J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.
http://dx.doi.org/10.7468/jksmeb.2014.21.3.183
Volume 21, Number 3 (August 2014), Pages 183–194

ISSN(Print) 1226-0657
ISSN(Online) 2287-6081

STRUCTURES INDUCED BY ALEXANDROV FUZZY TOPOLOGIES

Yong Chan Kim

Abstract. In this paper, we investigate the properties of Alexandrov fuzzy topologies and meet-join approximation operators. We study fuzzy preorder, Alexandrov topologies and meet-join approximation operators induced by Alexandrov fuzzy topologies. We give their examples.

1. Introduction

Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Höhle [3] introduced L-fuzzy topologies and L-fuzzy interior operators on complete residuated lattices. Pawlak [8, 9] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Radzikowska [10] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Zhang [6, 7] introduced Alexandrov L-topologies induced by fuzzy rough sets. Kim [5] investigated the properties of Alexandrov topologies in complete residuated lattices.

In this paper, we investigate the properties of Alexandrov fuzzy topologies and meet-join approximation operators in a sense as Höhle [3]. We study fuzzy preorder, Alexandrov topologies and meet-join approximation operators induced by Alexandrov fuzzy topologies. We give their examples.

2. Preliminaries

Definition 2.1 ([1-3])**.** A structure $(L, \vee, \wedge, \odot, \rightarrow, \perp, \top)$ is called a *complete residuated lattice* iff it satisfies the following properties:

Received by the editors March 26, 2014. Accepted June 3, 2014.
2010 *Mathematics Subject Classification.* 03E72, 03G10, 06A15, 54F05.
Key words and phrases. complete residuated lattices, fuzzy preorder, meet-join approximation operators, Alexandrov (fuzzy) topologies.

(L1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice where $\bot$ is the bottom element and $\top$ is the top element;

(L2) $(L, \odot, \top)$ is a monoid;

(L3) It has an adjointness,i.e.

$$x \le y \to z \text{ iff } x \odot y \le z.$$

An operator $^* : L \to L$ defined by $a^* = a \to \bot$ is called *strong negations* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases} \quad \top_x^*(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2 ([6, 7])**.** Let X be a set. A function $e_X : X \times X \to L$ is called a *fuzzy preorder* if it satisfies the following conditions

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \le e_X(x, z)$, for all $x, y, z \in X$'

Example 2.3. (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then e_L is a fuzzy preorder on L.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then e_{L^X} is a fuzzy preorder from Lemma 2.4 (9).

Lemma 2.4 ([1, 2])**.** *Let $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.*

(1) *If $y \le z$, then $x \odot y \le x \odot z$.*

(2) *If $y \le z$, then $x \to y \le x \to z$ and $z \to x \le y \to x$.*

(3) *$x \to y = \top$ iff $x \le y$.*

(4) *$x \to \top = \top$ and $\top \to x = x$.*

(5) *$x \odot y \le x \wedge y$.*

(6) *$x \odot (\bigvee_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma}(x \odot y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \odot y = \bigvee_{i \in \Gamma}(x_i \odot y)$.*

(7) *$x \to (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \to y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \to y = \bigwedge_{i \in \Gamma}(x_i \to y)$.*

(8) *$\bigvee_{i \in \Gamma} x_i \to \bigvee_{i \in \Gamma} y_i \ge \bigwedge_{i \in \Gamma}(x_i \to y_i)$ and $\bigwedge_{i \in \Gamma} x_i \to \bigwedge_{i \in \Gamma} y_i \ge \bigwedge_{i \in \Gamma}(x_i \to y_i)$.*

(9) *$(x \to y) \odot x \le y$ and $(y \to z) \odot (x \to y) \le (x \to z)$.*

(10) *$x \to y \le (y \to z) \to (x \to z)$ and $x \to y \le (z \to x) \to (z \to y)$.*

(11) *$\bigwedge_{i \in \Gamma} x_i^* = (\bigvee_{i \in \Gamma} x_i)^*$ and $\bigvee_{i \in \Gamma} x_i^* = (\bigwedge_{i \in \Gamma} x_i)^*$.*

(12) *$(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.*

(13) $x^* \to y^* = y \to x$ *and* $(x \to y)^* = x \odot y^*$.
(14) $y \to z \leq x \odot y \to x \odot z$.

Definition 2.5 ([5]). A map $\mathcal{M} : L^X \to L^Y$ is called an *meet-join approximation operator* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(M1) $\mathcal{M}(\alpha \to A) = \alpha \odot \mathcal{M}(A)$, where $(\alpha \to A)(x) = \alpha \to A(x)$ for each $x \in X$,
(M2) $\mathcal{M}(\bigwedge_{i \in I} A_i) = \bigvee_{i \in I} \mathcal{M}(A_i)$,
(M3) $A^* \leq \mathcal{M}(A)$,
(M4) $\mathcal{M}(\mathcal{M}^*(A)) \leq \mathcal{M}(A)$.

Definition 2.6 ([4]). An operator $\mathbf{T} : L^X \to L$ is called an *Alexandrov fuzzy topology* on X iff it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(T1) $\mathbf{T}(\alpha_X) = \top$, where $\alpha_X(x) = \alpha$ for each $x \in X$,
(T2) $\mathbf{T}(\bigwedge_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \mathbf{T}(A_i)$,
(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$, where $(\alpha \odot A)(x) = \alpha \odot A(x)$ for each $x \in X$,
(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

Definition 2.7 ([5]). A subset $\tau \subset L^X$ is called an *Alexandrov topology* if it satisfies satisfies the following conditions.

(O1) $\alpha_X \in \tau$.
(O2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i \in \Gamma} A_i, \bigwedge_{i \in \Gamma} A_i \in \tau$.
(O3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.
(O4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Remark 2.8. (1) If $\mathbf{T} : L^X \to L$ is an Alexandrov fuzzy topology. Define $\mathbf{T}^*(A) = \mathbf{T}(A^*)$. Then $\mathbf{T}^*$ is an Alexandrov fuzzy topology.

