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COMMON n-TUPLED FIXED POINT FOR HYBRID PAIR OF MAPPINGS UNDER NEW CONTRACTIVE CONDITION

Bhavana Deshpande [a,*] and Amrish Handa [b]

Abstract. We establish a common n-tupled fixed point theorem for hybrid pair of mappings under new contractive condition. It is to be noted that to find n-tupled coincidence point, we do not use the condition of continuity of any mapping involved. An example supporting to our result has also been cited. We improve, extend and generalize several known results.

1. Introduction and Preliminaries

Let $(X,\ d)$ be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Let $D(x,\ A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d, that is,

$$\begin{aligned} D(x, A) &= \inf_{a\in A} d(x, a) \\ \text{and } H(A, B) &= \max\left\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\right\} \text{ for all } A, B \in CB(X). \end{aligned}$$

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [10]. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to [3, 4, 6, 7, 12] and the reference therein. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics.

In [1], Bhaskar and Lakshmikantham established some coupled fixed point theorems and apply these results to study the existence and uniqueness of solution

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*Corresponding author.

for periodic boundary value problems. Lakshmikantham and Ciric [9] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces, extended and generalized the results of Bhaskar and Lakshmikantham [1].

Chandok, Sintunavarat and Kumam [2] established some coupled coincidence point and coupled common fixed point theorems for a pair of mappings having a mixed g-monotone property in partially ordered G-metric spaces. Kumam et al. [8] proved some tripled fixed point theorems in fuzzy normed spaces. Rahimi, Radenovic, Soleimani Rad [11] introduced some new definitions about quadrupled fixed point and obtained some new quadrupled fixed point results in abstract metric spaces.

Imdad, Soliman, Choudhury and Das [5] introduced the concept of $n-$tupled fixed point, $n-$tupled coincidence point and proved some $n-$tupled coincidence point and $n-$tupled fixed point results for single valued mapping.

These concepts was extended by Deshpande and Handa [4] to multivalued mappings and obtained $n-$tupled coincidence points and common $n-$tupled fixed point theorems involving hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction. In [4], Deshpande and Handa introduced the following for multivalued mappings:

Definition 1.1. Let X be a nonempty set, $F : X^r \to 2^X$ (a collection of all nonempty subsets of X) and g be a self-mapping on X. An element $(x^1, x^2, ..., x^r) \in X^r$ is called

(1) an *$r-$tupled fixed point* of F if $x^1 \in F(x^1, x^2, ..., x^r)$, $x^2 \in F(x^2, ..., x^r, x^1)$, ..., $x^r \in F(x^r, x^1, ..., x^{r-1})$.

(2) an *$r-$tupled coincidence point of hybrid pair* $\{F, g\}$ if $g(x^1) \in F(x^1, x^2, ..., x^r)$, $g(x^2) \in F(x^2, ..., x^r, x^1)$, ..., $g(x^r) \in F(x^r, x^1, ..., x^{r-1})$.

(3) a *common $r-$tupled fixed point of hybrid pair* $\{F, g\}$ if $x^1 = g(x^1) \in F(x^1, x^2, ..., x^r)$, $x^2 = g(x^2) \in F(x^2, ..., x^r, x^1)$, ..., $x^r = g(x^r) \in F(x^r, x^1, ..., x^{r-1})$.

We denote the set of $r-$tupled coincidence points of mappings F and g by $C\{F, g\}$. Note that if $(x^1, x^2, ..., x^r) \in C\{F, g\}$, then $(x^2, ..., x^r, x^1)$, ..., $(x^r, x^1, ..., x^{r-1})$ are also in $C\{F, g\}$.

Definition 1.2. Let $F : X^r \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called *$w-$compatible* if $g(F(x^1, x^2, ..., x^r)) \subseteq F(g(x^1), g(x^2), ..., g(x^r))$ whenever $(x^1, x^2, ..., x^r) \in C\{F, g\}$.

Definition 1.3. Let $F : X^r \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called $F-$ *weakly commuting at some point* $(x^1, x^2, ..., x^r) \in X^r$ if $g^2(x^1) \in F(g(x^1), g(x^2), ..., g(x^r))$, $g^2(x^2) \in F(g(x^2), ..., g(x^r), g(x^1))$, ..., $g^2(x^r) \in F(g(x^r), g(x^1), ..., g(x^{r-1}))$.

Lemma 1.1. Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b\in B} d(a, b)$.

In this paper, we establish a common $n-$tupled fixed point theorem for hybrid pair of mappings satisfying new contractive condition. It is to be noted that to find $n-$tupled coincidence point, we do not use the condition of continuity of any mapping involved. Our result improves, extend, and generalize the results of Bhaskar and Lakshmikantham [1] and Lakshmikantham and Ciric [9]. An example is also given to validate our result.

2. Main Results

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying
(i_φ) φ is non-decreasing,
(ii_φ) $\varphi(t) < t$ for all $t > 0$,
(iii_φ) $\lim_{r\to t+} \varphi(r) < t$ for all $t > 0$
and Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfies
(i_ψ) ψ is continuous,
(ii_ψ) $\psi(t) < t$, for all $t > 0$.
Note that, by (i_ψ) and (ii_ψ) we have that $\psi(t) = 0$ if and only if $t = 0$.
For simplicity, we define the following:

$$\begin{aligned}
&(A) M\left(x^1, ..., x^r, y^1, ..., y^r\right) \\
= \quad & \min\left\{\begin{array}{c} D(gx^1, F(x^1, ..., x^r)), D(gy^1, F(y^1, ..., y^r)), \\ ..., D(gx^r, F(x^r, ..., x^{r-1})), D(gy^r, F(y^r, ..., y^{r-1})), \\ \frac{D(gx^1, F(y^1, ..., y^r)) + D(gy^1, F(x^1, ..., x^r))}{2}, \\ ..., \frac{D(gx^r, F(y^r, ..., y^{r-1})) + D(gy^r, F(x^r, ..., x^{r-1}))}{2} \end{array}\right\}.
\end{aligned}$$

$$\begin{aligned}
&(B) m\left(x^1, ..., x^r, y^1, ..., y^r\right) \\
= \quad & \min\left\{\begin{array}{c} D(x^1, F(x^1, ..., x^r)), D(y^1, F(y^1, ..., y^r)), \\ ..., D(x^r, F(x^r, ..., x^{r-1})), D(y^r, F(y^r, ..., y^{r-1})), \\ \frac{D(x^1, F(y^1, ..., y^r)) + D(y^1, F(x^1, ..., x^r))}{2}, \\ ..., \frac{D(x^r, F(y^r, ..., y^{r-1})) + D(y^r, F(x^r, ..., x^{r-1}))}{2} \end{array}\right\}.
\end{aligned}$$

Theorem 2.1. *Let (X, d) be a metric space. Assume $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings satisfying*

$$
\begin{aligned}
& H(F(x^1,x^2,...,x^r),F(y^1,y^2,...,y^r)) \\
(2.1) \qquad \leq \ & \varphi\left[\max\left\{d(gx^1,gy^1),d(gx^2,gy^2),...,d(gx^r,gy^r)\right\}\right] \\
& +\psi\left[M(x^1,x^2,...,x^r,y^1,y^2,...,y^r)\right],
\end{aligned}
$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $\varphi \in \Phi$ *and* $\psi \in \Psi$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds:*
(a) F *and* g *are* $w-$*compatible.* $\lim_{i\to\infty} g^ix^1 = y^1$, $\lim_{i\to\infty} g^ix^2 = y^2$, ..., $\lim_{i\to\infty} g^ix^r = y^r$, *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$ *and* g *is continuous at* y^1, y^2, ..., y^r.
(b) g *is* $F-$*weakly commuting for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$, gx^1, gx^2, ..., gx^r *are fixed points of* g, *that is,* $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.
(c) g *is continuous at* x^1, x^2, ..., x^r. $\lim_{i\to\infty} g^iy^1 = x^1$, $\lim_{i\to\infty} g^iy^2 = x^2$, ..., $\lim_{i\to\infty} g^iy^r = x^r$ *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$.
(d) $g(C\{F, g\})$ *is a singleton subset of* $C\{F, g\}$.

Proof. Let x_0^1, x_0^2, ..., $x_0^r \in X$ be arbitrary. Then $F(x_0^1, x_0^2, ..., x_0^r)$, ..., $F(x_0^r, x_0^1, ..., x_0^{r-1})$ are well defined. Choose $gx_1^1 \in F(x_0^1, x_0^2, ..., x_0^r)$, ..., $gx_1^r \in F(x_0^r, x_0^1, ..., x_0^{r-1})$ because $F(X^r) \subseteq g(X)$. Since $F : X^r \to CB(X)$, therefore by Lemma 1.1, there exist $z^1 \in F(x_1^1, x_1^2, ..., x_1^r)$, ..., $z^r \in F(x_1^r, x_1^1, ..., x_1^{r-1})$ such that

$$
\begin{aligned}
d(gx_1^1,z^1) &\leq H(F(x_0^1,x_0^2,...,x_0^r),F(x_1^1,x_1^2,...,x_1^r)), \\
d(gx_1^2,z^2) &\leq H(F(x_0^2,...,x_0^r,x_0^1),F(x_1^2,...,x_1^r,x_1^1)), \\
...,d(gx_1^r,z^r) &\leq H(F(x_0^r,x_0^1,...,x_0^{r-1}),F(x_1^r,x_1^1,...,x_1^{r-1})).
\end{aligned}
$$

Since $F(X^r) \subseteq g(X)$, there exist x_2^1, x_2^2, ..., $x_2^r \in X$ such that $z^1 = gx_2^1$, $z^2 = gx_2^2$, ..., $z^r = gx_2^r$. Thus

$$
\begin{aligned}
d(gx_1^1,gx_2^1) &\leq H(F(x_0^1,x_0^2,...,x_0^r),F(x_1^1,x_1^2,...,x_1^r)), \\
d(gx_1^2,gx_2^2) &\leq H(F(x_0^2,...,x_0^r,x_0^1),F(x_1^2,...,x_1^r,x_1^1)), \\
...,d(gx_1^r,gx_2^r) &\leq H(F(x_0^r,x_0^1,...,x_0^{r-1}),F(x_1^r,x_1^1,...,x_1^{r-1})).
\end{aligned}
$$

Continuing this process, we obtain sequences $\{x_i^1\} \subset X$, $\{x_i^2\} \subset X$, ..., $\{x_i^r\} \subset X$ such that for all $i \in N$, we have $x_{i+1}^1 \in F(x_i^1, x_i^2, ..., x_i^r)$, $x_{i+1}^2 \in F(x_i^2, ..., x_i^r, x_i^1)$,

