EXISTENCE OF POSITIVE SOLUTIONS FOR SINGULAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study the existence of positive solutions for singular impulsive differential equations with integral boundary conditions

$$\begin{cases} u''(t) + q(t)f(t, u(t), u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = I_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -L_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ u(0) = \int_0^1 g(t)u(t)dt, \ u'(1) = 0, \end{cases}$$

where the nonlinearity $f(t, u, v)$ may be singular at $v = 0$. The proof is based on the theory of Leray-Schauder degree, together with a truncation technique. Some recent results in the literature are generalized and improved.

1. INTRODUCTION

Impulsive differential equations are basic instruments to study the dynamics of processes that are subjected to abrupt changes in their states. Recent development in this field has been focused by many applied problems, such as control theory [8,9], population dynamics [19] and medicine [4,5]. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [14].

During the last two decades, impulsive differential equations have been studied by many authors [1-3, 10, 13, 15-16, 20-25]. Many of them are on impulsive differential equation boundary value problems (BVPs for short). In recent years, there have been many studies related to impulsive multi-point boundary value problems [6-7, 11-12, 17, 26]. They include three, four, multi-point impulsive BVPs and impulsive

BVPs with integral boundary conditions. However, very few papers consider singular impulsive differential equations with integral boundary conditions.

In [2], using the Schauder's fixed point theorem, Agarwal et al. investigated the existence of at least one positive solution for singular BVPs for first and second order impulsive differential equations. In [18], using the Schauder's fixed point theorem, Miao et al. studied a singular BVP with integral boundary condition for a first-order impulsive differential equation. Motivated by [2], we extend the results in [18] to a second order singular impulsive differential equation. In this paper, we consider the following singular impulsive BVP

(1.1)
$$\begin{cases} u''(t) + q(t)f(t, u(t), u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = I_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -L_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ u(0) = \int_0^1 g(t)u(t)dt, \ u'(1) = 0, \end{cases}$$

where $0 < t_1 < t_2 < \cdots < t_p < 1$, $\mathbb{J}' = \mathbb{J} \setminus \{t_1, t_2, \cdots, t_p\}$, $\mathbb{J} = [0,1]$, $\Delta u(t_k)$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u(t_k) = u(t_k+0) - u(t_k-0)$, $u(t_k+0)$ and $u(t_k-0)$) represent the right and left limits of $u(t)$ at $t = t_k$, $\Delta u'(t_k)$ denotes the jump of $u'(t)$ at $t = t_k$, i.e., $\Delta u'(t_k) = u'(t_k + 0) - u'(t_k - 0)$, $u'(t_k + 0)$ and $u'(t_k - 0)$) represent the right and left derivative of $u(t)$ at $t = t_k$. We are mainly interested in the case that $f(t, u, v)$ may be singular at $v = 0$.

The method used in this paper mainly depends on the theory of Leray-Schauder degree. We first consider the existence of positive solutions for a constructed nonsingular BVP. Then, using Arzelà-Ascoli theorem, we obtain positive solutions for the singular problem that is approximated by the family of solutions to the nonsingular BVPs.

The following hypotheses are adopted throughout this paper:

(H_1) $q \in C[\mathbb{J}]$, $q(t) > 0$, $t \in (0,1)$, $f : \mathbb{J} \times [0, \infty) \times (0, \infty) \to (0, \infty)$ is continuous, I_k, $L_k : [0, \infty) \times [0, \infty) \to [0, \infty)(k = 1, 2, \cdots, p)$ are continuous, $g \in L^1[\mathbb{J}]$, $g(t) \geq 0$, $t \in \mathbb{J}$ and $0 \leq \sigma := \int_0^1 g(t)dt < 1$.

(H_2) $f(t, u, v) \leq h(u)[f_1(v) + f_2(v)]$, $(t, u, v) \in \mathbb{J} \times [0, \infty) \times (0, \infty)$, where $f_1(u) > 0$ is continuous, nonincreasing on $(0, \infty)$; $h(u)$, $f_2(u) \geq 0$ are continuous on $[0, \infty)$.

(H_3) For any given constants $K > 0$, $N > 0$, there is a constant $\gamma \in [0, 1)$ and a continuous function $\psi_{K,N} : \mathbb{J} \to (0, \infty)$ such that $f(t, u, v) \geq \psi_{K,N}(t)u^\gamma$, $(t, u, v) \in \mathbb{J} \times [0, K] \times (0, N]$.