(2) If $\mathbf{T}$ be an Alexandrov fuzzy topology on X, $\tau_T^r = \{A \in L^X \mid \mathbf{T}(A) \geq r\}$ is an Alexandrov topology on X and $\tau_T^r \subset \tau_T^s$ for $s \leq r \in L$.

3. Structures Induced by Alexandrov Fuzzy Topologies

Theorem 3.1. *If $\mathcal{M}$ is a meet-join approximation operator, then $\tau_{\mathcal{M}} = \{A \in L^X \mid \mathcal{M}(A) = A^*\}$ is an Alexandrov topology on X.*

Proof. (O1) Since $\top_X \leq \mathcal{M}(\bot_X)$ and $\mathcal{M}(\top_X) = \mathcal{M}(\bot_X \to A) = \bot_X \odot \mathcal{M}(A) = \bot$, $\bot_X = \mathcal{M}(\top_X)$ and $\top_X = \mathcal{M}(\bot_X)$. Then $\bot_X, \top_X \in \tau_{\mathcal{M}}$.

(O2) For $A_i \in \tau_{\mathcal{M}}$ for each $i \in \Gamma$, by (M2), $\mathcal{M}(\bigwedge_{i \in \Gamma} A_i) = \bigvee_{i \in \Gamma} \mathcal{M}(A_i) = \bigvee_{i \in \Gamma} A_i^*$. So, $\bigwedge_{i \in \Gamma} A_i \in \tau_{\mathcal{M}}$. Since $\bigwedge_{i \in \Gamma} A_i^* \leq \mathcal{M}(\bigvee_{i \in \Gamma} A_i) \leq \bigwedge_{i \in \Gamma} \mathcal{M}(A_i) = \bigwedge_{i \in \Gamma} A_i^*$, Thus, $\bigvee_{i \in \Gamma} A_i \in \tau_{\mathcal{M}}$.

(O3) For $A \in \tau_{\mathcal{M}}$, since $\alpha \odot \mathcal{M}(\alpha \odot A) = \mathcal{M}(\alpha \to (\alpha \odot A)) \geq \mathcal{M}(A)$, $\mathcal{M}(\alpha \odot A) \geq \alpha \to \mathcal{M}(A) = (\alpha \odot A)^*$. Then $\alpha \odot A \in \tau_{\mathcal{M}}$.

(O4) For $A \in \tau_{\mathcal{M}}$, by (M4), $\mathcal{M}(\alpha \to A) = \alpha \odot \mathcal{M}(A) = \alpha \odot A^*$. Hence $\alpha \to A \in \tau_{\mathcal{M}}$.

Theorem 3.2. *Let* $\mathbf{T}$ *be an Alexandorv fuzzy topology on* X. *Define*

$$R_T^r(x, y) = \bigwedge\{A(x) \to A(y) \mid \mathbf{T}(A) \geq r^*\}$$

$$R_T^{-r}(x, y) = \bigwedge\{B(y) \to B(x) \mid \mathbf{T}(B) \geq r^*\}.$$

We have the following properties.

(1) R_T^r *is a fuzzy preorder with* $R_T^r \leq R_T^s$ *for each* $s \leq r$.

(2) R_T^{-r} *is a fuzzy preorder with* $R_T^{-r} \leq R_T^{-s}$ *for each* $s \leq r$ *and*

$$R_T^{-r}(x, y) = R_{T^*}^r(x, y).$$

(3) *Define* $\mathcal{M}_{R_T^r} : L^X \to L^X$ *as follows*

$$\mathcal{M}_{R_T^r}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot R_T^r(x, y)).$$

Then $\mathcal{M}_{R_T^r}$ *is a meet-join approximation operator on* X *with* $\mathcal{M}_{R_T^r} \leq \mathcal{M}_{R_T^s}$ *for each* $s \leq r$.

(4) $\tau_T^{r*} = \tau_{\mathcal{M}_{R_{T^*}^r}}$.

(5) $\mathcal{M}_{R_T^{-r}}$ *is a meet-join approximation operator on* X *such that*

$$\mathcal{M}_{R_T^{-r}}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot R_T^{-r}(x, y)) = \bigvee_{x \in X} (A^*(x) \odot R_{T^*}(x, y)).$$

(6) $(\tau_T^{r*})^* = \tau_{\mathcal{M}_{R_T^r}}$.

(7) $\mathcal{M}_{R_{T^*}^r}(A) = \bigwedge\{A_i \mid A^* \leq A_i,\ \mathbf{T}(A_i) \geq r^*\}$ *for all* $A \in L^X$ *and* $r \in L$. *Moreover,* $R_T^r(x, y) = \mathcal{M}_{R_{T^*}^r}(\top_x)(y)$, *for each* $x, y \in X$.

(8) $\mathcal{M}_{R_T^r}(A) = \bigwedge\{A_i \mid A^* \leq A_i,\ \mathbf{T}^*(A_i) \geq r^*\}$ *for all* $A \in L^X$ *and* $r \in L$. *Moreover,* $R_T^{-r}(x, y) = R_{T^*}^r(x, y) = \mathcal{M}_{R_{T^*}^r}(\top_x)(y)$, *for each* $x, y \in X$.

(9) *If* $\mathcal{M}_{R_T^{r_i}}(A) = B$ *for all* $i \in \Gamma \neq \emptyset$, *then* $\mathcal{M}_{R_T^s}(A) = B$ *with* $s = \bigwedge_{i \in \Gamma} r_i$.

(10) *If* $\mathcal{M}_{R_T^{-r_i}}(A) = B$ *for all* $i \in \Gamma \neq \emptyset$, *then* $\mathcal{M}_{R_T^{-s}}(A) = B$ *with* $s = \bigwedge_{i \in \Gamma} r_i$.

