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Multiuser Heterogeneous-SNR MIMO Systems

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Abstract

Previous studies on multiuser multiple-input multiple-output (MIMO) mostly assume a homogeneous signal-to-noise ratio (SNR), where each user has the same average SNR. However, real networks are more likely to feature heterogeneous SNRs (a random-valued average SNR). Motivated by this fact, we analyze a multiuser MIMO downlink with a zero-forcing (ZF) receiver in a heterogeneous SNR environment. A transmitter with M antennas constructs M orthonormal beams and performs the SNR-based proportional fairness (S-PF) scheduling where data are transmitted to users each with the highest ratio of the SNR to the average SNR per beam. We develop a new analytical expression for the sum throughput of the multiuser MIMO system. Furthermore, simply modifying the expression provides the sum throughput for important special cases such as homogeneous SNR, max-rate scheduling, or high SNR. From the analysis, we obtain new insights (lemmas): i) S-PF scheduling maximizes the sum throughput in the homogeneous SNR and ii) under high SNR and a large number of users, S-PF scheduling yields the same multiuser diversity for both heterogeneous SNRs and homogeneous SNRs. Numerical simulation shows the interesting result that the sum throughput is not always proportional to M for a small number of users.

Keywords: multiuser MIMO, heterogeneous SNR, homogeneous SNR, proporitional fairness scheduling, max-rate scheduling, zero-forcing receiver

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1. Introduction

There is a need for developing wireless communications systems with an enhanced system capacity in order to meet the increasing demands of wireless services with high data rate and various types of devices. This has led to extensive research on multiple-input multiple-output (MIMO) systems. In particular, due to the higher multiuser diversity gain in wireless packet systems, many research groups have investigated the multiuser MIMO scheme also known as spatial division multiple access (SDMA) [1][2][3][4][5][6][7][8][9]. A specific area of research has explored the capacity region of the Gaussian MIMO broadcast channel [1][2]. It has been shown that dirty-paper coding (DPC) with perfect channel state information (CSI) is capacity achieving in a MIMO broadcast channel. In many applications, however, the DPC scheme is difficult to implement owing to its considerable computational complexity and feedback.

Clearly, it is of interest to develop a SDMA with imperfect CSI. Sharif et al. proposed a popular SDMA system in which a transmitter with M antennas constructs M random orthonormal beams and transmits data streams to receivers with the highest signal-to-interference-plus-noise ratio (SINR) for each beam [3], where M denotes the number of transmitting antennas. Opportunistic SDMA can asymptotically obtain the optimal scaling law of sum capacity when the number of users is large. Nonetheless, the analysis does not extend to a multiuser MIMO system with linear receivers for a finite number of users. In [4], on the other hand, an asymptotic capacity analysis shows that the capacity of a multiuser MIMO system with a zero-forcing (ZF) receiver increases linearly with the number of transmitting antennas; however, it is limited to the case of a large number of users. In real environments, unfortunately, the capacity of a multiuser MIMO system does not agree with the asymptotic results when the number of users is finite. Therefore, it is important to analyze the specific capacity of a multiuser MIMO system for a finite number of users in order to create high-quality applications in real environments.

There have been many approaches to the analysis of multiuser MIMO systems with a finite number of users and a ZF receiver [10][11][12][13][14][15]. The sum rate of three-dimensional MIMO uplink is analyzed in [10]. Several multiuser schedulings are analyzed for a system with the equal numbers of transmitting and receiving antennas [11]. Moreover, [12] compares single-user MIMO amd multi-user MIMO under the same antenna configuration. The authors in [13][14][15] then analyzed a system with an arbitrary number of transmitting and receiving antennas. Moreover, they considered maximal ratio combining (MRC) and minimum mean square error (MMSE) receivers [13], and the channel estimation error [14][15]. All the studies provide many useful analytical frameworks for and technical insights into the common system model of max-sum rate scheduling (which selects the user with the maximum signal-to-noise ratio (SNR)) in the homogeneous-SNR scenario, where each user has the same average SNR as shown in **Fig. 1**. However, real multiuser MIMO systems usually have heterogeneous SNRs (where the user SNRs are not equal) rather than a homogeneous SNR. Moreover, max-sum rate scheduling is not desirable in the case of heterogeneous SNRs because it can only benefit some users with high average SNR; i.e., fairness is not guaranteed. The authors in [15] mention the normalization approach to scheduling in heterogeneous scenarios but they do not analyze the scheduling performance in detail. Thus, an in-depth study on the sum throughput of a multiuser MIMO system with a proportional-fairness guaranteed scheduling in the heterogeneous SNR scenario is desirable.

1.1 Contributions and Organization

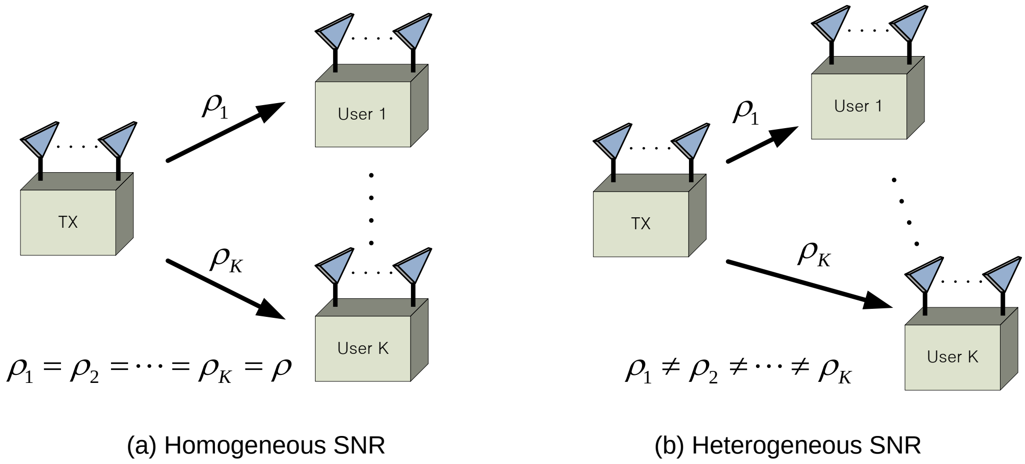

Fig. 1. Homogeneous and heterogeneous scenario. The average SNR of k -th user is denoted by ρ_k

In the heterogeneous-SNR scenario, we develop a new analytical expression for the sum throughput offered by a multiuser MIMO with a ZF receiver. Spatially uncorrelated channels are assumed. A multiuser MIMO transmitter constructs orthonormal beams and conducts a scheduling where the transmitter sends data to users with the highest normalized SNR, i.e., the ratio of the SNR to the mean SNR for each beam. Since the scheduling satisfies the proportional fairness principle, we call this scheduling SNR-based proportional fairness (S-PF) scheduling.

