{n-2}}-z_{n-3}$$

$$=\frac{\left(z_{n-2}+z_{n-4}\right)\left(1-z_{n-3}^{2}\right)}{1+z_{n-2}z_{n-3}+z_{n-2}z_{n-4}+z_{n-3}z_{n-4}}\ \text{for } n\geq 2.$$

(v) $z_{n+1}-z_{n-4}=\dfrac{z_{n-1}+z_{n-2}}{1+z_{n-1}z_{n-2}}-z_{n-4}$

$$=\frac{\left(\dfrac{z_{n-3}+z_{n-4}}{1+z_{n-3}z_{n-4}}\right)+\left(\dfrac{z_{n-4}+z_{n-5}}{1+z_{n-4}z_{n-5}}\right)}{1+\left(\dfrac{z_{n-3}+z_{n-4}}{1+z_{n-3}z_{n-4}}\right)\left(\dfrac{z_{n-4}+z_{n-5}}{1+z_{n-4}z_{n-5}}\right)}-z_{n-4}$$

$$=\frac{\left(z_{n-3}+z_{n-4}+z_{n-5}+z_{n-3}z_{n-4}z_{n-5}\right)\left(1-z_{n-4}^{2}\right)}{\left(1+z_{n-3}z_{n-4}\right)\left(1+z_{n-4}z_{n-5}\right)+\left(z_{n-3}+z_{n-4}\right)\left(z_{n-4}+z_{n-5}\right)}\ \text{for } n\geq 3.$$

Then, the proof is complete. □

Theorem 5.4. *Let* $\{z_n\}_{n=-2}^{\infty}$ *be a solution of Eq.(5.2), then the following statements are true*

(i) If $z_{n_0}=\overline{z_2}=1$, *for some* $n_0\in\{-1,0,1,2,...\}$, *then* $z_n=\overline{z_2}=1$, *for all* $n\geq n_0+2$.

Also if $z_{-2}=\overline{z_2}=1$, *then* $z_n=\overline{z_2}=1$, *for all* $n\geq 3$.

(ii) If $z_{n_0},z_{n_0+1},z_{n_0+2}<\overline{z_2}=1$, *for some* $n_0\in\{-2,-1,0,1,2,...\}$, *then* $z_n<\overline{z_2}=1$, *for all* $n\geq n_0$.

(iii) If (i) and (ii) are not satisfied, then $\{z_n\}_{n=-2}^{\infty}$ *oscillates about* $\overline{z_2}=1$, *with positive semicycles of length at most three, and negative semicycles of length at most two.*

Proof. (i) Let $z_{n_0}=\overline{z_2}=1$, for some $n_0\in\{-1,0,1,2,...\}$, then from Eq.(5.3) we have $z_n=\overline{z_2}=1$, for all $n\geq n_0+2$.

If $z_{-2}=\overline{z_2}=1$, then from Eq.(5.3) we have $z_1=\overline{z_2}=1$, which implies that $z_n=\overline{z_2}=1$, for all $n\geq 3$.

(ii) Let $z_{n_0},z_{n_0+1},z_{n_0+2}<\overline{z_2}=1$, for some $n_0\in\{-2,-1,0,1,2,...\}$, then from Eq.(5.3) we have $z_n<\overline{z_2}=1$, for all $n\geq n_0$.

(iii) Suppose without loss of generality that there exists $n_0\in\{-2,-1,0,1,2,...\}$, such that $z_{n_0},z_{n_0+1},z_{n_0+2}>\overline{z_2}=1$. Then from Eq.(5.3) we have $z_{n_0+3},z_{n_0+4}<1$, $z_{n_0+5}>1$, $z_{n_0+6}<1$ and $z_{n_0+7},z_{n_0+8},z_{n_0+9}>\overline{z_2}=1$. The proofs of the other possibilities are similar, and will be omitted. □

Theorem 5.5. *The equilibrium point* $\overline{z_2}=1$ *of Eq.(5.2) is globally asymptotically stable.*