Proof. (1) Since $\mathbf{T}(B) \geq r^*$ iff $B \in \tau_T^{r*}$, then $R_T^r(x, y) = \bigwedge_{B \in \tau_T^{r*}}(B(x) \to B(y))$. Since $R_T^r(x, x) = \bigwedge_{B \in \tau_T^{r*}}(B(x) \to B(x)) = \top$ and

$$\begin{aligned} R_T^r(x, y) \odot R_T^r(y, z) &= \textstyle\bigwedge_{B \in \tau_T^{r*}}(B(x) \to B(y)) \odot \bigwedge_{B \in \tau_T^{r*}}(B(y) \to B(z)) \\ &\leq \textstyle\bigwedge_{B \in \tau_T^{r*}}(B(x) \to B(y)) \odot (B(y) \to B(z)) \\ &\leq \textstyle\bigwedge_{B \in \tau_T^{r*}}(B(x) \to B(z)) = R_T^r(x, y). \end{aligned}$$

Hence R^r_T is a fuzzy preorder.

For $s \le r$, since $\mathbf{T}(B) \ge s^* \ge r^*$, we have $R^r_T \le R^s_T$.

(2) By a similar method as (1), R^{-r}_T is a fuzzy preorder. Moreover,

$$\begin{aligned} R^{-r}_T(x,y) &= \bigwedge\{B(y) \to B(x) \mid \mathbf{T}(B) \ge r^*\} \\ &= \bigwedge\{B^*(x) \to B^*(y) \mid \mathbf{T}(B^*) = \mathbf{T}^*(B) \ge r^*\} \\ &= R^r_{T^*}(x,y). \end{aligned}$$

(3) (M1) $\mathcal{M}_{R^r_T}(\bigwedge_{i\in\Gamma} A_i)(y) = \bigvee_{x\in X}((\bigwedge_{i\in\Gamma} A_i)^*(x) \odot R^r_T(x,y)) = \bigvee_{i\in\Gamma} \mathcal{M}_{R^r_T}(A_i)(y)$.

(M2)

$$\begin{aligned} &\mathcal{M}_{R^r_T}(\alpha \to A)(y) = \bigvee_{x\in X}((\alpha \to A)^*(x) \odot R^r_T(x,y)) \\ &= \bigvee_{x\in X}(\alpha \odot (A^*(x) \odot R^r_T(x,y))) = \alpha \odot \mathcal{M}_{R^r_T}(A)(y). \end{aligned}$$

(M3) $\mathcal{M}_{R^r_T}(A)(y) = \bigvee_{x\in X}(A^*(x) \odot R^r_T(x,y)) \ge A^*(y) \odot R^r_T(y,y) = A^*(y)$.

(M4)

$$\begin{aligned} \mathcal{M}_{R^r_T}(\mathcal{M}^*_{R^r_T}(A))(x) &= \bigvee_{y\in X}(\mathcal{M}_{R^r_T}(A)(y) \odot R^r_T(y,x)) \\ &= \bigvee_{y\in X}(\bigvee_{z\in X}(A^*(z) \odot R^r_T(z,y)) \odot R^r_T(y,x)) \\ &= \bigvee_{z\in X}(A^*(z) \odot \bigvee_{y\in X}(R^r_T(z,y) \odot R^r_T(y,x))) \\ &\le \bigvee_{z\in X}(A^*(z) \odot R^r_T(z,x)) \\ &= \mathcal{M}_{R^r_T}(A)(x). \end{aligned}$$

For $s \le r$, since $R^r_T \le R^s_T$, then $\mathcal{M}_{R^r_T} \le \mathcal{M}_{R^s_T}$.

(4) Since $A \in \tau^{r*}_T$;i.e. $\mathbf{T}(A) \ge r^*$, $R^r_{T^*}(x,y) \odot A^*(x) = \bigwedge_{B\in\tau^{r*}_{T^*}}(B(x) \to B(y)) \odot A^*(x) \le (A^*(x) \to A^*(y)) \odot A^*(x) \le A^*(y)$, by M(3), $\mathcal{M}_{R^r_{T^*}}(A) = A^*$. So, $A \in \tau_{\mathcal{M}_{R^r_{T^*}}}$. Thus $\tau^{r*}_T \subset \tau_{\mathcal{M}_{R^r_{T^*}}}$. Let $A \in \tau_{\mathcal{M}_{R^r_{T^*}}}$;i.e. Let $\mathcal{M}_{R^r_{T^*}}(A) = A^*$. Then

$$\begin{aligned} A &= \mathcal{M}^*_{R^r_{T^*}}(A) = (\bigvee_{y\in X}(A^*(y) \odot R^r_{T^*}(y,-)))^* \\ &= \bigwedge_{y\in X}(A(y) \to \bigvee_{B\in\tau^{r*}_{T^*}}(B(y) \odot B^*)) \end{aligned}$$

Since $\bigvee_{B\in\tau^{r*}_{T^*}}(B(y) \odot B^*) \in \tau^{r*}_T$ and $\bigwedge_{y\in X}(A(y) \to \bigvee_{B\in\tau^{r*}_{T^*}}(B(y) \odot B^*)) \in \tau^{r*}_T$, we have $A \in \tau^{r*}_T$. Hence $\tau_{\mathcal{M}_{R^r_{T^*}}} \subset \tau^{r*}_T$.

(5) It is similarly proved as (4).

(6) Let $A \in (\tau^{r*}_T)^*$. Since $A \in \tau^{r*}_T$,

$$\begin{aligned} R^r_T(x,y) \odot A^*(x) &= \bigwedge_{B\in\tau^{r*}_T}(B(x) \to B(y)) \odot A^*(x) \\ &\le (A^*(x) \to A^*(y)) \odot A^*(x) \le A^*(y). \end{aligned}$$

Hence $\mathcal{M}_{R^r_T}(A) = A^*$;i.e. $A \in \tau_{\mathcal{M}_{R^r_T}}$. Thus $(\tau^{r*}_T)^* \subset \tau_{\mathcal{M}_{R^r_T}}$.