From the analytical expression, two new lemmas are found. First, in the homogeneous -SNR case, S-PF scheduling and max-rate scheduling yield the same sum throughput. In other words, S-PF scheduling maximizes the sum throughput. Second, in the high SNR regime, S-PF scheduling for finite heterogeneous SNRs yields the same multiuser diversity gain compared with max-rate scheduling (or S-PF scheduling) for homogeneous SNRs.

Simulation results are also presented in order to verify the analytical approach for the sum throughput of multiuser MIMO systems. The analysis results show that the sum throughput of multiuser MIMO systems is not always proportional to M for a finite number of users, which does not agree with the asymptotic capacity results of previous studies [3][4]. Furthermore, the analysis shows that the number of transmitting antennas can be optimized to maximize the average sum throughput depending on the number of users, their average SNRs, and the number of receive antennas.

The rest of this paper is organized as follows. In Section 2, we describe the system model for the multiuser MIMO system with a ZF receiver. Section 3 presents S-PF scheduling details and analyzes the sum throughput of S-PF scheduling. We give numerical results in Section 4 and present concluding remarks in Section 5.

2. System Model

We consider a typical point-to-multipoint wireless communication system comprising a transmitter with M antennas and K mobile receivers (us6ers) with $N(\geq M)$ antennas. A frequency flat block Rayleigh fading channel is applied to the channel model for each link between a transmitter and the receivers. For each time slot, each user feeds back the partial CSI of the SNRs of the M streams to the transmitter. The transmitter selects a collection of users according to a scheduling policy, and then simultaneously transmits M independent streams to the selected users. The received signals at N antennas of the k th receiver are expressed using an $N\times 1$ complex vector as

$$\mathbf{y}_k = \sqrt{\frac{PG_k}{M}}\mathbf{H}_k\mathbf{x} + \mathbf{w}_k, \tag{1}$$

where $\mathbf{x}$ is an $M\times 1$ complex vector with the covariance matrix $\mathbb{E}\left\{\mathbf{x}\mathbf{x}^H\right\} = \mathbf{I}_M$, where $\mathbb{E}\{\cdot\}$ is the expectation operation and $(\cdot)^H$ is the transpose conjugate. Each element of $\mathbf{x}$ represents the signal at the M transmitting antennas. P is the transmitted power and an equal power P/M is allocated to each transmitting antenna. The large-scale channel gain including the path loss and shadowing is denoted by G_k. $\mathbf{H}_k$ is an $N\times M$ complex matrix of channel coefficient between the transmitter and the k th receiver, whose entries are independent and identically distributed (iid) complex Gaussian random variables with zero-mean and unit variance on the assumption of a flat Rayleigh fading channel without spatial correlation. $\mathbf{w}_k$ is an $N\times 1$ zero-mean white Gaussian noise vector with the covariance matrix $\mathbb{E}\left\{\mathbf{w}\mathbf{w}^H\right\} = N_0\mathbf{I}_N$.

A multiuser MIMO system that constructs M orthonormal beams $\{\mathbf{v}_m\}_{m=1,\cdots,M}$ is considered. Here, z_m and $\mathbf{v}_m$ denote the m th symbol intended for a receiver and the corresponding unitary beamforming vector, respectively; i.e., $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_M]$ is a unitary matrix[1]. The transmitted signal is then

$$\mathbf{x} = \mathbf{V}\mathbf{z} = \sum_{m=1}^{M}\mathbf{v}_m z_m \ , \tag{2}$$

where $\mathbf{z} = [z_1\ z_2 \cdots z_M]^T$ and $(\cdot)^T$ is the transpose. We assume that the original symbols z_m are restored by a ZF receiver[2], where the receiver output is

$$\hat{\mathbf{z}} = \sqrt{\frac{M}{PG_k}}\mathbf{N}_k^{\dagger}\mathbf{y}_k = \mathbf{z} + \sqrt{\frac{M}{PG_k}}\mathbf{N}_k^{\dagger}\mathbf{w}_k \ , \tag{3}$$

[1] This means the per-antenna power constraint is assumed here. The unitary matrix does not hold the power constraints if the number of streams is not equal to M. Nevertheless, the unitary matrix simplifies the sum rate analysis and clarifies technical insights into S-PF scheduling. A general formulation of the per-antenna power constraint is well explained in [16].

