

THE ROLE OF INSTANT NUTRIENT REPLENISHMENT ON PLANKTON SPECIES IN A CLOSED SYSTEM

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ABSTRACT. In this paper, we formulate two chemostat type models of phytoplankton and zooplankton population dynamics with instant nutrient recycling to study the role of viral infection on phytoplankton. The infection is transmitted only among phytoplankton population and it makes them more vulnerable to predation by zooplankton. It is observe that the chemostat system is very stable in the absence of viral infection but the presence of viral infection make the chemostat system sensitive with respect to the grazing rate of infected-phytoplankton by zooplankton. Further, if the grazing rate is less than certain threshold the system remain stable and exhibits Hopf-bifurcation after crossing it.

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Key words and phrases : Plankton dynamics, viral infection, infected phytoplankton, nutrient recycling, stability, Hopf-Bifurcation.

1. Introduction

Phytoplankton and micro zooplankton are one celled organisms those drift with the current on the surface of open ocean. They are the staple item for the food web and are producer and recycler of most of the energy that flow through the oceanic eco-system. Plankton play important role not only in maintaining the fish stock but stabilize environment by consuming half of the universe Carbon dioxide and release huge Oxygen. Pollution of fresh water in marine system by anthropogenic sources has become a concern over the last decades. Researchers have found out that each tea spoon of ocean water contains 10 million to 100 million of viruses. Again, viral infection is known to cause a cell lysis in phytoplankton and biomass is the common criterion for non-toxic species but the measurable level of toxin due to harmful species is responsible for the bloom dynamics. In coastal area viral disease can infect bacteria and phytoplankton

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[1]. Virus like particle is found in many eukastic algae [23] and natural phytoplankton community[20]. These virus like particle have also been described in many eukaryotic algae [7, 8]. The parasite modifying behavior has also been exhibited by the infected individuals of host population. This may happen by reducing stamina, disorientation and altering responses[9]. Kill fish(fundulus parvipinnes) tend to come closer to the surface of the sea on contracting disease, which make them more vulnerable to predation by birds [21]. Viruses have been held responsible for the collapse of Emiliaxia Huxleye bloom in mesocosms [13, 23]. Since virus have major role in shaping the dynamics of plankton. Many researchers have developed a epidemiological model and concluded that infected pollution does not persist if infection rate is below some threshold value [3, 4, 5, 14, 15, 16, 17, 21, 22]. Recently, the role of toxic producing phytoplankton and delay on planktonic ecosystem in the presence of a toxic producing phytoplankton are studied [24, 25].

These kind of transmissible disease cause death and behavior change in aquatic species. As a result of this many planktonic species show bloom in their population (i.e., the rapid growth and decay in their population). The blooms are of two type, one is spring bloom and other is red bloom. The spring bloom is seasonal. It is due to change in temperature and nutrient level associated with season. On the other hand Red bloom is localized out break associated with change in water temperature and with greater stability of water column [10]. Different criteria have been used in the literature to define and classify bloom [11, 22]. Recently, pattern formation have been studied for instant nutrient replenishment on plankton dynamics with diffusion in a closed system [6]. A paper in American Museum of Nature History (2000) reveals that when a virus injects its DNA into a cell, it hijacks that cells replication machinery and produce hundreds of new particles. These rupture the host and are released into the environment to find new victims. Hence, the infected particle itself becomes the source of infection and this rate of infection is directly proportional to the number of infected phytoplankton and the number of susceptible phytoplankton. Therefore, in this paper, we have assumed that infected phytoplankton themselves are the source of infection.

Keeping in view the above discussion, in this paper, we shell study the role of virus in phytoplankton, zooplankton interaction by developing two chemostat model of phytoplankton and zooplankton species one without infected phytoplankton and other with infected phytoplankton class. In section 2 we shell be developing and analyzing chemostat model of phytoplankton and zooplankton species without infected phytoplankton and in section 3 will have the development and analysis of model of phytoplankton and zooplankton with infected phytoplankton class in instantaneous nutrient recycling environment. In section 4 the bifurcation analysis of nutrient plankton system with infected phytoplankton class will be carried out and section 5 will be devoted to the conclusion of analysis carried out in the previous sections.