Proof. We proved that $\overline{z_2}=1$ of Eq.(5.2) is locally asymptotically stable, and so it suffices to show that $\lim_{n\to\infty} z_n=1$. If there exists $n_0\in\{-2,-1,0,1,2,...\}$, such that $z_{n_0}=\overline{z_2}=1$, then from Theorem 5.4 we have $\lim_{n\to\infty} z_n=1$. Also, if $z_{-2},z_{-1},z_0<\overline{z_2}=1$, then by Theorem 5.4 we have $z_n<\overline{z_2}=1$, for all $n\geq -2$, and from Eq.(5.4), we have $z_{n+1}>z_{n-1}$, for $n\geq 0$. So the sequences $\{z_{2n}\}_{n=0}^{\infty}$ and $\{z_{2n-1}\}_{n=0}^{\infty}$ are increasing and bounded, which implies that the even terms $\{z_{2n}\}_{n=0}^{\infty}$ converge to a limit (say $M_1>0$) and the odd terms $\{z_{2n-1}\}_{n=0}^{\infty}$ converge to a limit (say $M_2>0$). Then

$$M_1=\frac{M_1+M_2}{1+M_1M_2}\quad\text{and}\quad M_2=\frac{M_1+M_2}{1+M_1M_2},$$

which implies that $M_1=M_2=1$.

Now, Suppose that $z_{-2},z_{-1},z_0>\overline{z_2}=1$, then from Eqs.(5.4) - (5.7) we have the following results

The sequence $\{z_{7n}\}_{n=0}^{\infty}$ is decreasing and bounded, and so converges to a limit (say $L_0>0$) .

The sequence $\{z_{7n+1}\}_{n=0}^{\infty}$ is increasing and bounded, and so converges to a limit (say $L_1>0$) .

The sequence $\{z_{7n+2}\}_{n=0}^{\infty}$ is increasing and bounded, and so converges to a limit (say $L_2>0$) .

The sequence $\{z_{7n+3}\}_{n=0}^{\infty}$ is decreasing and bounded, and so converges to a limit (say $L_3 > 0$) .

The sequence $\{z_{7n+4}\}_{n=0}^{\infty}$ is increasing and bounded, and so converges to a limit (say $L_4 > 0$) .

The sequence $\{z_{7n+5}\}_{n=0}^{\infty}$ is decreasing and bounded, and so converges to a limit (say $L_5 > 0$) .

The sequence $\{z_{7n+6}\}_{n=0}^{\infty}$ is decreasing and bounded, and so converges to a limit (say $L_6 > 0$) .

So we have from Eq.(5.2) that

$$\begin{aligned} L_0 &= \frac{L_4+L_5}{1+L_4L_5}, \quad L_1=\frac{L_5+L_6}{1+L_5L_6}, \quad L_2=\frac{L_0+L_6}{1+L_0L_6}, \\ L_3 &= \frac{L_0+L_1}{1+L_0L_1}, \quad L_4=\frac{L_1+L_2}{1+L_1L_2}, \quad L_5=\frac{L_2+L_3}{1+L_2L_3}, \\ L_6 &= \frac{L_3+L_4}{1+L_3L_4}. \end{aligned}$$

The solution of this system is either $L_i = -1, i = 0, 1, ...6$, or $L_i = 0, i = 0, 1, ...6$, or $L_i = 1,\ i = 0, 1, ...6$. Since $L_i > 0,\ i = 0, 1, ...6$, we have $\lim_{n\to\infty}\ z_n = 1$.

The proofs for the other cases are as follows.