[2] Although MMSE transceivers are better for the sum rate improvement [17],[18], they make SINR (and sum rate) expressions complex and intractable. The sum rate analysis for MMSE transceivers would be a topic for future investigation.

where $\mathbf{N}^{\dagger} = (\mathbf{N}^H\mathbf{N})^{-1}\mathbf{N}^H$ is the pseudo-inverse of $\mathbf{N}$, and $\mathbf{N}_k = \mathbf{H}_k\mathbf{V}$. The output symbol vector of a ZF receiver can be seen as the original symbol vector $\mathbf{z}$ plus the noise vector $\mathbf{q}_k = \sqrt{\frac{M}{PG_k}}\mathbf{N}_k^{\dagger}\mathbf{w}_k$. Since $\mathbb{E}\{\mathbf{z}\mathbf{z}^H\} = \mathbf{I}_M$ and $\mathbb{E}\{\mathbf{q}_k\mathbf{q}_k^H\} = \frac{MN_0}{PG_k}\left(\mathbf{N}_k^H\mathbf{N}_k\right)^{-1}$ (see Appendix 6.1 for details), we obtain that the SNR of the m th decoded stream at the k th user $\gamma_{k,m}$ is given by [19]

$$\gamma_{k,m} = \frac{\left[\mathbb{E}\{\mathbf{z}\mathbf{z}^H\}\right]_{m,m}}{\left[\mathbb{E}\{\mathbf{q}_k\mathbf{q}_k^H\}\right]_{m,m}} = \frac{\rho_k}{M\left[\left(\mathbf{N}_k^H\mathbf{N}_k\right)^{-1}\right]_{m,m}} = \rho_k\alpha_{k,m} \quad , \tag{4}$$

where $[\mathbf{A}]_{m,m}$ denotes the (m,m)-th entry of a matrix $\mathbf{A}$ and $\rho_k = \frac{PG_k}{N_0}$ denotes the average SNR of the k th user. The probability density function (PDF) of $\gamma_{k,m}$ is given by Theorem 1 in [19]:

$$f_{\gamma_{k,m}}(x) = \frac{\sigma_{k,m}^2 M e^{-\frac{\sigma_{k,m}^2 M}{\rho_k}x}}{\rho_k (N-M)!}\left(\frac{\sigma_{k,m}^2 M}{\rho_k}x\right)^{N-M}, \tag{5}$$

where $\sigma_{k,m}^2$ is the m th diagonal entry of $\left(\mathbb{E}\{\mathbf{N}_k^H\mathbf{N}_k\}\right)^{-1}$. Since we assume a spatially uncorrelated channel and unitary beamforming, we obtain

$$\begin{aligned}\mathbb{E}\{\mathbf{N}_k^H\mathbf{N}_k\} &= \mathbb{E}\{\mathbf{V}^H\mathbf{H}_k^H\mathbf{H}_k\mathbf{V}\} \\ &= \mathbf{V}^H\mathbb{E}\{\mathbf{H}_k^H\mathbf{H}_k\}\mathbf{V} \\ &= \mathbf{V}^H\mathbf{V} \\ &= \mathbf{I}_M \ .\end{aligned} \tag{6}$$

Thus, $\sigma_{k,m}^2 = 1$ for all and m and k. The symbol $\alpha_{k,m}$ denotes the instantaneous channel gain on the m th stream of the k th receiver, whose PDF is

$$\begin{aligned}f_{\alpha_{k,m}}(x) &\overset{(a)}{=} \rho_k f_{\gamma_{k,m}}(\rho_k x) \\ &= \frac{Me^{-Mx}}{(N-M)!}(Mx)^{N-M},\end{aligned} \tag{7}$$

where (a) follows from the fact that the PDF of $Y = aX$, $a \ge 0$ for random variables X and Y is $f_Y(y) = \frac{1}{a} f_X\left(\frac{y}{a}\right)$.

3. Average Sum Throughput

3.1 General Case

We derive an analytical expression for the average sum throughput achieved by a multiuser MIMO system with S-PF scheduling in the heterogeneous-SNR scenario. The scheduler sends a packet to the k_m^* th user with the largest ratio of the SNR to the average SNR for the m th stream transmitted via $\mathbf{v}_m$ at each time slot. The multiuser diversity gain is thus achieved from a selection of the users in relatively better channel conditions regardless of the average SNR. This also achieves a certain degree of fairness because the user selection is independent of the average SNR, i.e., the path loss and shadowing due to the distance and obstacles between a transmitter and the receivers. The scheduling selects the users that satisfy the following equation:

$$k_m^* = \arg \max_{k \in \{1,2,\cdots,K\}} \frac{\gamma_{k,m}}{\rho_k}, \;\; m = 1,\cdots,M. \tag{8}$$

Substituting (4) in (8) yields

$$k_m^* = \arg \max_{k \in \{1,2,\cdots,K\}} \alpha_{k,m}, \;\; m = 1,\cdots,M \;, \tag{9}$$

which implies that the user selection is conducted on the basis of the instantaneous channel gain only. On the reasonable assumption that the instantaneous channel gain of all users is the same, every user has equal service (access) time guaranteeing the proportional fairness principle. Note that proportional fairness (PF) scheduling [20] adopts a user selection scheme that is similar that adopted in (8). PF scheduling selects the user with the largest ratio of the instantaneous data rate to the mean data rate, and thus it offers identical access time for users. Thus, S-PF scheduling in (8) can be regarded as another type of PF scheduling.

The SNR of the user scheduled for the m th stream γ_m (which can also be denoted by $\gamma_{k_m^*,m}$ when using the SNR symbol $\gamma_{k,m}$) is expressed as a product of two independent random variables

$$\gamma_m = \rho \alpha_m \,, \tag{10}$$

where α_m (which can be also denoted by $\alpha_{k_m^*,m}$ when using the SNR symbol $\alpha_{k,m}$) is the normalized channel gain at the user scheduled for the m th stream using the rule in (9), and ρ is the average SNR of the scheduled user, which is unrelated to m. On the generall assumption of identical instantaneous (small-scale fading) channel statistics over users, the random variable ρ is identical in terms of statistics to a sample mean of the average SNR of K users $\{\rho_k\}_{k=1,\cdots,K}$, because every user has equal access time. Thus, the PDF of ρ is given by

$$f_\rho(x) = \frac{1}{K}\sum_{k=1}^{K}\delta\left(x-\rho_k\right), \tag{11}$$

where $\delta(\cdot)$ denotes Dirac's delta function. According to the selection rule in (9), the normalized channel gain of the scheduled user α_m is given by $\alpha_m = \max\left\{\alpha_{1,m},\cdots,\alpha_{K,m}\right\}$. Since the PDF $f_{\alpha_{k,m}}(\cdot)$ are identical for all users in (7), the cumulative distribution function (CDF) α_m is given by the largest order statistics [21]:

$$\begin{aligned} F_{\alpha_m}(x) &= P\left[\alpha_m < x\right] \\ &= P\left[\max\left\{\alpha_{1,m},\cdots,\alpha_{K,m}\right\} < x\right] \\ &= \prod_{k=1}^{K} P\left[\alpha_{k,m} < x\right] \\ &= P\left[\alpha_{k,m} < x\right]^K \\ &= F_{\alpha_{k,m}}(x)^K \\ &= \left(\frac{\Gamma^{(l)}\left(1+N-M, Mx\right)}{(N-M)!}\right)^K, \end{aligned} \tag{12}$$

where $F_{\alpha_{k,m}}(\cdot)$ denotes the CDF of $\alpha_{k,m}$ and $\Gamma^{(l)}(a,x) = \int_0^x t^{(a-1)}\, e^{-t}\, dt$ denotes the lower incomplete gamma function. Thus the PDF of α_m is given as

$$\begin{aligned} f_{\alpha_m}(x) &= \frac{dF_{\alpha_m}(x)}{dx} \\ &\overset{(a)}{=} \frac{dF_{\alpha_{k,m}}(x)^K}{dx} \\ &= Kf_{\alpha_{k,m}}(x)\left(F_{\alpha_{k,m}}(x)\right)^{K-1} \\ &= K\frac{Me^{-x}}{(N-M)!}\left(Mx\right)^{N-M}\left(\frac{\Gamma^{(l)}\left(1+N-M, Mx\right)}{(N-M)!}\right)^{K-1}. \end{aligned} \tag{13}$$

This equation shows the PDF of the maximum random variable for a given K iid random variables. Combining (10), (11), and (13), we finally obtain the PDF of γ_m as

$$\begin{aligned} f_{\gamma_m}(\gamma) &\overset{(a)}{=} \int_{-\infty}^{\infty} \frac{1}{|x|} f_\rho(x)\, f_{\alpha_m}\left(\frac{\gamma}{x}\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{|x|} \frac{1}{K} \sum_{k=1}^{K} \delta(x-\rho_k) f_{\alpha_m}\left(\frac{\gamma}{x}\right) dx \\ &= \frac{1}{K} \sum_{k=1}^{K} \int_{-\infty}^{\infty} \frac{1}{|x|} \delta(x-\rho_k) f_{\alpha_m}\left(\frac{\gamma}{x}\right) dx \\ &= \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\rho_k} f_{\alpha_m}\left(\frac{\gamma}{\rho_k}\right) \int_{-\infty}^{\infty} \delta(x-\rho_k) dx \\ &\overset{(b)}{=} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{\rho_k} f_{\alpha_m}\left(\frac{\gamma}{\rho_k}\right), \end{aligned} \tag{14}$$

where (a) is given from Rohatgi's well-known results [22]: the pdf of $Z = XY$, where X and Y are independent, is $f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x)\, f_Y\left(\frac{z}{x}\right) dx$. (b) follows from the definition of Dirac's delta function: $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$. Since $f_{\alpha_m}(x)$ is identical for all m in (13), $f_{\gamma_m}(\gamma)$ is also the same for all m, i.e. $\left\{ f_{\gamma_m}(\gamma) \right\}_{m=1,\cdots,M} = f_\gamma(\gamma)$, from which (a) of the following equation is given.

The final expression of the sum throughput of the S-PF scheduler is given by

$$\begin{aligned} C_{\text{S-PF}}^{\text{HE}} &= \int_0^{\infty} \sum_{m=1}^{M} \log_2(1+\gamma)\, f_{\gamma_m}(\gamma)\, d\gamma \\ &\overset{(a)}{=} M \int_0^{\infty} \log_2(1+\gamma)\, f_\gamma(\gamma)\, d\gamma \\ &= \frac{M}{(N-M)!} \sum_{k=1}^{K} \frac{1}{\rho_k} \int_0^{\infty} \log_2(1+\gamma) e^{-\frac{M\gamma}{\rho_k}} \left(\frac{M\gamma}{\rho_k}\right)^{N-M} \left(\frac{\Gamma^{(l)}\left(1+N-M, \frac{M\gamma}{\rho_k}\right)}{(N-M)!} \right)^{K-1} d\gamma \\ &\overset{(b)}{=} \frac{M}{\left((N-M)!\right)^K} \sum_{k=1}^{K} \int_0^{\infty} \log_2\left(1+\frac{\rho_k}{M} t\right) e^{-t} t^{N-M} \Gamma^{(l)}(1+N-M, t)^{K-1}\, dt, \end{aligned} \tag{15}$$

where (b) follows from the change of variables $t = \gamma / \rho_k$.

3.2 Special Case: Homogeneous SNR

We derive the sum throughput in the homogeneous-SNR scenario where each user has the same average SNR, $\rho_k = \rho$ for all k. Substituting $\rho_k = \rho$ into (15), we obtain

$$C_{\text{S-PF}}^{\text{HO}} = \frac{KM}{\left((N-M)!\right)^K} \int_0^\infty \log_2\left(1+\frac{\rho}{M}t\right) e^{-t} t^{N-M} \Gamma^{(l)}\left(1+N-M,t\right)^{K-1} dt . \quad (16)$$