2. Nutrient-Plankton System without Infected Phytoplankton Class

Here we propose a prey-predator model of phytoplankton in the absence of infected phytoplankton with instantaneous nutrient recycling. Let $N(t)$, $P_s(t)$ and $Z(t)$ are nutrient, susceptible phytoplankton and zooplankton population densities at any time t respectively. Again, d_1 and d_3 , are the per capita death rate of susceptible phytoplankton and zooplankton respectively. In a natural plankton system, water flowing into the system brings in nutrient and out going water carries away nutrient from the system. Further, it is assumed that water is carrying away nutrient, susceptible phytoplankton along with flow in the same rate. Here N_0 is the constant nutrient input concentration at any time and D is the water influx rate or washout rate along with nutrient, susceptible phytoplankton and zooplankton. The nutrient uptake and grazing rate of phytoplankton by zooplankton is assumed to follow law of mass action. Let a_1 is the nutrient uptake rate by susceptible phytoplankton and c^1 is the grazing rate of susceptible phytoplankton by zooplankton. The conversional rate of the nutrient into susceptible phytoplankton is given by a^1 and c^{11} is the conversional rate of susceptible phytoplankton into zooplankton. Model equation for the above model are given by

$$\frac{dN}{dt} = D(N_0 - N) - a_1NP_s + (1 - a^1)a_1NP_s + (1 - c^{11})c^1P_sZ + d_1P_s + D_3P_z, \tag{1}$$

$$\frac{dP_s}{dt} = a^1a_1NP_s - c^1P_sZ - d_1P_s - DP_s, \tag{2}$$

$$\frac{dZ}{dt} = c^{11}c^1P_sZ - d_3Z - DZ, \tag{3}$$

$$N(0) > 0, \quad P_s(0) > 0, \quad Z(0) > 0.$$

Here all the conversion rates lies between $(0,1)$ and rest of the parameters are positive. Again, the positive feed back term's $(1 - a^1)a_1NP_s$, $(1 - c^{11})c^1P_sZ$, d_1P_s and d_3Z will be recycled into nutrients, i.e., all the losses are being are being replenished into nutrients.

Suppose $W = N + P_s + Z$, then $\dot{W} = \dot{N} + \dot{P}_s + \dot{Z}$. Using (1)-(3), we get $\dot{W} + DW = DN_0$. Clearly, $W \rightarrow N_0$ as $t \rightarrow \infty$. Hence, the solutions of the system (1)-(3) exist and bounded for all time to come.

Again, $\dot{N} = DN_0 > 0$ at $N = 0$, which shows that $N(t) > 0, \forall t > 0$. Similarly for P_s and Z . Therefore, we can state the following lemma:

Lemma 2.1. *Solutions of equations (1)-(3) are non-negative and bounded.*

Now we shell study the asymptotic dynamic of the above system (1)-(3), using $N + P_s + Z = N_0$, our model reduces to

$$\frac{dP_s}{dt} = a^1a_1(N_0 - P_s - Z)P_s - c^1P_sZ - d_1P_s - DP_s, \tag{4}$$

$$\frac{dZ}{dt} = c^{11}c^1P_sZ + d_3Z - DZ. \tag{5}$$

Putting $P_s = x$, $Z(t) = z$, $a^1 a_1 = a$, $c^1 = c$, $d_1 + D = \delta_1$, $d^1 = d$, $c^1 c^1 = \alpha$ and $d_3 + D = \delta_3$, then system (4)-(5) become

$$\frac{dx}{dt} = a(N_0 - x - z)x - cxz - \delta_1 x, \quad (6)$$

$$\frac{dz}{dt} = \alpha xz - \delta_3 z. \quad (7)$$

All possible equilibrium of the system are, (i) $E_0(0, 0)$, (ii) $E_1(\frac{aN_0 - \delta_1}{a}, 0)$, exist if $aN_0 > \delta_1$ and (iii) $E_2(\frac{\delta_3}{\alpha}, \frac{a(\alpha N_0 - \delta_1) - \alpha \delta_1}{\alpha(a+c)})$, exist if $a\alpha N_0 > a\delta_3 + \alpha\delta_1$.

Clearly the dynamic of the above system (6)-(7) can be stated as follows:

- (1) $E_0(0, 0)$ always exist and stable if $aN_0 < \delta_1$, then all solution will converges to E_0 .
- (2) The instability of $E_0(0, 0)$, i.e., $aN_0 > \delta_1$ leads to the existence of E_1 and all flow converges to E_1 if $a\alpha N_0 < a\delta_3 + \alpha\delta_1$.
- (3) The instability of E_1 , i.e., $a\alpha N_0 > a\delta_3 + \alpha\delta_1$ leads to the existence of E_2 .