Let $A \in \tau_{\mathcal{M}_{R^r_T}}$;i.e. $\mathcal{M}_{R^r_T}(A) = A^*$. Then

$$\begin{aligned} A &= \mathcal{M}^*_{R^r_T}(A) = (\bigvee_{y\in X}(A(y) \odot R^r_T(y,-)))^* \\ &= \bigwedge_{y\in X}(A^*(y) \to \bigvee_{B\in\tau^{r*}_T}(B(y) \odot B^*)) \end{aligned}$$

Since $\bigvee_{B\in\tau_T^{r*}}(B(y)\odot B^*) \in (\tau_T^{r*})^*$ and $\bigwedge_{y\in X}(A^*(y) \to \bigvee_{B\in\tau_T^{r*}}(B(y)\odot B^*)) \in (\tau_T^{r*})^*$, we have $A \in (\tau_T^{r*})^*$. Hence $(\tau_T^{r*})^* = \tau_{\mathcal{M}_{R_T^r}}$.

(7) For each $A \in L^X$ with $A^* \le A_i$, $\mathbf{T}(A_i) \ge r^*$, since $A_i \in \tau_T^{r*} = \tau_{\mathcal{M}_{R_{T^*}^r}}$, then

$$\bigwedge_i A_i \le \mathcal{M}_{R_{T^*}^r}(\bigvee_i A_i^*) \le A_i = \mathcal{M}_{R_{T^*}^r}(A_i^*).$$

So, $\mathcal{M}_{R_{T^*}^r}(\bigvee_i A_i^*) = \bigwedge_i A_i$. Since $A \ge \bigvee A_i^*$,

$$\mathcal{M}_{R_{T^*}^r}(A) \le \mathcal{M}_{R_{T^*}^r}(\bigvee_i A_i^*) = \bigwedge_i A_i = \bigwedge\{A_i \mid A^* \le A_i,\ \mathbf{T}(A_i) \ge r^*\}.$$

Since $\mathcal{M}_{R_{T^*}^r}(\mathcal{M}^*_{R_{T^*}^r}(A)) = \mathcal{M}_{R_{T^*}^r}(A) \ge A^*$ and $\mathcal{M}_{R_{T^*}^r}(A) \in \tau_{\mathcal{M}_{R_{T^*}^r}} = \tau_T^{r*}$. So,, $\bigwedge\{A_i \mid A^* \le A_i,\ \mathbf{T}(A_i) \ge r^*\} \le \mathcal{M}_{R_{T^*}^r}(A)$. Hence $\bigwedge\{A_i \mid A^* \le A_i,\ \mathbf{T}(A_i) \ge r^*\} = \mathcal{M}_{R_{T^*}^r}(A)$ for all $A \in L^X$ and $r \in L$.

(8) It is proved in a similar way as (7).

(9) Let $\mathcal{M}_{R_T^{r_i}}(A) = B$ for all $i \in \Gamma \neq \emptyset$. Since

$$\begin{aligned}\mathcal{M}_{R_T^{r_i}}(A) &= \textstyle\bigvee_{x\in X}(A^*(x) \odot R_T^{r_i}(x,-))\\ &= \textstyle\bigvee_{x\in X}(A^*(x) \odot \bigwedge_{D\in\tau_T^{r_i^*}}(D(x) \to D)) \in \tau_T^{r_i^*}\end{aligned}$$

$\mathbf{T}(B) = \mathbf{T}(\mathcal{M}_{R_T^{r_i}}(A)) \ge r_i^*$, then $\mathbf{T}(B) \ge \bigvee_{i\in\Gamma} r_i^* = (\bigwedge_{i\in\Gamma} r_i)^* = s^*$ where $s = \bigwedge_{i\in\Gamma} r_i$. Since $B^* \in (\tau_T^{s^*})^* = \tau_{\mathcal{M}_{R_T^s}}$, then $\mathcal{M}_{R_T^s}(B^*) = B = \mathcal{M}_{R_T^{r_i}}(A) \ge A^*$. So, $A \ge \mathcal{M}^*_{R_T^s}(B^*) = B^*$. Thus

$$\mathcal{M}_{R_T^s}(A) \le \mathcal{M}_{R_T^s}(\mathcal{M}^*_{R_T^s}(B^*) = \mathcal{M}_{R_T^s}(B^*) = B.$$

Since $s \le r_i$, $\mathcal{M}_{R_T^s}(A) \ge \mathcal{M}_{R_T^{r_i}}(A) = B$. Thus $\mathcal{M}_{R_T^s}(A) = B$. □

Theorem 3.3. *Let $\mathbf{T}$ be an Alexandorv fuzzy topology on X. We have the following properties.*

(1) *Define $\mathbf{T}_{M_T} : L^X \to L$ as*

$$\mathbf{T}_{M_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{M}_{R_T^{r_i}}(A) = A^*\}.$$

Then $\mathbf{T}_{M_T} = \mathbf{T}^*$ is an Alexandrov fuzzy topology on X.

(2) *Define $\mathbf{T}_{M_{T^*}} : L^X \to L$ as*

$$\mathbf{T}_{M_{T^*}}(A) = \bigvee\{r_i^* \in L \mid \mathcal{M}_{R_T^{-r_i}}(A) = A^*\}$$

Then $\mathbf{T}_{M_{T^*}} = \mathbf{T}$ is an Alexandrov fuzzy topology on X.

(3) *$e_{L^X}(\mathcal{M}_{R_T^r}(A), B) = e_{L^X}(\mathcal{M}_{R_{T^*}^r}(B), A)$ for all $A, B \in L^X$.*

(4) *There exists an Alexandrov fuzzy topology $\mathbf{T}^r$ such that*

$$\mathbf{T}^r(A) = e_{L^X}(\mathcal{M}_{R_T^r}(A), A^*).$$

If $r \leq s$*, then* $\mathbf{T}^r \leq \mathbf{T}^s$ *for all* $A \in L^X$*.*

(5) *There exists an Alexandrov fuzzy topology* $\mathbf{T}^{*r}$ *such that*

$$\mathbf{T}^{*r}(A) = e_{L^X}(\mathcal{M}_{R_T^{-r}}(A), A^*).$$

Moreover, $\mathbf{T}^{*r}(A) = \mathbf{T}^r(A^*)$ *for all* $A \in L^X$*. If* $r \leq s$*, then* $\mathbf{T}^{*r} \leq \mathbf{T}^{*s}$ *for all* $A \in L^X$*.*