We next directly derive the sum throughput of the max-rate scheduling in the homogeneous-SNR case. Substituting $\rho_k = \rho$ into the SNR PDF expression in (5), we obtain the SNR PDF in the homogeneous-SNR case as

$$f_{\tilde{\gamma}_{k,m}}(x) = \frac{M e^{-\frac{M}{\rho}x}}{\rho (N-M)!} \left(\frac{M}{\rho}x\right)^{N-M}, \quad (17)$$

whose CDF is

$$\begin{aligned} F_{\tilde{\gamma}_{k,m}}(x) &\overset{(a)}{=} 1 - e^{-\frac{M}{\rho}x} \sum_{s=0}^{N-M} \frac{\left(\frac{M}{\rho}x\right)^s}{s!} \\ &= \frac{1}{(N-M)!}\left((N-M)! - (N-M)!\, e^{-\frac{M}{\rho}x} \sum_{s=0}^{N-M} \frac{\left(\frac{M}{\rho}x\right)^s}{s!}\right) \\ &\overset{(b)}{=} \frac{1}{(N-M)!}\left(\Gamma\left(N-M+1\right) - \Gamma^{(u)}\left(N-M+1, \tfrac{M}{\rho}x\right)\right) \\ &\overset{(c)}{=} \frac{\Gamma^{(l)}\left(N-M+1, \tfrac{M}{\rho}x\right)}{(N-M)!}, \end{aligned} \quad (18)$$

where (a) follows from [13, Eq.(7)] and (b) is obtained from the gamma function $\Gamma(a) = (a-1)!$ for an integer a and the upper incomplete gamma function $\Gamma^{(u)}(a,x) = (a-1)!\, e^{-x} \sum_{s=0}^{a-1} \frac{x^s}{s!}$ for an integer a [23]. (c) is obtain from $\Gamma(a) = \Gamma^{(l)}(a,x) + \Gamma^{(u)}(b,x)$ [23]. The output SNR of max-rate scheduling is

$$\gamma_m^{\max} = \max_{k \in \{1,2,\cdots,K\}} \tilde{\gamma}_{k,m}, \quad m = 1,\cdots,M \ . \quad (19)$$

Since the SNR PDF given in (17) is identical for all k and m, the output SNR $\gamma_m^{\max} = \gamma^{\max}$ is for all m, from which (a) in the following equation is given. The sum throughput of max-rate scheduling is given as

$$\begin{aligned} C_{\text{max-rate}}^{\text{HO}} &\overset{(a)}{=} MK\int_0^\infty \log_2(1+r) f_{\gamma_{k,m}}(r)\left(F_{\gamma_{k,m}}(r)\right)^{K-1} dr, \\ &\overset{(b)}{=} \frac{KM}{\left((N-M)!\right)^K}\int_0^\infty \log_2(1+r)\tfrac{M}{\rho} e^{-\frac{M}{\rho}r}\left(\tfrac{M}{\rho}r\right)^{N-M}\Gamma^{(l)}\left(1+N-M,\tfrac{M}{\rho}r\right)^{K-1} dr \\ &\overset{(c)}{=} \frac{KM}{\left((N-M)!\right)^K}\int_0^\infty \log_2\left(1+\tfrac{\rho}{M}t\right)e^{-t}t^{N-M}\Gamma^{(l)}\left(1+N-M,t\right)^{K-1} dt, \end{aligned} \tag{20}$$

where (b) is given by subtituting (17) and (18), and (c) follows from the change of variables $t=\frac{M}{\rho}r$.

It is notable that the sum throughput expressions of S-PF scheduling (given in (16)) and max-rate scheduling (given in (20)) are identical. This result confirms the following lemma.

Lemma 1: *In the homogeneous-SNR case, S-PF scheduling and max-rate scheduling yield the same sum throughput. In other words, S-PF scheduling maximizes the sum throughput.*

3.3 Special Case: Equal Number of Antennas

We assume that a transmitter and receiver have the same number of antennas: $N=M$. The throughput expression of (15) can be rewritten as

$$\begin{aligned} C_{\text{S-PF}}^{\text{HE}} &= M\sum_{k=1}^{K}\int_0^\infty \log_2\left(1+\frac{\rho_k}{M}t\right)e^{-t}\left(1-e^{-t}\right)^{K-1}dt \\ &\overset{(a)}{=} M\sum_{j=0}^{K-1}\binom{K-1}{j}(-1)^j\sum_{k=1}^{K}\int_0^\infty \log_2\left(1+\frac{\rho_k}{M}t\right)e^{-(j+1)t}dt \\ &\overset{(b)}{=} \frac{M}{\ln 2}\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{j+1}\sum_{k=1}^{K}e^{\frac{M(j+1)}{\rho_k}}\text{Ei}\left(\frac{M(j+1)}{\rho_k}\right), \end{aligned} \tag{21}$$

where (a) is obtained by the binomial series

$$\begin{aligned} \left(1-e^{-t}\right)^{K-1}e^{-t} &= \sum_{j=0}^{K-1}\binom{K-1}{j}(-1)^j e^{-jt}e^{-t} \\ &= \sum_{j=0}^{K-1}\binom{K-1}{j}(-1)^j e^{-(j+1)t} \end{aligned} \tag{22}$$

and (b) follows from the integral equality [24]: $\int_0^\infty \ln(1+at)e^{-bt}dt = e^{b/a}\text{Ei}(b/a)/b$, where $\mathrm{Ei}(x)=\int_x^{\infty} e^{-t}t^{-1}dt$ is the exponential integral function.

We next derive the sum throughput for the case of an equal number of transmitting and receiving antennas in the homogeneous-SNR scenario. Setting $\rho_k=\rho$ in (21), we obtain

$$C_{\text{S-PF}}^{\text{HO}}=C_{\text{max-rate}}^{\text{HO}}=\frac{KM}{\ln 2}\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{j+1}e^{\frac{M(j+1)}{\rho}}\mathrm{E}_1\left(\frac{M(j+1)}{\rho}\right). \tag{23}$$

This is the same as the results for max-rate scheduling for homogeneous SNRs given in [11, Eq. (43)] and in [12, Eq. (6)], which also confirms Lemma 1.