..., $x_{i+1}^r \in F(x_i^r, x_i^1, ..., x_i^{r-1})$ such that

$$\begin{aligned} & d(gx_i^1, gx_{i+1}^1) \\ \leq\ & H(F(x_{i-1}^1, x_{i-1}^2, ..., x_{i-1}^r), F(x_i^1, x_i^2, ..., x_i^r)) \\ \leq\ & \varphi\left[\max\left\{d(gx_{i-1}^1, gx_i^1), ..., d(gx_{i-1}^r, gx_i^r)\right\}\right] \\ & +\psi\left[M(x_{i-1}^1, x_{i-1}^2, ..., x_{i-1}^r, x_i^1, x_i^2, ..., x_i^r)\right] \\ \leq\ & \varphi\left[\max\left\{d(gx_{i-1}^1, gx_i^1), ..., d(gx_{i-1}^r, gx_i^r)\right\}\right]. \end{aligned}$$

Thus

$$d(gx_i^1, gx_{i+1}^1) \leq \varphi\left[\max\left\{d(gx_{i-1}^1, gx_i^1), ..., d(gx_{i-1}^r, gx_i^r)\right\}\right].$$

Similarly

$$\begin{aligned} d(gx_i^2, gx_{i+1}^2) &\leq \varphi\left[\max\left\{d(gx_{i-1}^1, gx_i^1), ..., d(gx_{i-1}^r, gx_i^r)\right\}\right], \\ ..., d(gx_i^r, gx_{i+1}^r) &\leq \varphi\left[\max\left\{d(gx_{i-1}^1, gx_i^1), ..., d(gx_{i-1}^r, gx_i^r)\right\}\right]. \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\left\{d(gx_i^1, gx_{i+1}^1), d(gx_i^2, gx_{i+1}^2), ..., d(gx_i^r, gx_{i+1}^r)\right\} \\ \leq\ & \varphi\left[\max\left\{d(gx_{i-1}^1, gx_i^1), d(gx_{i-1}^2, gx_i^2), ..., d(gx_{i-1}^r, gx_i^r)\right\}\right], \end{aligned} \tag{2.2}$$

which implies, by (ii_φ), that

$$\begin{aligned} & \max\left\{d(gx_i^1, gx_{i+1}^1), d(gx_i^2, gx_{i+1}^2), ..., d(gx_i^r, gx_{i+1}^r)\right\} \\ <\ & \max\left\{d(gx_{i-1}^1, gx_i^1), d(gx_{i-1}^2, gx_i^2), ..., d(gx_{i-1}^r, gx_i^r)\right\}. \end{aligned}$$

This shows that the sequence $\{\delta_i\}_{i=0}^{\infty}$ defined by $\delta_i = \max\{d(gx_i^1,\ gx_{i+1}^1),\ d(gx_i^2,\ gx_{i+1}^2),\ ...,\ d(gx_i^r,\ gx_{i+1}^r)\}$ is a decreasing sequence of positive numbers. Then there exists $\delta \geq 0$ such that

$$\lim_{i\to\infty} \delta_i = \lim_{i\to\infty} \max\left\{d(gx_i^1, gx_{i+1}^1), d(gx_i^2, gx_{i+1}^2), ..., d(gx_i^r, gx_{i+1}^r)\right\} = \delta. \tag{2.3}$$

We shall prove that $\delta = 0$. Suppose that $\delta > 0$. Letting $i \to \infty$ in (2.2), by using (2.3) and (iii_φ), we get

$$\delta \leq \lim_{i\to\infty} \varphi(\delta_{i+1}) = \lim_{\delta_{i+1}\to\delta+} \varphi(\delta_{i+1}) < \delta,$$

which is a contradiction. Hence

$$\lim_{i\to\infty} \delta_i = \lim_{i\to\infty} \max\left\{d(gx_i^1, gx_{i+1}^1), d(gx_i^2, gx_{i+1}^2), ..., d(gx_i^r, gx_{i+1}^r)\right\} = 0. \tag{2.4}$$

We now prove that $\{gx_i^1\}_{i=0}^{\infty}$, $\{gx_i^2\}_{i=0}^{\infty}$, ..., $\{gx_i^r\}_{i=0}^{\infty}$ are Cauchy sequences in (X, d). Suppose, to the contrary, that one of the sequences is not a Cauchy sequence.

Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx^1_{i(k)}\}$, $\{gx^1_{j(k)}\}$ of $\{gx^1_i\}_{i=0}^{\infty}$, $\{gx^2_{i(k)}\}$, $\{gx^2_{j(k)}\}$ of $\{gx^2_i\}_{i=0}^{\infty}$, ..., $\{gx^r_{i(k)}\}$, $\{gx^r_{j(k)}\}$ of $\{gx^r_i\}_{i=0}^{\infty}$ such that

$$\max\left\{d(gx^1_{i(k)}, gx^1_{j(k)}), ..., d(gx^r_{i(k)}, gx^r_{j(k)})\right\} \geq \varepsilon . k = 1, 2, ... \tag{2.5}$$

We can choose $i(k)$ to be the smallest positive integer satisfying (2.5). Then

$$\max\left\{d(gx^1_{i(k)-1}, gx^1_{j(k)}), ..., d(gx^r_{i(k)-1}, gx^r_{j(k)})\right\} < \varepsilon. \tag{2.6}$$

By (2.5), (2.6) and triangle inequality, we have

$$\begin{aligned}
\varepsilon \ \leq \ & r_k = \max\left\{d(gx^1_{i(k)}, gx^1_{j(k)}), ..., d(gx^r_{i(k)}, gx^r_{j(k)})\right\} \\
\leq \ & \max\left\{d(gx^1_{i(k)}, gx^1_{i(k)-1}), ..., d(gx^r_{i(k)}, gx^r_{i(k)-1})\right\} \\
& + \max\left\{d(gx^1_{i(k)-1}, gx^1_{j(k)}), ..., d(gx^r_{i(k)-1}, gx^r_{j(k)})\right\} \\
< \ & \max\left\{d(gx^1_{i(k)}, gx^1_{i(k)-1}), ..., d(gx^r_{i(k)}, gx^r_{i(k)-1})\right\} + \varepsilon.
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.4), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \max\left\{d(gx^1_{i(k)}, gx^1_{j(k)}), ..., d(gx^r_{i(k)}, gx^r_{j(k)})\right\} = \varepsilon. \tag{2.7}$$