Hence the system persists uniformly. Again, the characteristic equation of above system around E^* is

$$p(\lambda) = \lambda^2 + \lambda C_1 + C_2 = 0, \quad (8)$$

where $C_1 = x^* a > 0$ and $C_2 = \alpha(a+c)x^* z^* > 0$. Hence, both the eigen values of characteristic equation having negative real parts. So, interior equilibrium E^* if exists, then it is always stable.

Remark 2.1. From the above discussion it is clear that as the nutrient supply (i.e., aN_0) is less than removal rate of nutrient (i.e., δ_1) then the system stays in trivial equilibrium. Again, when the nutrient supply is greater than removal rate of nutrient δ_1 but less than $\delta_1 + a\delta_3/\alpha$, then system converges to the zooplankton free equilibrium. On the other hand if the nutrient supply is greater than $\delta_1 + a\delta_3/\alpha$, then the interior equilibrium exists and stable.

3. Nutrient Plankton System with Infected Phytoplankton Class

In this model we propose a prey-predator model for phytoplankton and zooplankton in the presence of viruses with instantaneous nutrient recycling. Let $N(t)$, $P_s(t)$, $P_i(t)$ and $Z(t)$ are nutrient concentration, susceptible phytoplankton, infected phytoplankton and zooplankton population densities at any time t respectively. Let d_1 , d_2 and d_3 , are the per capita death rate of susceptible phytoplankton, infected phytoplankton and zooplankton respectively. Here, N_0 is the constant nutrient input concentration and it is assumed that water is carrying away nutrient, susceptible phytoplankton, infected phytoplankton along with flow in the same rate $'D'$. The nutrient-phytoplankton and phytoplankton-zooplankton interactions are assumed to follow law of mass action. Here a_1 is the nutrient uptake rate by susceptible phytoplankton and c^1 , d^1 are the grazing rate of susceptible and infected phytoplankton by zooplankton respectively.

The conversional rate of the nutrient into susceptible phytoplankton is given by a^1 , whereas c^{11} and d^{11} are the conversional rate of susceptible phytoplankton and infected phytoplankton into zooplankton respectively. Here b^1 is the disease contact rate. Then the model equations for the above system are given as:

$$\begin{aligned} \frac{dN}{dt} &= D(N_0 - N) - a_1NP_s + (1 - a^1)a_1NP_s + (1 - c^{11})e^1P_sZ \\ &+ (1 - d^{11})d^1P_iZ + d_1P_s + d_2P_i + d_3P_z, \end{aligned}$$

$$\frac{dP_s}{dt} = a^1a_1NP_s - b^1P_sP_i - c^1P_sZ - d_1P_s - DP_s, \tag{9}$$

$$\frac{dP_i}{dt} = b^1P_sP_i - d^1P_iZ - d_2P_i - DP_i, \tag{10}$$

$$\frac{dZ}{dt} = c^{11}c^1P_sZ + d^{11}d^1P_iZ - d_3Z - DZ, \tag{11}$$

$$N(0) > 0, \quad P_s(0) > 0, \quad P_i(0) > 0, \quad Z(0) > 0.$$

Here all the conversional rates lies between (0, 1) and all other parameters are positive. Moreover, feed back term's $(1 - a^1)a_1NP_s$, $(1 - c^{11})c^1P_sZ$, $(1 - d^{11})d^1P_iZ$, d_1P_s , d_2P_i and d_3Z are being recycled into nutrients, i.e, all the losses are being replenished into nutrients. Suppose $W = N + P_s + P_i + Z$, then $\dot{W} = \dot{N} + \dot{P}_s + \dot{P}_i + \dot{Z}$. Using (9)-(11), we get $\dot{W} + DW = DN_0$. Clearly, $W = N_0$ as $t \Rightarrow \infty$. Hence, the solution of the system exists and bounded for all time to come. Again, $\dot{N}_{N=0} = DN_0 > 0$, Which shows that $N(t) > 0, \forall t > 0$. Similarly P_s, P_i and Z . Therefore, we can state the following lemma:

Lemma 3.1. *Solution of system (9)-(11) are non-negative and bounded.*

Now we shell discus the asymptotic dynamic of the above system, using, $N + P_s + P_i + Z = N_0$ in equations (9)-(11), our model reduces to

$$\frac{dP_s}{dt} = a^1a_1(N_0 - P_s - P_i - Z)P_s - b^1P_sP_i - c^1P_sZ - d_1P_s - DP_s, \tag{12}$$

$$\frac{dP_i}{dt} = b^1P_sP_i - d^1P_iZ - d_2P_i - DP_i, \tag{13}$$

$$\frac{dZ}{dt} = c^{11}c^1P_sZ + d^{11}d^1P_iZ - d_3Z - DZ. \tag{14}$$

Putting $P_s = x, P_i(t) = y, Z(t) = z, a^1a_1 = a, b^1 = b, c^1 = c, d_1 + D = \delta_1, d^1 = d, d_2 + D = \delta_2, c^{11}c^1 = \alpha, d^{11}d^1 = \beta$ and $d_3 + D = \delta_3$.

Then equations (12)-(14) become

$$\frac{dx}{dt} = a(N_0 - x - y - z)x - bxy - cxz - \delta_1x, \tag{15}$$

$$\frac{dy}{dt} = bxy - dyz - \delta_2y, \tag{16}$$

$$\frac{dz}{dt} = \alpha xz + \beta yz - \delta_3z. \tag{17}$$

All the feasible equilibrium of the system are (i) $E_0(0, 0, 0)$, (ii) $E_1(\frac{aN_0-\delta_1}{a}, 0, 0)$, exist if $aN_0 > \delta_1$, (iii) $E_2\left(\frac{\delta_2}{b}, \frac{a(bN_0-\delta_2)-b\delta_1}{b(a+b)}, 0\right)$, exist if $abN_0 > a\delta_2 + b\delta_1$, (iv) $E_3\left(\frac{\delta_3}{\alpha}, 0, \frac{a(\alpha N_0-\delta_3)-\alpha\delta_1}{\alpha(a+c)}\right)$, exist if $a\alpha N_0 > a\delta_3 + \alpha\delta_1$ and (v) $E_4(x^*, y^*, z^*)$, where $y^* = \frac{\delta_3-\alpha x^*}{\beta}$, $z^* = \frac{bx^*-\delta_2}{d}$, $x^* = \frac{d(aN_0-\delta_1)\beta+(a+c)\beta\delta_2-(a+b)d\delta_3}{ad\beta+(a+c)b\beta-(a+b)d\alpha}$, exist if $\beta > \max\left\{\frac{(a+b)\delta_3}{aN_0-\delta_1}, \frac{\alpha(a+b)}{a}, \frac{(a+b)(b\delta_3-\alpha\delta_2)}{b(aN_0-\delta_1)-a\delta_2}\right\}$ and $d < \hat{d} = \frac{(a+c)(b\delta_3-\alpha\delta_2)}{\alpha(aN_0-\delta_1)-a\delta_3}$.

Now the dynamic of the above system (15)-(17) can be stated as follow:

- (1) The trivial steady state $E_0(0, 0, 0)$ always exist and stable if $aN_0 < \delta_1$, i.e., the total nutrient supply is less than the wash out rate. Then all solution will converges to E_0 .
- (2) The instability of $E_0(0, 0, 0)$,i.e., $aN_0 > \delta_1$ leads to the existence of boundary steady state E_1 and all flow converges to it if $abN_0 < a\delta_2 + b\delta_1$ and $a\alpha N_0 < a\delta_3 + \alpha\delta_1$.
- (3) The instability of E_1 , for $abN_0 > a\delta_2 + b\delta_1$, leads to the existence of E_2 , i.e., zooplankton free equilibrium and all flow will converge toward it if $\beta < \hat{\beta} = \frac{(a+b)(b\delta_3-\alpha\delta_2)}{b(aN_0-\delta_1)-a\delta_2}$.
- (4) The instability of E_1 , for $a\alpha N_0 > a\delta_3 + \alpha\delta_1$ leads to the existence infection free steady state E_3 and all solution converges to it if $d > \hat{d} = \frac{(a+c)(b\delta_3-\alpha\delta_2)}{\alpha(aN_0-\delta_1)-a\delta_3}$.
- (5) Endemic equilibrium $E^* = (x^*, y^*, z^*)$, where $x^* = \frac{d(aN_0-\delta_1)\beta+(a+c)\beta\delta_2-(a+b)\beta_2\delta_2}{ad\beta+(a+c)b\beta-(a+b)ab\alpha}$, $y^* = \frac{(a+c)(b\delta_3-\alpha\delta_2)-d(\alpha((aN_0-\delta_1)-a\delta_3))}{ad\beta+(a+c)b\beta-(a+b)d\alpha}$ and $z^* = \frac{\beta(b(aN_0-\delta_1)-a\delta_2)-(a+b)(b\delta_3-\alpha\delta_2)}{ad\beta+(a+c)b\beta-(a+b)d\alpha}$ exist and stable only if the equilibria points E_0, E_1, E_2 , and E_3 are unstable and $\beta > \hat{\beta} = \frac{(a+b)\alpha d}{ad+b(a+c)}$. Now, the endemic equilibrium, i.e., E^* exist If $\beta > \max\left\{\frac{(a+b)\delta_3}{aN_0-\delta_1}, \frac{\alpha(a+b)}{a}, \frac{(a+b)(b\delta_3-\alpha\delta_2)}{b(aN_0-\delta_1)-a\delta_2}, \frac{(a+b)\alpha d}{ad+b(a+c)}\right\}$ and $d < \frac{(a+c)(b\delta_3-\alpha\delta_2)}{\alpha(aN_0-\delta_1)-a\delta_3}$. Since $dd^{11} = \beta$ and hence $\max\left\{\frac{(a+b)\delta_3}{aN_0-\delta_1}, \frac{\alpha(a+b)}{a}, \frac{(a+b)(b\delta_3-\alpha\delta_2)}{b(aN_0-\delta_1)-a\delta_2}, \frac{(a+b)\alpha d}{ad+b(a+c)}\right\} < dd^{11} < \frac{d^{11}(a+c)(b\delta_3-\alpha\delta_2)}{\alpha(aN_0-\delta_1)-a\delta_3}$, is the range of values of d for which E^* exist.

Now, we will examine the local behavior of the system around the endemic equilibrium $E^* = (x^*, y^*, z^*)$ with respect to the change in parameter d , i.e., the grazing rate of infected-phytoplankton by zooplankton. The possibility of bifurcation of the solution of dynamical system will be explored by taking, the grazing rate of infected-phytoplankton by zooplankton as a control parameter. The characteristic equation of system around E^* is as follow:

$$P(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \tag{18}$$

where

$$\begin{aligned} A_1 &= ax^*, \\ A_2 &= \beta dy^*z^* + (a+b)by^*x^* + (a+c)\alpha x^*z^*, \\ A_3 &= (ad\beta + (a+c)b\beta - (a+b)d\alpha)x^*y^*z^*. \end{aligned}$$

On substitution the values of x^* and z^* , it can be easily verified that $A_i > 0$, for $i = 1, 2, 3$. Now, from Routh-Hurwitz criterion a set of necessary and sufficient conditions for all the roots of the above equation (18) having negative real part are $A_i > 0, i = 1, 2, 3$ and $A_1A_2 > A_3$. Now solving the above inequality, we get a sufficient condition for the stability.

Hence, we can state the following theorem:

Theorem 3.2. *If the endemic equilibrium exist, then the sufficient condition for the system (15)-(17) to be locally stable around the endemic equilibrium point E^* is $d > \frac{b(a+c)\beta}{(a+b)\alpha}$.*

4. Hopf-bifurcation Analysis

Further, we will study the Hopf-bifurcation of above system (15)-(18), taking d (i.e., the grazing rate of infected-phytoplankton by zooplankton) as a bifurcation parameter. Now, the necessary and sufficient conditions for the existence of the Hopf-bifurcation, if there exists $d = d_0$ such that (i) $A_i(d_0) > 0, i = 1, 2, 3$, (ii) $A_1(d_0)A_2(d_0) - A_3(d_0) = 0$ and (iii) if we consider the eigen values of the characteristic equation (18) of the form $\lambda_i = u_i + iv_i$, then $\frac{d}{dd}(u_i) \neq 0, i = 1, 2, 3$. After substitution of the values, the condition $A_1A_2 - A_3 = 0$ becomes

$$d^2B_1 + dB_2 + B_3 = 0, \tag{19}$$

where

$$\begin{aligned} B_1 &= (a+b)\alpha m_1 l_2 - (a+b)abk_1 l_2, \\ B_2 &= (a+b)ab(k_1 l_1 - K_2 l_2) + (a+c)\alpha ak_1 m_1 + (a+c)\beta bl_2 m_1 - (a+b)\alpha m_1 l_1, \\ B_3 &= (a+b)abk_2 l_1 + (a+c)\alpha ak_2 m_1 - (a+c)\beta bl_1 m_1. \end{aligned}$$