3.4 Special Case: Equal Number of Antennas and High SNR

We assume a high average SNR, i.e., $\rho_k>>1$, and thus the relation $\log(1+\frac{\rho_k}{M}t)=\log(\frac{\rho_k}{M}t)$ holds. The throughput expression of (15) can be then rewritten as

$$\begin{aligned} C_{\text{S-PF}}^{\text{HE}} &= M\sum_{j=0}^{K-1}\binom{K-1}{j}(-1)^j\sum_{k=1}^{K}\int_0^{\infty}\log_2\left(\frac{\rho_k}{M}t\right)e^{-(j+1)t}dt \\ &\overset{(a)}{=}\frac{M}{\ln 2}\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{j+1}\sum_{k=1}^{K}\left(-\ln\left(\frac{M(j+1)}{\rho_k}\right)-\varepsilon\right), \end{aligned} \tag{24}$$

where $\varepsilon=0.5772\ldots$ is the Euler constant.

Next, we derive an interesting result that is one of main contributions of this paper. By applying $\{\rho_k\}_{k=1,\cdots K}=\rho$ to (24), the sum throughput for homogeneous SNRs is given as

$$C_{\text{S-PF}}^{\text{HO}}=C_{\text{max-rate}}^{\text{HO}}=\frac{KM}{\ln 2}\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{j+1}\left(-\ln\left(\frac{M(j+1)}{\rho}\right)-\varepsilon\right). \tag{25}$$

From (24) and (25), we obtain the following lemma.

Lemma 2: *Under high SNR and a large number of users, S-PF scheduling for finite heterogeneous SNRs yields the same multiuser diversity gain compared with max-rate scheduling (or S-PF scheduling) for homogeneous SNRs*

Proof: See Appendix 6.2.

In other words, Lemma 2 indicates that S-PF scheduling provides the same multiuser diversity gain for both homogeneous and heterogeneous SNRs.

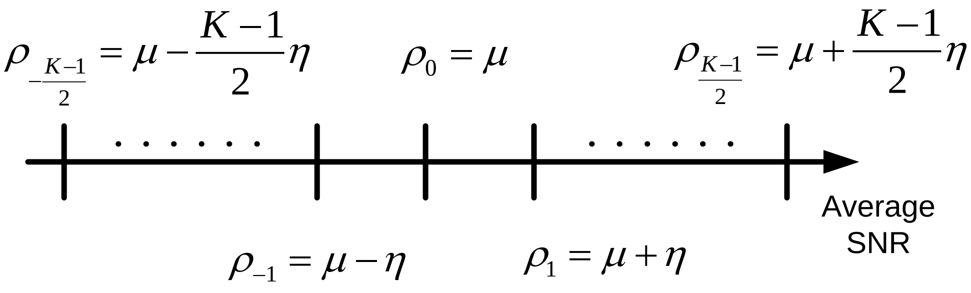

Fig. 2. Average SNR distribution.

4. Numerical Results

In this section, the sum throughput of the multiuser MIMO system is numerically evaluated. Here, we assume that ρ_k is equally distributed with interval η and mean μ, and an odd number of users K as shown in **Fig. 2**. The average SNR of K users can then be represented as

$$\rho_k = \mu + k\eta, \text{ for } k = 0, \pm 1, \cdots, \pm \tfrac{K-1}{2}, \tag{26}$$

from which the standard deviation of ρ_k is computed as

$$\begin{aligned} \sigma &= \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left(\rho_k - \mu\right)^2} \\ &= \sqrt{\frac{1}{K}\left(\sum_{k=1}^{\frac{K-1}{2}} 2\left(\mu + k\eta - \mu\right)^2\right)} \\ &= \eta\sqrt{\frac{2}{K}\frac{\left(\frac{K-1}{2}\right)\left(\frac{K+1}{2}\right)K}{6}} = \eta\sqrt{\frac{K^2-1}{12}}. \end{aligned} \tag{27}$$

By combining (26) with (27), the averaged SNR can be finally rewritten as a function of μ, σ, and K:

$$\rho_k = \mu + k\sigma\sqrt{\frac{12}{K^2-1}}, k = 0, \pm 1, \cdots, \pm\frac{K-1}{2}, \tag{28}$$

The receiver position above follows the uniform distribution, which could be fairly reasonable. Other types of random distribution could be handled in a straightforward manner.

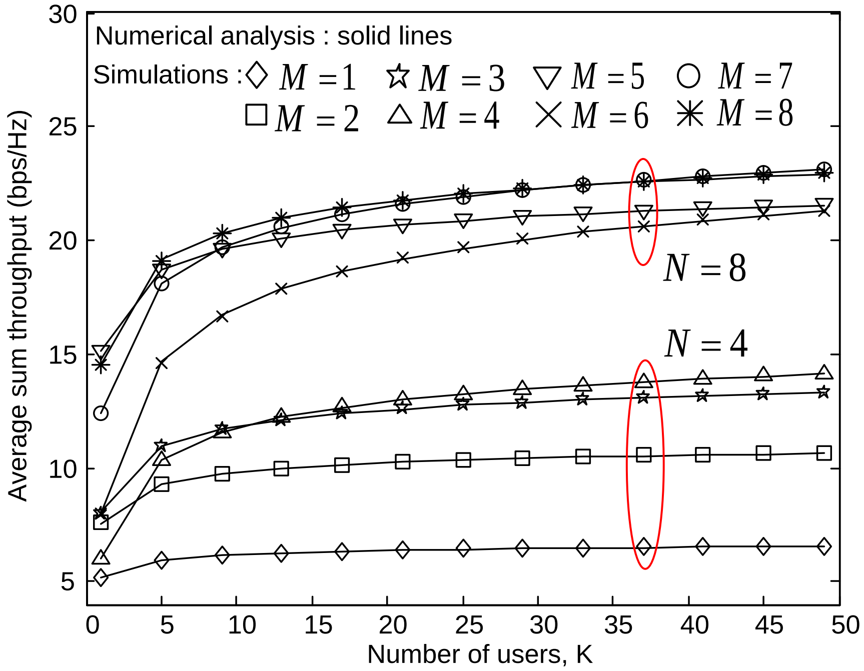

Fig. 3. Average sum throughput dependence on to M and N when $\mu = 10.0$ and $\sigma = 2.0$.