By triangle inequality, we have

$$\begin{aligned}
r_k \ = \ & \max\left\{d(gx^1_{i(k)}, gx^1_{j(k)}), ..., d(gx^r_{i(k)}, gx^r_{j(k)})\right\} \\
\leq \ & \max\left\{d(gx^1_{i(k)}, gx^1_{i(k)+1}), ..., d(gx^r_{i(k)}, gx^r_{i(k)+1})\right\} \\
& + \max\left\{d(gx^1_{i(k)+1}, gx^1_{j(k)+1}), ..., d(gx^r_{i(k)+1}, gx^r_{j(k)+1})\right\} \\
& + \max\left\{d(gx^1_{j(k)+1}, gx^1_{j(k)}), ..., d(gx^r_{j(k)+1}, gx^r_{j(k)})\right\} \\
\leq \ & \delta_{i(k)} + \delta_{j(k)} + \max\left\{d(gx^1_{i(k)+1}, gx^1_{j(k)+1}), ..., d(gx^r_{i(k)+1}, gx^r_{j(k)+1})\right\}.
\end{aligned}$$

Thus

$$r_k \leq \delta_{i(k)} + \delta_{j(k)} + \max\left\{d(gx^1_{i(k)+1}, gx^1_{j(k)+1}), ..., d(gx^r_{i(k)+1}, gx^r_{j(k)+1})\right\}. \tag{2.8}$$

Since $gx^1_{i(k)+1} \in F(x^1_{i(k)}, ..., x^r_{i(k)})$, ..., $gx^r_{i(k)+1} \in F(x^r_{i(k)}, ..., x^{r-1}_{i(k)})$, $gx^1_{j(k)+1} \in F(x^1_{j(k)}, ..., x^r_{j(k)})$, ..., $gx^r_{j(k)+1} \in F(x^r_{j(k)}, ..., x^{r-1}_{j(k)})$, therefore by (2.1) and by triangle inequality, we have

$$
\begin{aligned}
& d(gx^1_{i(k)+1}, gx^1_{j(k)+1}) \\
\leq \ & H(F(x^1_{i(k)}, ..., x^r_{i(k)}), F(x^1_{j(k)}, ..., x^r_{j(k)})) \\
\leq \ & \varphi\left[\max\left\{d(gx^1_{i(k)}, gx^1_{j(k)}), ..., d(gx^r_{i(k)}, gx^r_{j(k)})\right\}\right] \\
& +\psi\left[M(x^1_{i(k)}, ..., x^r_{i(k)}, x^1_{j(k)}, ..., x^r_{j(k)})\right] \\
\leq \ & \varphi(r_k) + \psi\left[M(x^1_{i(k)}, ..., x^r_{i(k)}, x^1_{j(k)}, ..., x^r_{j(k)})\right].
\end{aligned}
$$

Thus

$$
d(gx^1_{i(k)+1}, gx^1_{j(k)+1}) \leq \varphi(r_k) + \psi\left[M(x^1_{i(k)}, ..., x^r_{i(k)}, x^1_{j(k)}, ..., x^r_{j(k)})\right].
$$

Similarly

$$
\begin{aligned}
d(gx^2_{i(k)+1}, gx^2_{j(k)+1}) \ & \leq \ \varphi(r_k) + \psi\left[M(x^1_{i(k)}, ..., x^r_{i(k)}, x^1_{j(k)}, ..., x^r_{j(k)})\right], \\
..., d(gx^r_{i(k)+1}, gx^r_{j(k)+1}) \ & \leq \ \varphi(r_k) + \psi\left[M(x^1_{i(k)}, ..., x^r_{i(k)}, x^1_{j(k)}, ..., x^r_{j(k)})\right].
\end{aligned}
$$

Combining them, we get

$$
\begin{aligned}
& \max\left\{d(gx^1_{i(k)+1}, gx^1_{j(k)+1}), ..., d(gx^r_{i(k)+1}, gx^r_{j(k)+1})\right\} \\
\leq \ & \varphi(r_k) + \psi\left[M(x^1_{i(k)}, ..., x^r_{i(k)}, x^1_{j(k)}, ..., x^r_{j(k)})\right].
\end{aligned}
\tag{2.9}
$$

By (2.8) and (2.9), we get

$$
r_k \leq \delta_{i(k)} + \delta_{j(k)} + \varphi(r_k) + \psi\left[M(x^1_{i(k)}, ..., x^r_{i(k)}, x^1_{j(k)}, ..., x^r_{j(k)})\right].
$$

Letting $k \to \infty$ in the above inequality, by using (2.4), (2.7), (A), (i_ψ), (ii_ψ) and (iii_φ), we get

$$
\varepsilon \leq 0 + 0 + \lim_{k\to\infty} \varphi(r_k) + 0 \ \leq \lim_{r_k \to \varepsilon+} \varphi(r_k) \quad < \varepsilon,
$$

which is a contradiction. This shows that $\{gx^1_i\}_{i=0}^\infty$, $\{gx^2_i\}_{i=0}^\infty$, ..., $\{gx^r_i\}_{i=0}^\infty$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, thus there exist x^1, x^2, ..., $x^r \in X$ such that

$$
\lim_{i\to\infty} gx^1_i = gx^1, \lim_{i\to\infty} gx^2_i = gx^2, ..., \lim_{i\to\infty} gx^r_i = gx^r.
\tag{2.10}
$$