$z_{-2}, z_{-1} > \overline{z_2} = 1, z_0 < \overline{z_2} = 1$, or $z_{-2}, z_{-1} < \overline{z_2} = 1, z_0 > \overline{z_2} = 1$, or $z_{-2} > \overline{z_2} = 1, z_{-1}, z_0 < \overline{z_2} = 1$, or $z_{-2} < \overline{z_2} = 1, z_{-1}, z_0 > \overline{z_2} = 1$, are similar to the proof of the last case, and will be omitted. Therefore the proof is complete. □

References

1. R. Abu-Saris, C. Cinar and I. Yalçinkaya, *On the asymptotic stability of* $x_{n+1} = \dfrac{a + x_n x_{n-k}}{x_n + x_{n-k}}$, Computers & Mathematics with Applications **56** (2008), 1172-1175.
2. A. Ahmed, *On the dynamics of a higher order rational difference equation,* Discrete Dynamics in Nature and Society, **Article ID 419789,** (2011), 8 pages, doi:10.1155/2011/419789.
3. A. Ahmed, H. El-Owaidy, A. Hamza and A. Youssef, *On the recursive sequence* $x_{n+1} = \dfrac{a + bx_{n-1}}{A + Bx_n^k}$, J. Appl. Math. & Informatics, **27, No. 1 - 2**, (2009), 275 - 289.
4. A. Ahmed and A. Youssef, *A Solution Form of a Class of Higher-Order Rational Difference Equations,* J. Egyptian Math. Soc., **21** (2013), 248 -?253.
5. C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \dfrac{x_{n-1}}{1 + x_n x_{n-1}}$, Appl. Math. Comp., **150**, (2004), 21-24.
6. C. Cinar, *On the difference equation* $x_{n+1} = \dfrac{x_{n-1}}{-1 + x_n x_{n-1}}$, Appl. Math. Comp., **158**, (2004), 813-816.
7. C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \dfrac{ax_{n-1}}{1 + bx_n x_{n-1}}$, Appl. Math. Comp., **156** , (2004), 587-590.
8. C. Cinar, R. Karatas and I. Yalcinkaya, *On solutions of the difference equation* $x_{n+1} = \dfrac{x_{n-3}}{-1 + x_n x_{n-1} x_{n-2} x_{n-3}}$, Mathematica Bohemica, **132 (3)**, (2007), 257-261.
9. E. Elsayed, *Dynamics of a rational recursive sequence,* International J. Difference Equations, **4 (2),** (2009), 185-200.

10. E. Elsayed, *Behavior of a rational recursive sequences,* Stud. Univ. Babes-Bolyai Math. **LVI (1)**, (2011), 27-42.
11. E. Elabbasy, H. El-Metwally and E. Elsayed, *On the difference equations* $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math., **5 (2),** (2007), 101-113.
12. E. Elabbasy, H. El-Metwally and E. Elsayed, *Qualitative behavior of higher order difference equation,* Soochow Journal of Mathematics, **33 (4),** (2007), 861-873.
13. E. Elabbasy, H. El-Metwally and E. Elsayed, *On the Difference Equation* $x_{n+1} = \frac{a_0x_n+a_1x_{n-1}+\ldots+a_kx_{n-k}}{b_0x_n+b_1x_{n-1}+\ldots+b_kx_{n-k}}$, Mathematica Bohemica, **133 (2),** (2008), 133-147.
14. H. El-Owaidy, A. Ahmed and A. Youssef, *The dynamics of the recursive sequence* $x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}$, Appl. Math. Lett., **18 (9)**, (2005), 1013–1018.
15. H. El-Owaidy, A. Ahmed and A. Youssef, *On the dynamics of* $x_{n+1} = \frac{bx_{n-1}^2}{A + Bx_{n-2}}$, Rostock. Math. Kolloq., **59**, (2004), 11-18.
16. V. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications,* Kluwer Academic Publishers, Dordrecht, 1993.

A.M. Ahmed obtained his B.Sc. (1997), M.Sc. (2000) and Ph. D. (2004) from Al-Azhar University. He obtained the rank associate professor from Al-Azhar University in 2012. His research interests in difference equations and its applications.

Mathematics Department, Faculty of Science, Al-Azhar University , Nasr City(11884), Cairo, Egypt.
e-mail: ahmedelkb@yahoo.com