(6) *Define* $\mathbf{T}_M : L^X \to L$ *as*

$$\mathbf{T}_M(A) = \bigvee\{r^* \in L \mid \mathbf{T}^r(A) = \top\}.$$

Then $\mathbf{T}_M = \mathbf{T}^* = \mathbf{T}_{M_T}$ *is an Alexandrov fuzzy topology on* X*.*

(7) *Define* $\mathbf{T}_{M^*} : L^X \to L$ *as*

$$\mathbf{T}_{M^*}(A) = \bigvee\{r^* \in L \mid \mathbf{T}^{*r}(A) = \top\}.$$

Then $\mathbf{T}_{M^*} = \mathbf{T} = \mathbf{T}_{M_{T^*}}$ *is an Alexandrov fuzzy topology on* X*.*

Proof. (1) We only show that $\mathbf{T}_{M_T} = \mathbf{T}^*$. Let $\mathcal{M}_{R_T^{r_i}}(A) = A^*$. Then $A \in \tau_{\mathcal{M}_{R_T^{r_i}}} = (\tau_T^{r*})^*$ from Theorem 3.3.(6). So, $\mathbf{T}^*(A) = \mathbf{T}(A^*) = \mathbf{T}(\mathcal{M}_{R_T^{r_i}}(A)) \geq r_i^*$. Thus,

$$\mathbf{T}_{M_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{M}_{R_T^{r_i}}(A) = A^*\} \leq \mathbf{T}^*(A).$$

Since $\mathbf{T}^*(A) \geq (\mathbf{T}(A))^*$, then $\tau_{T^*}^{s*} = \tau_{\mathcal{M}_{R_T^s}}$ with $s = \mathbf{T}(A)$. Thus,

$$\mathbf{T}_{M_T}(A) = \bigvee\{r_i^* \in L \mid \mathcal{M}_{R_T^{r_i}}(A) = A^*\} \geq s^* = \mathbf{T}^*(A).$$

Hence $\mathbf{T}_{M_T} = \mathbf{T}^*$.

(3)
$$\begin{aligned}
e_{L^X}(\mathcal{M}_{R_T^r}(A), B) &= \textstyle\bigwedge_{y\in X}(\mathcal{M}_{R_T^r}(A)(y) \to B(y)) \\
&= \textstyle\bigwedge_{y\in X}(\bigvee_{x\in X}(A^*(x) \odot R_T^r(x,y)) \to B(y)) \\
&= \textstyle\bigwedge_{y\in X}\bigwedge_{x\in X}(R_T^r(x,y) \to (A^*(x) \to B(y))) \\
&= \textstyle\bigwedge_{y\in X}\bigwedge_{x\in X}(R_T^r(x,y) \to (B^*(y) \to A(x))) \\
&= \textstyle\bigwedge_{x\in X}(\bigvee_{y\in X}(B^*(y) \odot R_T^r(x,y)) \to A(x)) \\
&= e_{L^X}(\mathcal{M}_{R_T^{-r}}(B), A)
\end{aligned}$$

(4) (T1) Since $\mathcal{M}_{R_T^r}(\alpha_X)(y) = \bigvee_{x\in X}(\alpha_X(x) \odot R_T^r(x,y)) = \alpha \odot \bigvee_{x\in X} R_T^r(x,y) = \alpha$, $\mathbf{T}^r(\alpha_X) = e_{L^X}(\mathcal{M}_{R_T^r}(\alpha_X), \alpha_X) = \top$.

(T2) Since $\mathcal{M}_{R_T^r}(\bigvee_{i\in\Gamma} A_i) \leq \bigwedge_{i\in\Gamma} \mathcal{M}_{R_T^r}(A_i)$, we have

$$\begin{aligned}
\mathbf{T}^r(\textstyle\bigvee_{i\in\Gamma} A_i) &= e_{L^X}(\mathcal{M}_{R_T^r}(\textstyle\bigvee_{i\in\Gamma} A_i), (\textstyle\bigvee_{i\in\Gamma} A_i)^*) \\
&\geq e_{L^X}(\textstyle\bigwedge_{i\in\Gamma} \mathcal{M}_{R_T^r}(A_i), \textstyle\bigwedge_{i\in\Gamma} A_i^*) \\
&\geq \textstyle\bigwedge_{i\in\Gamma} e_{L^X}(\mathcal{M}_{R_T^r}(A_i), A_i) = \textstyle\bigwedge_{i\in\Gamma} \mathbf{T}^r(A_i)
\end{aligned}$$

$$\begin{array}{l}\mathbf{T}^r(\bigwedge_{i\in\Gamma} A_i) = e_{L^X}(\mathcal{M}_{R^r_T}(\bigwedge_{i\in\Gamma} A_i), (\bigwedge_{i\in\Gamma} A_i)^*)\\ = e_{L^X}(\bigvee_{i\in\Gamma} \mathcal{M}_{R^r_T}(A_i), \bigvee_{i\in\Gamma} A_i^*)\\ \geq \bigwedge_{i\in\Gamma} e_{L^X}(\mathcal{M}_{R^r_T}(A_i), A_i^*) = \bigwedge_{i\in\Gamma} \mathbf{T}^r(A_i)\end{array}$$

(T3) Since $\alpha\odot\mathcal{M}_{R^r_T}(\alpha\odot A) = \mathcal{M}_{R^r_T}(\alpha\to(\alpha\odot A)) \leq \mathcal{M}_{R^r_T}(A)$, then $\mathcal{M}_{R^r_T}(\alpha\odot A) \leq \alpha\to\mathcal{M}_{R^r_T}(A)$. Thus

$$\begin{array}{l}\mathbf{T}^r(\alpha\odot A) = e_{L^X}(\mathcal{M}_{R^r_T}(\alpha\odot A), (\alpha\odot A)^*)\\ \geq e_{L^X}(\alpha\to\mathcal{M}_{R^r_T}(A), \alpha\to A^*)\\ \geq e_{L^X}(\mathcal{M}_{R^r_T}(A), A^*) = \mathbf{T}^r(A)\end{array}$$