Now, since $gx^1_{i+1} \in F(x^1_i, ..., x^r_i)$, ..., $gx^r_{i+1} \in F(x^r_i, ..., x^{r-1}_i)$, therefore by using condition (2.1), we get

$$\begin{aligned}
& D(gx^1_{i+1}, F(x^1, x^2, ..., x^r)) \\
\leq\ & H(F(x^1_i, x^2_i, ..., x^r_i), F(x^1, x^2, ..., x^r)) \\
\leq\ & \varphi\left[\max\left\{d(gx^1_i, gx^1), d(gx^2_i, gx^2), ..., d(gx^r_i, gx^r)\right\}\right] \\
& +\psi\left[M\left\{x^1_i, x^2_i, ..., x^r_i, x^1, x^2, ..., x^r\right\}\right].
\end{aligned}$$

Letting $i \to \infty$ in the above inequality, by using (2.10), (A), (i_ψ), (ii_ψ) and (iii_φ), we get

$$D(gx^1, F(x^1, x^2, ..., x^r)) \leq \lim_{t\to 0+} \varphi(t) + 0 = 0 + 0 = 0.$$

Thus

$$D(gx^1, F(x^1, x^2, ..., x^r)) = 0.$$

Similarly

$$D(gx^2, F(x^2, ..., x^r, x^1)) = 0, ..., D(gx^r, F(x^r, x^1, ..., x^{r-1})) = 0,$$

which implies that

$$gx^1 \in F(x^1, x^2, ..., x^r), ..., gx^r \in F(x^r, x^1, ..., x^{r-1}),$$

that is, $(x^1, x^2, ..., x^r)$ is an $r-$tupled coincidence point of F and g.

Suppose now that (a) holds. Assume that for some $(x^1, x^2, ..., x^r) \in C\{F, g\}$,

$$\lim_{i\to\infty} g^i x^1 = y^1,\ \lim_{i\to\infty} g^i x^2 = y^2, ..., \lim_{i\to\infty} g^i x^r = y^r where y^1, y^2, ..., y^r \in X. \tag{2.11}$$

Since g is continuous at y^1, y^2, ..., y^r, we have, by (2.11), that y^1, y^2, ..., y^r are fixed points of g, that is,

$$gy^1 = y^1, gy^2 = y^2, ..., gy^r = y^r. \tag{2.12}$$

As F and g are $w-$compatible, so for all $i \geq 1$,

$$\begin{aligned}
g^i x^1 &\in F(g^{i-1}x^1, g^{i-1}x^2, ..., g^{i-1}x^r), \\
g^i x^2 &\in F(g^{i-1}x^2, ..., g^{i-1}x^r, g^{i-1}x^1), \\
..., g^i x^r &\in F(g^{i-1}x^r, g^{i-1}x^1, ..., g^{i-1}x^{r-1}).
\end{aligned} \tag{2.13}$$

By using (2.1) and (2.13), we obtain

$$\begin{aligned}
& D(g^i x^1, F(y^1, y^2, ..., y^r)) \\
\leq\ & H(F(g^{i-1}x^1, g^{i-1}x^2, ..., g^{i-1}x^r), F(y^1, y^2, ..., y^r)) \\
\leq\ & \varphi\left[\max\left\{d(g^i x^1, gy^1), d(g^i x^2, gy^2), ..., d(g^i x^r, gy^r)\right\}\right] \\
& +\psi\left[M\left\{g^{i-1}x^1, g^{i-1}x^2, ..., g^{i-1}x^r, y^1, y^2, ..., y^r\right\}\right].
\end{aligned}$$

On taking limit as $i \to \infty$ in the above inequality, by using (2.11), (2.12), (A), (i_ψ), (ii_ψ) and (iii_φ), we get

$$D(gy^1, F(y^1, y^2, ..., y^r)) \le \lim_{t\to 0+} \varphi(t) + 0 = 0 + 0 = 0,$$

which implies that

$$D(gy^1, F(y^1, y^2, ..., y^r)) = 0.$$

Similarly

$$D(gy^2, F(y^2, ..., y^r, y^1)) = 0, ..., D(gy^r, F(y^r, y^1, ..., y^{r-1})) = 0.$$

Thus

$$gy^1 \in F(y^1, y^2, ..., y^r), ..., gy^r \in F(y^r, y^1, ..., y^{r-1}). \tag{2.14}$$

Thus, by (2.12) and (2.14), we get

$$y^1 = gy^1 \in F(y^1, y^2, ..., y^r), ..., y^r = gy^r \in F(y^r, y^1, ..., y^{r-1}),$$