The sum throughput is obtained by averaging the sum throughput for each drop. A MIMO channel matrix and a noise vector are randomly generated in every time slot for each drop, from which the sum throughput per drop is computed. In this simulation, 10,000 time slots and more than 1000 drops are considered. For a channel realization, K MIMO channel matrices with uncorrelated complex Gaussian entries are generated.

Fig. 3 plots the sum throughput obtained by simulation and by an analytical expression in (15) versus the number of users K. Here, μ and σ are set to 10.0 and 2.0, respectively. For a given values of M and N, the analytical curves correspond with the simulation curves, which confirms the accuracy of the proposed analytical approach. The results show that a larger M increases the sum throughput for a larger number of users. On the other hand, the results also show that the sum throughput is not always proportional to M for a smaller number of users. Specifically, the sum throughput for $M = 5$ is higher than that for $M = 8$ when $K \leq 49$. For a small number of users, where the multiuser diversity gain is limited, the decrease in the average SNR per beam due to an increase in M leads to a loss in the average sum throughput. This relationship between the average SNR per beam and the optimal M achieving the maximum average sum throughput is demonstrated by the following results.

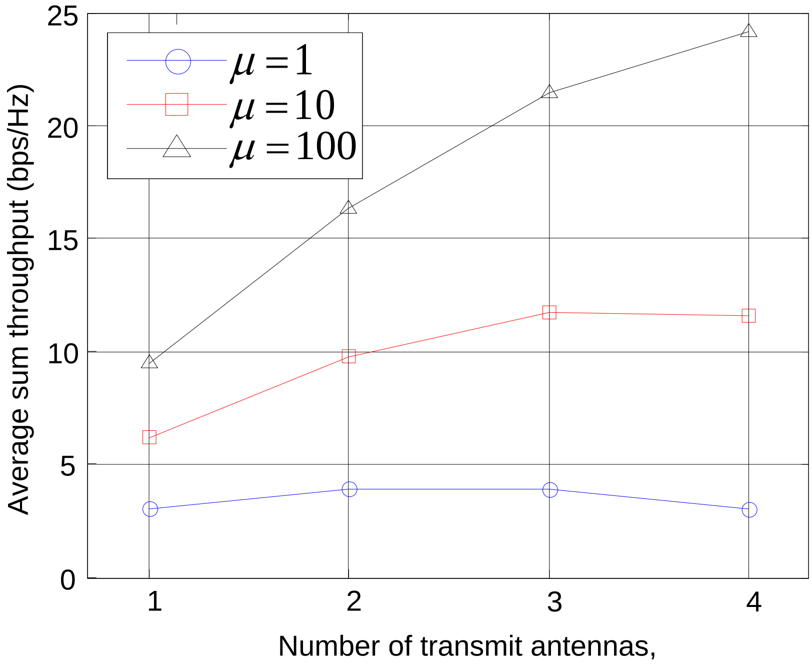

Fig. 4. Average sum throughput according to μ and M when $N = 4$ and $K = 9$.

Fig. 4 shows the effect of $M(\leq N)$ and μ on the sum throughput, especially for a small number of users. For $\mu = 100$, the result shows that the sum throughput increases as M increases; that is, the maximum sum throughput is achieved when $M = 4$, which is the famous result given in [3]. For a lower value of μ, however, it is shown that the maximum sum throughput is achieved with a smaller value of M. Specifically, the maximum sum throughputs for $\mu = 10$ and $\mu = 1$ are achieved when $M = 3$ and $M = 2$, respectively. This is because at lower μ, an increase in average SNR due to a smaller M is increasingly superior to a decrease in the multiuser diversity gain due to a smaller K. Consequently, an adaptive selection of the number of antennas could increase the sum throughput for the case of a small number of users.

The results that lower number of transmit antennas yields higher sum throughput, shown in **Fig 4** and **Fig 5**, is also explained by the following simple mathematical approach. The sum throughput can be roughly expressend as $C \approx M \log_2\left(\frac{\mu}{M}\gamma(K)\right)$ where $\gamma(K)$ is the normalized SNR of scheduled user and a function of the number of users K. We now define the probability of the event that lower M yields higher C as

$$\Pr\left[M_1 \log_2\left(\frac{\mu}{M_1}\gamma(K)\right) > M_2 \log_2\left(\frac{\mu}{M_2}\gamma(K)\right)\right], \ \ for\ M_1 < M_2$$

This probability can be rewritten as

$$
\begin{aligned}
&\Pr\left[\left(\frac{\mu}{M_1}\gamma(K)\right)^{M_1} > \left(\frac{\mu}{M_2}\gamma(K)\right)^{M_2}\right] \\
&= \Pr\left[\frac{\gamma(K)^{M_1}}{\gamma(K)^{M_2}} > \frac{M_1{}^{M_1}}{M_2{}^{M_2}}\frac{\mu^{M_2}}{\mu^{M_1}}\right] \\
&= \Pr\left[\frac{\gamma(K)^{M_1}}{\gamma(K)^{M_1}\gamma(K)^{M_2-M_1}} > \frac{M_1{}^{M_1}}{M_2{}^{M_1}M_2{}^{M_2-M_1}}\frac{\mu^{M_1}\mu^{M_2-M_1}}{\mu^{M_1}}\right] \\
&= \Pr\left[\gamma(K)^{M_2-M_1} < \left(\frac{M_2}{M_1}\right)^{M_1}\left(\frac{M_2}{\mu}\right)^{M_2-M_1}\right] \\
&= \Pr\left[\gamma(K) < \left(\frac{M_2}{M_1}\right)^{\frac{M_1}{M_2-M_1}}\frac{M_2}{\mu}\right]
\end{aligned} \tag{29}
$$

This equation indicates that the event occurs more frequently for lower SNR μ or lower number of users K (lower K yields lower $\gamma(K)$). For example, $(M_2/M_1)^{\frac{M_1}{M_2-M_1}}(M_2/\mu)=9.48$ for $\mu=1, M_1=3, M_2=4$, and thus the event that $\gamma(K)<9.48$ is perfectly possible; especially for smaller K.