(T4)

$$\begin{array}{l}\mathbf{T}^r(\alpha\to A) = e_{L^X}(\mathcal{M}_{R^r_T}(\alpha\to A), (\alpha\to A)^*)\\ = e_{L^X}(\alpha\odot\mathcal{M}_{R^r_T}(A), \alpha\odot A^*)\\ \geq e_{L^X}(\mathcal{M}_{R^r_T}(A), A^*) = \mathbf{T}^r(A)\end{array}$$

Hence $\mathbf{T}^r$ is an Alexandrov fuzzy topology. Since $\mathcal{M}_{R^s_T} \leq \mathcal{M}_{R^r_T}$ for $r\leq s$, $\mathbf{T}^s(A) = e_{L^X}(\mathcal{M}_{R^s_T}(A), A^*) \geq e_{L^X}(\mathcal{M}_{R^r_T}(A), A^*) = \mathbf{T}^r(A)$.

(5) From a similar method as (4), $\mathbf{T}^{*r}$ is an Alexandrov fuzzy topology. By (3), $\mathbf{T}^r(A^*) = e_{L^X}(\mathcal{M}_{R^r_T}(A^*), A) = e_{L^X}(\mathcal{M}_{R^r_{T^*}}(A), A^*) = \mathbf{T}^{*r}(A)$ for all $A\in L^X$.

(6) Since $\mathbf{T}^r(A) = e_{L^X}(\mathcal{M}_{R^r_T}(A), A^*) = \top$ iff $A^* = \mathcal{M}_{R^r_T}(A)$, by (9),

$$\begin{array}{ll}\mathbf{T}_M(A) & = \bigvee\{r^*\in L \mid \mathbf{T}^r(A) = \top\}\\ & = \bigvee\{r^*\in L \mid \mathcal{M}_{R^r_T}(A) = A^*\}\\ & = \mathbf{T}_{M_T}(A) = \mathbf{T}^*(A).\end{array}$$

(2) and (7) are similarly proved as (1) and (6), respectively. □

Example 3.4. Let $(L = [0,1], \odot, \to, ^*)$ be a complete residuated lattice with a strong negation.

(1) Let $X = \{x, y, z\}$ be a set. Define a map $\mathbf{T} : [0,1]^X \to [0,1]$ as

$$\mathbf{T}(A) = A(x)\to A(z).$$

Trivially, $\mathbf{T}(\alpha_X) = 1$

Since $\alpha\odot A(x) \to \alpha\odot A(z) \geq A(x)\to A(z)$ from Lemma 2.4 (14), $\mathbf{T}(\alpha\odot A) \geq \mathbf{T}(A)$. Since $(\alpha\to A(x))\to(\alpha\to A(z)) \geq A(x)\to A(z)$ from Lemma 2.4 (10), $\mathbf{T}(\alpha\to A) \geq \mathbf{T}(A)$. By Lemma 2.4 (8), $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma}\mathbf{T}(A_i)$ and $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma}\mathbf{T}(A_i)$. Hence $\mathbf{T}$ is an Alexandrov fuzzy topology.

If $\mathbf{T}(A) = A(x)\to A(z) \geq r^*$, then $A(z) \geq A(x)\odot r^*$. Put $A(x) = 1, A(y) = 0$. So, $R^r_T(x,y) = \bigwedge\{A(x)\to A(y) \mid \mathbf{T}(A)\geq r^*\} = 0$ and $R^r_T(x,z) = \bigwedge\{A(x)\to A(z) \mid$

$\mathbf{T}(A) \geq r^*\} = r^*$, similarly, we can obtain

$$\begin{pmatrix} R_T^r(x,x)=1 & R_T^r(x,y)=0 & R_T^r(x,z)=r^* \\ R_T^r(y,x)=0 & R_T^r(y,y)=1 & R_T^r(y,z)=0 \\ R_T^r(z,x)=0 & R_T^r(z,y)=0 & R_T^r(z,z)=1 \end{pmatrix}$$

By Theorem 3.2(3), we obtain $\mathcal{M}_{R_T^r}(A)(y) = \bigvee_{x\in X}(A^*(x) \odot R_T^r(x,y))$ such that

$$\mathcal{M}_{R_T^r}(A) = (A^*(x), A^*(y), A^*(z) \vee (A^*(x) \odot r^*))$$

If $A^*(x) \odot r^* \leq A^*(z)$, then $\mathcal{M}_{R_T^r}(A) = A^*$. Thus $A \in \tau_{\mathcal{M}_{R_T^r}}$. Moreover, since $\mathbf{T}^*(A) = A^*(x) \to A^*(z) \geq r^*$ iff $A^*(z) \geq A^*(x) \odot r^*$, $A \in \tau_{T^*}^{r^*}$ iff $A \in \tau_{\mathcal{M}_{R_T^r}}$. So, $\tau_{T^*}^{r^*} = \tau_{\mathcal{M}_{R_T^r}}$. From Theorem 3.3(1), we have

$$\begin{aligned} \mathbf{T}_{M_T}(A) &= \bigvee\{r^* \in L \mid \mathcal{M}_{R_T^r}(A) = A\} \\ &= A^*(x) \to A^*(z) = \mathbf{T}(A^*) = \mathbf{T}^*(A). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{T}^r(A) &= \bigwedge_{x\in X}(\mathcal{M}_{R_T^r}(A)(x) \to A^*(x)) \\ &= (A^*(x) \odot r^*) \to A^*(z) = r^* \to (A^*(x) \to A^*(z)). \end{aligned}$$

$$\begin{aligned} \mathbf{T}_M(A) &= \bigvee\{r^* \in L \mid \mathbf{T}^r(A) = 1\} \\ &= A^*(x) \to A^*(z). \end{aligned}$$

Hence $\mathbf{T}_M = \mathbf{T}_{M_T} = \mathbf{T}^*$.