that is, $(y^1, y^2, ..., y^r)$ is a common $r-$tupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x^1, x^2, ..., x^r) \in C\{F, g\}$, g is $F-$weakly commuting, that is, $g^2x^1 \in F(gx^1, gx^2, ..., gx^r)$, $g^2x^2 \in F(gx^2, ..., gx^r, gx^1)$, ..., $g^2x^r \in F(gx^r, gx^1, ..., gx^{r-1})$ and $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$. Thus $gx^1 = g^2x^1 \in F(gx^1, gx^2, ..., gx^r)$, $gx^2 = g^2x^2 \in F(gx^2, ..., gx^r, gx^1)$, ..., $gx^r = g^2x^r \in F(gx^r, gx^1, ..., gx^{r-1})$, that is, $(gx^1, gx^2, ..., gx^r)$ is a common $r-$tupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x^1, x^2, ..., x^r) \in C\{F, g\}$ and for some $y^1, y^2, ..., y^r \in X$, $\lim_{i\to\infty} g^iy^1 = x^1$, $\lim_{i\to\infty} g^iy^2 = x^2$, ..., $\lim_{i\to\infty} g^iy^r = x^r$. Since g is continuous at $x^1, x^2, ..., x^r$. We have that $x^1, x^2, ..., x^r$ are fixed points of g, that is, $gx^1 = x^1$, $gx^2 = x^2$, ..., $gx^r = x^r$. Since $(x^1, x^2, ..., x^r) \in C\{F, g\}$, therefore, we obtain $x^1 = gx^1 \in F(x^1, x^2, ..., x^r)$, $x^2 = gx^2 \in F(x^2, ..., x^r, x^1)$, ..., $x^r = gx^r \in F(x^r, x^1, ..., x^{r-1})$, that is, $(x^1, x^2, ..., x^r)$ is a common $r-$tupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C\{F, g\}) = \{(x^1, x^1, ..., x^1)\}$. Then $\{x^1\} = \{gx^1\} = F(x^1, x^1, ..., x^1)$. Hence $(x^1, x^1, ..., x^1)$ is a common $r-$tupled fixed point of F and g. □

Example 2.1. Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X^r \to$

$CB(X)$ be defined as

$$F(x^1, x^2, ..., x^r) = \begin{cases} \{0\}, \text{ for } x^1, x^2, ..., x^r = 1 \\ \left[0, \frac{1}{2r} \sum_{n=1}^{r} (x^n)^2\right], \text{ for } x^1, x^2, ..., x^r \in [0, 1) \end{cases}$$

and $g : X \to X$ be defined as

$$g(x) = x^2, \text{ for all } x \in X.$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{2}, \text{ for } t \neq 1 \\ \frac{3}{4}, \text{ for } t = 1 \end{cases}$$

and $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \frac{t}{4}, \text{ for } t \geq 0.$$

Now, for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in [0, 1)$.

If $(x^1)^2 + (x^2)^2 + ... + (x^r)^2 = (y^1)^2 + (y^2)^2 + ... + (y^r)^2$, then

$$\begin{aligned} & H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\ = & \frac{1}{2r} \sum_{n=1}^{r} (y^n)^2 \\ \leq & \frac{1}{2r} \sum_{n=1}^{r} \max\{(x^n)^2, (y^n)^2\} \\ \leq & \frac{1}{2r} \sum_{n=1}^{r} d(gx^n, gy^n) \\ \leq & \frac{1}{2} \max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\} \\ \leq & \varphi\left[\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\}\right] \\ & +\psi\left[M(x^1, x^2, ..., x^r, y^1, y^2, ..., y^r)\right]. \end{aligned}$$

But If $(x^1)^2 + (x^2)^2 + ... + (x^r)^2 < (y^1)^2 + (y^2)^2 + ... + (y^r)^2$, then

$$\begin{aligned} & H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\ = & \frac{1}{2r} \sum_{n=1}^{r} (y^n)^2 \\ \leq & \frac{1}{2r} \sum_{n=1}^{r} \max\{(x^n)^2, (y^n)^2\} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2r}\sum_{n=1}^{r} d(gx^n, gy^n) \\
&\leq \frac{1}{2}\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\} \\
&\leq \varphi\left[\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\}\right] \\
&\quad +\psi\left[M(x^1, x^2, ..., x^r, y^1, y^2, ..., y^r)\right].
\end{aligned}
$$

Similarly, we obtain the same result for $(y^1)^2 + (y^2)^2 + ... + (y^r)^2 < (x^1)^2 + (x^2)^2 + ... + (x^r)^2$. Thus the contractive condition (2.1) is satisfied for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in [0, 1)$. Again, for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., $x^r \in [0, 1)$ and y^1, y^2, ..., $y^r = 1$, we have

$$
\begin{aligned}
&H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\
&= \frac{1}{2r}\sum_{n=1}^{r} (x^n)^2 \\
&\leq \frac{1}{2r}\sum_{n=1}^{r} \max\{(x^n)^2, (y^n)^2\} \\
&\leq \frac{1}{2r}\sum_{n=1}^{r} d(gx^n, gy^n) \\
&\leq \frac{1}{2}\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\} \\
&\leq \varphi\left[\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\}\right] \\
&\quad +\psi\left[M(x^1, x^2, ..., x^r, y^1, y^2, ..., y^r)\right].
\end{aligned}
$$

Thus the contractive condition (2.1) is satisfied for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., $x^r \in [0, 1)$ and y^1, y^2, ..., $y^r = 1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$ with x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r = 1$. Hence, the hybrid pair $\{F, g\}$ satisfy the contractive condition (2.1), for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0, 0, ..., 0)$ is a common $r-$tupled fixed point of hybrid pair $\{F, g\}$. The function $F : X^r \to CB(X)$ involved in this example is not continuous on X^r.

Corollary 2.2. *Let (X, d) be a metric space. Assume $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings satisfying*

$$H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r))$$

$$\leq \varphi\left[\frac{1}{r}\sum_{n=1}^{r} d(gx^n, gy^n)\right] + \psi\left[M(x^1, ..., x^r, y^1, ..., y^r)\right],$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $\varphi \in \Phi$ *and* $\psi \in \Psi$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds:*
(*a*) F *and* g *are* $w-$*compatible.* $\lim_{i\to\infty} g^i x^1 = y^1$, $\lim_{i\to\infty} g^i x^2 = y^2$, ..., $\lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$ *and* g *is continuous at* y^1, y^2, ..., y^r.
(*b*) g *is* $F-$*weakly commuting for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$, gx^1, gx^2, ..., gx^r *are fixed points of* g, *that is,* $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.
(*c*) g *is continuous at* x^1, x^2, ..., x^r. $\lim_{i\to\infty} g^i y^1 = x^1$, $\lim_{i\to\infty} g^i y^2 = x^2$, ..., $\lim_{i\to\infty} g^i y^r = x^r$ *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$.
(*d*) $g(C\{F, g\})$ *is a singleton subset of* $C\{F, g\}$.