5. Conclusions

We developed a new analytical framework for evaluating the sum throughput of a multiuser MIMO system with S-PF scheduling using the PF principle. The framework is based on the realistic assumption that each user has a different average SNR, i.e., different large-scale fading such as path loss and shadowing. Moreover, the effect of S-PF scheduling for a finite number of users is captured by the framework. The analytical expressions are remarkably close to the simulation results. S-PF scheduling maximizes the sum throughput for homogeneous SNRs and provides the same multiuser diversity gain for both heterogeneous SNRs and homogeneous SNRs. It is noteworthy that the sum throughput does not always increase with the number of transmitting antennas for a small number of users. This implies that we need to adjust (reduce) the number of transmitting antennas depending on the number of users and receiving antennas, or the average SNR over all users. Future extensions of this approach could include spatially correlated channels, inter-cell interference, or other types of receivers.

6. Appendix

6.1 The covariance matrix of the signal and noise vector

The covariance matrix of the signal vector is

$$\begin{aligned}\mathbb{E}\left\{\mathbf{z}\mathbf{z}^H\right\} &= \mathbb{E}\left\{\mathbf{V}^H\mathbf{x}\mathbf{x}^H\mathbf{V}\right\} \\ &= \mathbf{V}^H\mathbb{E}\left\{\mathbf{x}\mathbf{x}^H\right\}\mathbf{V} \\ &= \mathbf{I}_M \ .\end{aligned} \tag{29}$$

The covariance matrix of the noise vector is

$$\begin{aligned}\mathbb{E}\left\{\mathbf{q}\mathbf{q}_k^H\right\} &= \frac{M}{PG_k}\mathbb{E}\left\{\left(\mathbf{N}_k^\dagger\right)^H \mathbf{w}\mathbf{w}^H\mathbf{N}_k^\dagger\right\} \\ &= \frac{M}{PG_k}\left(\mathbf{N}_k^\dagger\right)^H \mathbb{E}\left\{\mathbf{x}\mathbf{x}^H\right\}\mathbf{N}_k^\dagger \\ &= \frac{M}{PG_k}\left(\mathbf{N}_k^\dagger\right)^H \mathbf{N}_k^\dagger \\ &= \frac{M}{PG_k}\left(\left(\mathbf{N}_k^H\mathbf{N}_k\right)^{-1}\mathbf{N}_k^H\right)^H \left(\mathbf{N}_k^H\mathbf{N}_k\right)^{-1}\mathbf{N}_k^H \\ &\overset{(a)}{=} \frac{M}{PG_k}\mathbf{N}_k\mathbf{N}_k^{-1}\left(\mathbf{N}_k^H\right)^{-1}\mathbf{N}_k^{-1}\left(\mathbf{N}_k^H\right)^{-1}\mathbf{N}_k^H \\ &= \frac{M}{PG_k}\left(\mathbf{N}_k^H\right)^{-1}\mathbf{N}_k^{-1} \\ &= \frac{M}{PG_k}\left(\mathbf{N}_k\mathbf{N}_k^H\right)^{-1},\end{aligned} \tag{30}$$

where (a) is given from the matrix operations $(\mathbf{AB})^H = \mathbf{B}^H\mathbf{A}^H$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, and $\left((\mathbf{A})^{-1}\right)^H = \left((\mathbf{A})^H\right)^{-1}$.

6.2 Proof of Lemma 2

By using the maximum SNR $\rho_{\max}$ and the minimum SNR $\rho_{\min}$, $\rho_{\min} \le \rho_k \le \rho_{\max}$, the sum throughput for heterogeneous SNRs in (24) is bounded as

$$\begin{aligned}B_L &= \frac{KM}{\ln 2}\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{j+1}\left(-\ln\left(\frac{M(j+1)}{\rho_{\min}}\right)-\varepsilon\right) \le \\ &C_{\text{S-PF}}^{\text{HE}} \le B_U = \frac{KM}{\ln 2}\sum_{j=0}^{K-1}\binom{K-1}{j}\frac{(-1)^j}{j+1}\left(-\ln\left(\frac{M(j+1)}{\rho_{\max}}\right)-\varepsilon\right).\end{aligned} \tag{31}$$

From (25) and (31), we obtain

$$\begin{aligned} B_L - C_{\text{max-rate}}^{\text{HO}} &= \frac{KM}{\ln 2} \sum_{j=0}^{K-1} \binom{K-1}{j} \frac{(-1)^j}{j+1} \left(-\ln\left(\frac{M(j+1)}{\rho_{\min}} \right) + \ln\left(\frac{M(j+1)}{\rho} \right) \right) \\ &= KM \log_2 \frac{\rho_{\min}}{\rho} \sum_{j=0}^{K-1} \binom{K-1}{j} \frac{(-1)^j}{j+1} \\ &= M \log_2 \frac{\rho_{\min}}{\rho}, \end{aligned} \tag{32}$$

where we use $\frac{1}{K} = \sum_{j=0}^{K-1} \binom{K-1}{j} \frac{(-1)^j}{j+1}$. Applying the same way, we obtain

$$B_U - C_{\text{max-rate}}^{\text{HO}} = M \log_2 \frac{\rho_{\max}}{\rho} . \tag{33}$$

Substituting (32) and (33) into (31), we obtain

$$C_{\text{max-rate}}^{\text{HO}} + M \log_2 \left(\frac{\rho_{\min}}{\rho} \right) \le C_{\text{S-PF}}^{\text{HE}} \le C_{\text{max-rate}}^{\text{HO}} + M \log_2 \left(\frac{\rho_{\max}}{\rho} \right) \tag{34}$$

The bounds above are not independent of the number of users K. This means for large values of K, the rate of increase in the sum throughput due to increasing K is the same for both heterogeneous SNRs and homogeneous SNRs, i.e., the same multiuser diversity gain is obtained.

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