$$\mathcal{M}_{R_T^r}(1_x^*)(z) = \bigwedge\{B(z) \mid B \geq 1_x,\ \mathbf{T}(B) \geq r^*\}$$

Since $B(x) = 1$ and $\mathbf{T}(B) = 1 \to B(z) = B(z) \geq r^*$, then $\mathcal{M}_{R_T^r}(1_x^*)(z) = r^*$.

$$\mathcal{M}_{R_T^r}(1_x^*)(x) = \bigwedge\{B(x) \mid B \geq 1_x,\ \mathbf{T}(B) \geq r^*\} = 1$$

$$\mathcal{M}_{R_T^r}(1_x^*)(y) = \bigwedge\{B(y) \mid B \geq 1_x,\ \mathbf{T}(B) \geq r^*\} = 0$$

$$\mathcal{M}_{R_T^r}(1_z^*)(x) = \bigwedge\{B(x) \mid B \geq 1_z,\ \mathbf{T}(B) \geq r^*\}$$

Since $B(z) = 1$ and $\mathbf{T}(B) = B(x) \to 1 = 1$, then $\mathcal{M}_{R_T^r}(1_z^*)(x) = 0$.

$$\begin{pmatrix} \mathcal{M}_{R_T^r}(1_x^*)(x)=1 & \mathcal{M}_{R_T^r}(1_x^*)(y)=0 & \mathcal{M}_{R_T^r}(1_x^*)(z)=r^* \\ \mathcal{M}_{R_T^r}(1_y^*)(x)=0 & \mathcal{M}_{R_T^r}(1_y^*)(y)=1 & \mathcal{M}_{R_T^r}(1_y^*)(z)=0 \\ \mathcal{M}_{R_T^r}(1_z^*)(x)=0 & \mathcal{M}_{R_T^r}(1_z^*)(y)=0 & \mathcal{M}_{R_T^r}(1_z^*)(z)=1 \end{pmatrix}$$

Then $R_T^r(x,y) = \mathcal{M}_{R_T^r}(1_x^*)(y)$.

(2) By (1), we obtain a map $\mathbf{T}^* : [0,1]^Y \to [0,1]$ as

$$\mathbf{T}^*(A) = A^*(x) \to A^*(z) = A(z) \to A(x).$$

Since $\mathbf{T}^*(A) = A(z) \to A(x) \geq r^*$, then $A(x) \geq A(z) \odot r^*$. Put $A(z) = 1, A(y) = 0$. So, $R^r_{T^*}(z,y) = \bigwedge\{A(z) \to A(y) \mid \mathbf{T}^*(A) \geq r^*\} = 0$ and $R^r_{T^*}(z,x) = \bigwedge\{A(z) \to A(x) \mid \mathbf{T}(A) \geq r^*\} = r^*$

$$\begin{pmatrix} R^r_{T^*}(x,x) = 1 & R^r_{T^*}(x,y) = 0 & R^r_{T^*}(x,z) = 0 \\ R^r_{T^*}(y,x) = 0 & R^r_{T^*}(y,y) = 1 & R^r_{T^*}(y,z) = 0 \\ R^r_{T^*}(z,x) = r^* & R^r_{T^*}(z,y) = 0 & R^r_{T^*}(z,z) = 1 \end{pmatrix}$$

Moreover, $R^r_{T^*}(x,y) = R^{-r}_T(x,y) = R^r_T(y,x)$ for all $x, y \in X$.

$$\mathcal{M}_{R^r_{T^*}}(A)(y) = \bigvee_{x\in X} (A(x) \odot R^r_{T*}(x,y)).$$

$$\mathcal{M}_{R^r_{T^*}}(A) = (A^*(x) \vee (A^*(z) \odot r^*), A^*(y), A^*(z))$$

If $A^*(z)\odot r^* \leq A^*(x)$, then $\mathcal{M}_{R^r_{T^*}}(A) = A^*$. If $\mathcal{M}_{R^r_{T^*}}(A) = A^*$, then $A^*(z)\odot r^* \leq A^*(x)$. Moreover, since $\mathbf{T}(A) = A(x) \to A(z) \geq r^*$ iff $A^*(z) \odot r^* \leq A^*(z)$, $A \in \tau^{r^*}_T$ iff $A \in \tau_{\mathcal{M}_{R^r_{T^*}}}$. Thus

$$\begin{aligned} \mathbf{T}_{M_{T^*}}(A) &= \bigvee\{r^* \in L \mid \mathcal{M}_{R^r_{T^*}}(A) = A\} \\ &= A^*(z) \to A^*(x) = \mathbf{T}(A) = A(x) \to A(z). \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{T}^{*r}(A) &= \textstyle\bigwedge_{x\in X}(\mathcal{M}_{R^r_{T^*}}(A)(x) \to A(x)) \\ &= (A^*(z) \odot r^*) \to A^*(x) = r^* \to (A^*(z) \to A^*(x)). \end{aligned}$$

$$\begin{aligned} \mathbf{T}_{M^*}(A) &= \bigvee\{r^* \in L \mid \mathbf{T}^{*r}(A) = 1\} \\ &= A^*(z) \to A^*(x) = \mathbf{T}(A). \end{aligned}$$

Hence $\mathbf{T}_{M^*} = \mathbf{T}_{M_{T^*}} = \mathbf{T}$.

$$\mathcal{M}_{R^r_{T^*}}(1^*_x)(z) = \bigwedge\{B(z) \mid B \geq 1_x,\ \mathbf{T}^*(B) \geq r^*\}$$

Since $B(x) = 1$ and $\mathbf{T}^*(B) = B(z) \to 1 = 1$, then $\mathcal{M}_{R^r_{T^*}}(1^*_x)(z) = 0$.