Proof. It suffices to remark that

$$\frac{1}{r}\sum_{n=1}^{r} d(gx^n, gy^n) \leq \max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\}.$$

Then, we apply Theorem 2.1, since φ is non-decreasing. □

If we put $g = I$ (the identity mapping) in the Theorem 2.1, we get the following result:

Corollary 2.3. *Let* (X, d) *be a complete metric space,* $F : X^r \to CB(X)$ *be a mapping satisfying*

$$\begin{aligned} & H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\ \leq\ & \varphi\left[\max\left\{d(x^1, y^1), ..., d(x^r, y^r)\right\}\right] + \psi\left[m(x^1, ..., x^r, y^1, ..., y^r)\right], \end{aligned}$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $\varphi \in \Phi$ *and* $\psi \in \Psi$. *Then* F *has an* $r-$*tupled fixed point.*

If we put $g = I$ (the identity mapping) in the Corollary 2.2, we get the following result:

Corollary 2.4. *Let* (X, d) *be a complete metric space,* $F : X^r \to CB(X)$ *be a mapping satisfying*

$$\begin{aligned} & H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\ \leq\ & \varphi\left[\frac{1}{r}\sum_{n=1}^{r} d(x^n, y^n)\right] + \psi\left[m(x^1, ..., x^r, y^1, ..., y^r)\right], \end{aligned}$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $\varphi \in \Phi$ *and* $\psi \in \Psi$. *Then* F *has an* $r-$*tupled fixed point.*

If we put $\psi(t) = 0$ in Theorem 2.1, we get the following result:

Corollary 2.5. *Let* $(X,\ d)$ *be a metric space. Assume* $F : X^r \to CB(X)$ *and* $g : X \to X$ *be two mappings satisfying*

$$\begin{aligned} & H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\ \leq\ & \varphi\left[\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\}\right], \end{aligned}$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $\varphi \in \Phi$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds:*

(a) F *and* g *are* $w-$*compatible.* $\lim_{i\to\infty} g^i x^1 = y^1$, $\lim_{i\to\infty} g^i x^2 = y^2$, ..., $\lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1,\ x^2,\ ...,\ x^r) \in C\{F,\ g\}$ *and for some* y^1, y^2, ..., $y^r \in X$ *and* g *is continuous at* y^1, y^2, ..., y^r.

(b) g *is* $F-$*weakly commuting for some* $(x^1,\ x^2,\ ...,\ x^r) \in C\{F,\ g\}$, gx^1, gx^2, ..., gx^r *are fixed points of* g, *that is,* $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.

(c) g *is continuous at* x^1, x^2, ..., x^r. $\lim_{i\to\infty} g^i y^1 = x^1$, $\lim_{i\to\infty} g^i y^2 = x^2$, ..., $\lim_{i\to\infty} g^i y^r = x^r$ *for some* $(x^1,\ x^2,\ ...,\ x^r) \in C\{F,\ g\}$ *and for some* y^1, y^2, ..., $y^r \in X$.

(d) $g(C\{F,\ g\})$ *is a singleton subset of* $C\{F,\ g\}$.

If we put $\psi(t) = 0$ in Corollary 2.2, we get the following result:

Corollary 2.6. *Let* $(X,\ d)$ *be a metric space. Assume* $F : X^r \to CB(X)$ *and* $g : X \to X$ *be two mappings satisfying*

$$H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \leq \varphi\left[\frac{1}{r}\sum_{n=1}^{r} d(gx^n, gy^n)\right],$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $\varphi \in \Phi$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds:*

(a) F and g are $w-$compatible. $\lim_{i\to\infty} g^i x^1 = y^1, \lim_{i\to\infty} g^i x^2 = y^2, ..., \lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1,\ x^2,\ ...,\ x^r) \in C\{F,\ g\}$ *and for some* $y^1,\ y^2,\ ...,\ y^r \in X$ *and g is continuous at* $y^1,\ y^2,\ ...,\ y^r$.

(b) g is $F-$weakly commuting for some $(x^1,\ x^2,\ ...,\ x^r) \in C\{F,\ g\}$, $gx^1,\ gx^2,\ ...,\ gx^r$ *are fixed points of g, that is,* $g^2x^1 = gx^1,\ g^2x^2 = gx^2,\ ...,\ g^2x^r = gx^r$.

(c) g is continuous at $x^1,\ x^2,\ ...,\ x^r$. $\lim_{i\to\infty} g^i y^1 = x^1, \lim_{i\to\infty} g^i y^2 = x^2, ..., \lim_{i\to\infty} g^i y^r = x^r$ *for some* $(x^1,\ x^2,\ ...,\ x^r) \in C\{F,\ g\}$ *and for some* $y^1,\ y^2,\ ...,\ y^r \in X$.

(d) $g(C\{F,\ g\})$ *is a singleton subset of* $C\{F,\ g\}$.