For example, taking $a = 0.4, n_0 = 50, b = 0.8, c = 1, \delta_1 = 0.4, \delta_2 = 0.2, \delta_3 = 0.5, \alpha = 0.05, \beta = 0.5$, we get a positive root $d = 0.367$ of the quadratic equation (19). Therefore, the eigen values of the characteristic equation (19) at $d = 0.367$ are of the form $\lambda_{1,2} = \pm iv$ and $\lambda_3 = -w$, where v and w are positive real number.

Now, we will verify the condition (iii) of hopf-bifurcation. Put $\lambda = u + iv$ in (19), we get

$$(u + iv)^3 + A_1(u + iv)^2 + A_2(u + iv) + A_3 = 0. \tag{20}$$

On separating the real and imaginary part and eliminating v between real and imaginary part, we get

$$8u^3 + 8A_1u^2 + 2(A_1^2 + A_2)u + A_1A_2 - A_3 = 0. \tag{21}$$

It is clear from the above that $u(d_0) = 0$ iff $A_1(d_0)A_2(d_0) - A_3(d_0) = 0$. Further, at $d = d_0, u(d_0)$ is the only root, since the discriminant $8u^2 + 8A_1u + 2(A_1^2 + A_2) =$

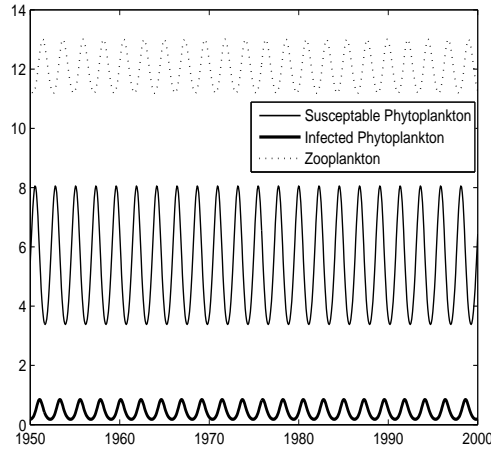


FIGURE 1. The phase plane representation of three species around the endemic equilibrium, taking $a = 0.4$, $n_0 = 50$, $b = 0.8$, $c = 1$, $\delta_1 = 0.4$, $\delta_2 = 0.2$, $\delta_3 = 0.5$, $\alpha = 0.05$, $\beta = 0.5$, and $d = 0.35$

0 is $64A_1^2 - 64(A_1^2 + A_2) < 0$. Again, differentiating (17) with respect to d , we have $(24u^2 + 16A_1u + 2(A_1^2 + A_2)) \frac{du}{dd} + (8u^2 + 4A_1u) \frac{dA_1}{dd} + 2u \frac{dA_2}{dd} + \frac{d}{dd} (A_1A_2 - A_3) = 0$. Now, since at $d = d_0$, $u(d_0) = 0$, we get $\left[\frac{du}{dd}\right]_{d=d_0} = \frac{-\frac{d}{dd} (A_1A_2 - A_3)}{2(A_1^2 + A_2)} \neq 0$, which will ensure that the above system has a hopf-bifurcation and it is shown graphically in figures 1-4.

5. Conclusion

In this paper, we have studied the effect of a viral infection which was spreading only among phytoplankton and rendered them more vulnerable to predation by zooplankton. It was observed that the dynamical system was more stable in the absence of viral infection. We established that in this dynamical system disease free equilibrium exist and stable only when the grazing rate of infected-phytoplankton by zooplankton, (i.e., "d") was greater than the threshold value $\hat{d} = \frac{(a+c)(b\delta_3 - \alpha\delta_2)}{\alpha(aN_0 - \delta_1) - a\delta_3}$. In other words we found that if the vulnerability of infected phytoplankton to the predation by the zooplankton was more than the certain threshold value \hat{d} , then the infected phytoplankton would extinct provided the boundary equilibrium (i.e., E_0 and E_1) were unstable. We have also analyzed that the above chemostat system would go without zooplankton population as long as the net conversion rate of infected phytoplankton into zooplankton, i.e, β was below the threshold $\hat{\beta} = \frac{(a+b)(b\delta_3 - \alpha\delta_2)}{b(aN_0 - \delta_1) - a\delta_2}$ provided the boundary equilibrium