$$\mathcal{M}_{R^r_{T^*}}(1^*_z)(y) = \bigwedge\{B(y) \in L^X \mid B \geq 1_z,\ \mathbf{T}^*(B) \geq r^*\} = 0$$

$$\mathcal{M}_{R^r_{T^*}}(1^*_y)(y) = \bigwedge\{B(y) \in L^X \mid B \geq 1_y,\ \mathbf{T}^*(B) \geq r^*\} = 1$$

$$\mathcal{M}_{R^r_{T^*}}(1^*_z)(x) = \bigwedge\{B(x) \in L^X \mid B \geq 1_z,\ \mathbf{T}^*(B) \geq r^*\}$$

Since $B(z) = 1$ and $\mathbf{T}^*(B) = 1 \to B(x) = B(x) \geq r^*$, then $B(x) \geq r^*$. We have $\mathcal{M}_{R^r_{T^*}}(1^*_z)(x) = r^*$.

$$\begin{pmatrix} \mathcal{M}_{R^r_{T^*}}(1^*_x)(x) = 1 & \mathcal{M}_{R^r_{T^*}}(1^*_x)(y) = 0 & \mathcal{M}_{R^r_{T^*}}(1^*_x)(z) = 0 \\ \mathcal{M}_{R^r_{T^*}}(1^*_y)(x) = 0 & \mathcal{M}_{R^r_{T^*}}(1^*_y)(y) = 1 & \mathcal{M}_{R^r_{T^*}}(1^*_y)(z) = 0 \\ \mathcal{M}_{R^r_{T^*}}(1^*_z)(x) = r^* & \mathcal{M}_{R^r_{T^*}}(1^*_z)(y) = 0 & \mathcal{M}_{R^r_{T^*}}(1^*_z)(z) = 1 \end{pmatrix}$$

Then $R^r_{T^*}(x,y) = \mathcal{M}_{R^r_{T^*}}(1^*_x)(y)$.

(3) Let $(L = [0,1], \odot, \rightarrow, ^*)$ be a complete residuated lattice with a strong negation defined by, for each $n \in N$,

$$x \odot y = ((x^n + y^n - 1) \vee 0)^{\frac{1}{n}}, \;\; x \rightarrow y = (1 - x^n + y^n)^{\frac{1}{n}} \wedge 1, \; x^* = (1 - x^n)^{\frac{1}{n}}.$$

By (1) and (2), we obtain

$$\mathbf{T}(A) = (1 - A(x)^n + A(z)^n)^{\frac{1}{n}} \wedge 1, \;\; \mathbf{T}^*(A) = (1 - A(z)^n + A(x)^n)^{\frac{1}{n}} \wedge 1.$$

$$R^r_T = \begin{pmatrix} 1 & 0 & (1-r^n)^{\frac{1}{n}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\; R^r_{T^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (1-r^n)^{\frac{1}{n}} & 0 & 1 \end{pmatrix}$$

$$\mathcal{M}_{R^r_T}(A) = (A^*(x), A^*(y), A^*(z) \vee (((A^*(x))^n - r^n) \vee 0)^{\frac{1}{n}})$$

$$\mathcal{M}_{R^r_{T^*}}(A) = (A^*(x) \vee (((A^*(z))^n - r^n) \vee 0)^{\frac{1}{n}}, A^*(y), A^*(z))$$

Since $\mathbf{T}(A) = (1 - A(x)^n + A(z)^n)^{\frac{1}{n}} \wedge 1 \geq (1 - x^n)^{\frac{1}{n}}$, we have

$$\begin{aligned} \tau^{r^*}_T = \tau_{\mathcal{M}_{R^r_{T^*}}} &= \{A \in L^X \mid A^n(x) - A^n(z) \leq r^n\} \\ \tau^{r^*}_{T^*} = \tau_{\mathcal{M}_{R^r_T}} &= \{A \in L^X \mid A^n(z) - A^n(x) \leq r^n\}. \end{aligned}$$

$$\begin{aligned} \mathbf{T}^r(A) &= (A^*(x) \odot r^*) \rightarrow A^*(z) = (r^n - (A^*(x))^n + (A^*(z))^n)^{\frac{1}{n}} \wedge 1 \\ \mathbf{T}^{*r}(A) &= (A(z) \odot r^*) \rightarrow A(x) = (r^n - (A^*(z))^n + (A^*(x))^n)^{\frac{1}{n}} \wedge 1. \end{aligned}$$

References

1. R. Bělohlávek: *Fuzzy Relational Systems.* Kluwer Academic Publishers, New York, 2002.
2. P. Hájek: *Metamathematices of Fuzzy Logic.* Kluwer Academic Publishers, Dordrecht, 1998.
3. U. Höhle & S.E. Rodabaugh: *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series 3.* Kluwer Academic Publishers, Boston, 1999.
4. Fang Jinming: I-fuzzy Alexandrov topologies and specialization orders. *Fuzzy Sets and Systems* **158** (2007), 2359-2374.
5. Y.C. Kim: Alexandrov L-topologies and L-join meet approximation operators. *International Journal of Pure and Applied Mathematics.* **91** (2014), no. 1, 113-129.
6. H. Lai & D. Zhang: Fuzzy preorder and fuzzy topology. *Fuzzy Sets and Systems* **157** (2006), 1865-1885.
7. ______: Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory. *Int. J. Approx. Reasoning* **50** (2009), 695-707.
8. Z. Pawlak: Rough sets. *Int. J. Comput. Inf. Sci.* **11** (1982), 341-356.

9. ______: Rough probability. *Bull. Pol. Acad. Sci. Math.* **32** (1984), 607-615.
10. A.M. Radzikowska & E.E. Kerre: A comparative study of fuzy rough sets. *Fuzzy Sets and Systems* **126** (2002), 137-155.
11. Y.H. She & G.J. Wang: An axiomatic approach of fuzzy rough sets based on residuated lattices. *Computers and Mathematics with Applications* **58** (2009), 189-201.
12. Zhen Ming Ma & Bao Qing Hu: Topological and lattice structures of L-fuzzy rough set determined by lower and upper sets. *Information Sciences* **218** (2013), 194-204.

DEPARTMENT OF MATHEMATICS, GANGNEUNG-WONJU NATIONAL UNIVERSITY, GANGNEUNG 210-702, KOREA
Email address: yck@gwnu.ac.kr