If we put $g = I$ (the identity mapping) in the Corollary 2.5, we get the following result:

Corollary 2.7. *Let $(X,\ d)$ be a complete metric space, $F : X^r \to CB(X)$ be a mapping satisfying*

$$\begin{aligned} & H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\ \leq\ & \varphi\left[\max\left\{d(x^1, y^1), d(x^2, y^2), ..., d(x^r, y^r)\right\}\right], \end{aligned}$$

for all $x^1,\ x^2,\ ...,\ x^r,\ y^1,\ y^2,\ ...,\ y^r \in X$, *where $\varphi \in \Phi$. Then F has an $r-$tupled fixed point.*

If we put $g = I$ (the identity mapping) in the Corollary 2.6, we get the following result:

Corollary 2.8. *Let $(X,\ d)$ be a complete metric space, $F : X^r \to CB(X)$ be a mapping satisfying*

$$H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \leq \varphi\left[\frac{1}{r}\sum_{n=1}^{r} d(x^n, y^n)\right],$$

for all $x^1,\ x^2,\ ...,\ x^r,\ y^1,\ y^2,\ ...,\ y^r \in X$, *where $\varphi \in \Phi$. Then F has an $r-$tupled fixed point.*

If we put $\varphi(t) = kt$ where $0 < k < 1$ in Corollary 2.5, we get the following result:

Corollary 2.9. *Let $(X,\ d)$ be a metric space. Assume $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} & H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \\ \leq\ & k\max\left\{d(gx^1, gy^1), d(gx^2, gy^2), ..., d(gx^r, gy^r)\right\}, \end{aligned}$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $0 < k < 1$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds:*

(a) F *and* g *are* $w-$*compatible.* $\lim_{i\to\infty} g^i x^1 = y^1$, $\lim_{i\to\infty} g^i x^2 = y^2$, ..., $\lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$ *and* g *is continuous at* y^1, y^2, ..., y^r.

(b) g *is* $F-$*weakly commuting for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$, gx^1, gx^2, ..., gx^r *are fixed points of* g, *that is,* $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.

(c) g *is continuous at* x^1, x^2, ..., x^r. $\lim_{i\to\infty} g^i y^1 = x^1$, $\lim_{i\to\infty} g^i y^2 = x^2$, ..., $\lim_{i\to\infty} g^i y^r = x^r$ *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$.

(d) $g(C\{F, g\})$ *is a singleton subset of* $C\{F, g\}$.

If we put $\varphi(t) = kt$ where $0 < k < 1$ in Corollary 2.6, we get the following result:

Corollary 2.10. *Let* (X, d) *be a metric space. Assume* $F : X^r \to CB(X)$ *and* $g : X \to X$ *be two mappings satisfying*

$$H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \leq k \left[\frac{1}{r} \sum_{n=1}^{r} d(gx^n, gy^n) \right],$$

for all x^1, x^2, ..., x^r, y^1, y^2, ..., $y^r \in X$, *where* $0 < k < 1$. *Furthermore assume that* $F(X^r) \subseteq g(X)$ *and* $g(X)$ *is a complete subset of* X. *Then* F *and* g *have an* $r-$*tupled coincidence point. Moreover,* F *and* g *have a common* $r-$*tupled fixed point, if one of the following conditions holds:*

(a) F *and* g *are* $w-$*compatible.* $\lim_{i\to\infty} g^i x^1 = y^1$, $\lim_{i\to\infty} g^i x^2 = y^2$, ..., $\lim_{i\to\infty} g^i x^r = y^r$, *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$ *and* g *is continuous at* y^1, y^2, ..., y^r.

(b) g *is* $F-$*weakly commuting for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$, gx^1, gx^2, ..., gx^r *are fixed points of* g, *that is,* $g^2x^1 = gx^1$, $g^2x^2 = gx^2$, ..., $g^2x^r = gx^r$.

(c) g *is continuous at* x^1, x^2, ..., x^r. $\lim_{i\to\infty} g^i y^1 = x^1$, $\lim_{i\to\infty} g^i y^2 = x^2$, ..., $\lim_{i\to\infty} g^i y^r = x^r$ *for some* $(x^1, x^2, ..., x^r) \in C\{F, g\}$ *and for some* y^1, y^2, ..., $y^r \in X$.

(d) $g(C\{F, g\})$ *is a singleton subset of* $C\{F, g\}$.

If we put $g = I$ (the identity mapping) in the Corollary 2.9, we get the following result:

Corollary 2.11. *Let* $(X,\, d)$ *be a complete metric space,* $F : X^r \to CB(X)$ *be a mapping satisfying*

$$H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \leq k \max \left\{ d(x^1, y^1), d(x^2, y^2), ..., d(x^r, y^r) \right\},$$

for all $x^1,\, x^2,\, ...,\, x^r,\, y^1,\, y^2,\, ...,\, y^r \in X$, *where* $0 < k < 1$. *Then* F *has an* $r-$*tupled fixed point.*

If we put $g = I$ (the identity mapping) in the Corollary 2.10, we get the following result:

Corollary 2.12. *Let* $(X,\, d)$ *be a complete metric space,* $F : X^r \to CB(X)$ *be a mapping satisfying*

$$H(F(x^1, x^2, ..., x^r), F(y^1, y^2, ..., y^r)) \leq k \left[\frac{1}{r} \sum_{n=1}^{r} d(x^n, y^n) \right],$$

for all $x^1,\, x^2,\, ...,\, x^r,\, y^1,\, y^2,\, ...,\, y^r \in X$, *where* $0 < k < 1$. *Then* F *has an* $r-$*tupled fixed point.*

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[a]Department of Mathematics, Govt. Arts & Science P.G. College, Ratlam- 457001 (M.P.) India
Email address: bhavnadeshpande@yahoo.com

[b]Department of Mathematics, Govt. P. G. Arts and Science College, Ratlam-457001 (MP), India
Email address: amrishhanda83@gmail.com