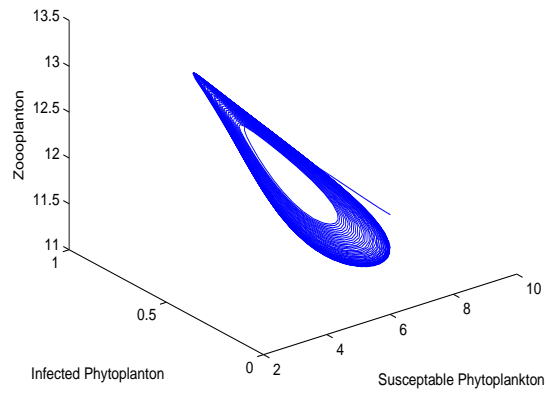


FIGURE 2. The phase plane representation of three species around the endemic equilibrium, taking $a = 0.4$, $n_0 = 50$, $b = 0.8$, $c = 1$, $\delta_1 = 0.4$, $\delta_2 = 0.2$, $\delta_3 = 0.5$, $\alpha = 0.05$, $\beta = 0.5$, and $d = 0.35$

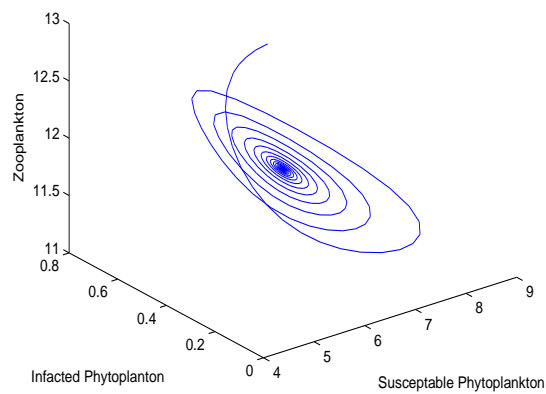


FIGURE 3. The phase plane representation of three species around the endemic equilibrium, taking $a = 0.4$, $n_0 = 50$, $b = 0.8$, $c = 1$, $\delta_1 = 0.4$, $\delta_2 = 0.2$, $\delta_3 = 0.5$, $\alpha = 0.05$, $\beta = 0.5$, and $d = 0.4$

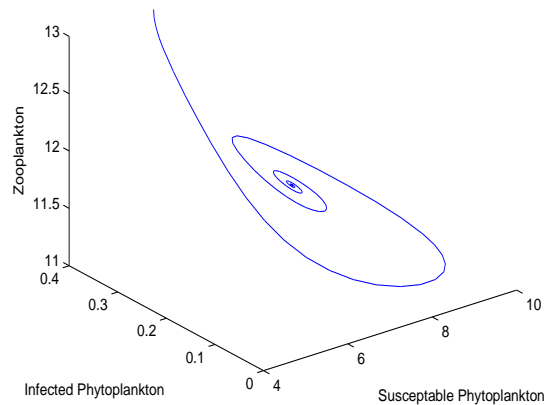


FIGURE 4. The phase plane representation of three species around the endemic equilibrium, taking $a = 0.4$, $n_0 = 50$, $b = 0.8$, $c = 1$, $\delta_1 = 0.4$, $\delta_2 = 0.2$, $\delta_3 = 0.5$, $\alpha = 0.05$, $\beta = 0.5$, and $d = 0.5$

(i.e., E_0 and E_1) were unstable. Further it has been established that endemic equilibrium $E^* = (x^*, y^*, z^*)$ became stable only in some range of values of d . In section 4, the possibility of bifurcation with respect to the grazing rate of infected-phytoplankton by zooplankton (i.e., d) was explored both analytically and numerically. It is found that viral infection has important role in shaping the dynamics of plankton and this kind of viral infection could give reason for the outbreak of bloom in phytoplankton species depending upon the rate of grazing of infected-phytoplankton by zooplankton.

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