(H_4) $\int_0^1 q(t)f_1(\rho(t))dt < \infty$, where $\rho(t) := \int_t^1 s^\gamma q(s)\psi_{K,N}(s)ds$, for any $K, N > 0$.

(H_5)

$$\sup_{c\in(0,\infty)} \frac{(1-\sigma)c}{\sum\limits_{k=1}^{p} \max\limits_{u,v\in[0,c]} I_k(u,v) + \Gamma^{-1}\big(\frac{\sum\limits_{k=1}^{p} \max\limits_{u,v\in[0,c]} L_k(u,v)}{f_1(c)} + \max\limits_{u\in[0,c]} h(u)\int_0^1 q(t)dt\big)} > 1,$$

where $\Gamma(\mu) := \int_0^\mu \frac{1}{f_1(z) + f_2(z)}dz, \ \mu > 0.$

2. Preliminaries

For convenience, we first give some notations:

(1) $\mathbb{J}_0 = [0, t_1], \ \mathbb{J}_k = (t_k, t_{k+1}], k = 1, 2, \cdots, p-1, \ \mathbb{J}_p = (t_p, 1].$

(2) $PC^1[\mathbb{J}] = \{u : \mathbb{J} \to \mathbb{R} \mid u'(t) \text{ is continuous in } \mathbb{J}' \text{ and there exist } u'(t_k - 0) = u'(t_k), \ u'(t_k + 0) < \infty(k = 1, 2, \cdots, p)\}.$

Obviously, $(PC^1[\mathbb{J}], ||u||_{PC^1})$ is a Banach space with the norm $||u||_{PC^1} = \max\{||u||, ||u'||\}$, here $||u|| = \sup\limits_{t\in\mathbb{J}} |u(t)|$. $(PC^1[\mathbb{J}], ||u||_{PC^1})$ is abbreviated as $PC^1[\mathbb{J}]$.

Definition 2.1. We say a function $u \in PC^1[\mathbb{J}]$ is a *positive solution* to problem (1.1) if u satisfies (1.1) and $u(t) > 0, \ t \in (0, 1).$

Definition 2.2 ([14]). A set $S \subset PC^1[\mathbb{J}]$ is said to be *quasiequicontinuous* if for all $u \in S$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $s, t \in \mathbb{J}_k(k = 1, 2, \cdots, p)$ and $|s-t| < \delta$ implies

$$|u(s) - u(t)| < \varepsilon \text{ and } |u'(s) - u'(t)| < \varepsilon.$$

We present the following result about relatively compact sets in $PC^1[\mathbb{J}]$ which is a consequence of the Arzelà-Ascoli Theorem.

Lemma 2.3 ([14]). *A set $S \subset PC^1[\mathbb{J}]$ is relatively compact in $PC^1[\mathbb{J}]$ if and only if S is bounded and quasiequicontinuous.*

Lemma 2.4. *Suppose that $e \in L^1[\mathbb{J}], \ e(t) > 0, \ t \in (0, 1), \ a_k, \ b_k \geq 0 \ (k = 1, 2, \cdots, p), a \geq 0$ are constants. Then, BVP*

(2.1)
$$\begin{cases} u''(t) + e(t) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = a_k, k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -b_k, k = 1, 2, \cdots, p, \\ u(0) = \int_0^1 g(t)u(t)dt, \ u'(1) = a \end{cases}$$

has a unique solution. Moreover, this solution can be expressed by
(2.2)
$$u(t) = \sum_{t_k < t} a_k + \sum_{k=1}^{p} G(t, t_k) b_k + \int_0^1 G(t, s) e(s) ds + \frac{1}{1 - \sigma} \left[\int_0^1 g(t) \sum_{t_k < t} a_k dt \right.$$
$$\left. + \int_0^1 g(t) \sum_{k=1}^{p} G(t, t_k) b_k dt + \int_0^1 g(t) \int_0^1 G(t, s) e(s) ds dt \right] + \frac{\int_0^1 t g(t) dt}{1 - \sigma} a,$$

where

$$G(t, s) = \begin{cases} t, & 0 < t \le s < 1, \\ s, & 0 < s < t < 1. \end{cases}$$

Proof. It is easy to verify that (2.2) is a solution of (2.1). On the other hand if u is a solution of (2.1), then

$$u''(t) = -e(t), \quad t \in \mathbb{J}'.$$

For any $t \in \mathbb{J}_k$, $k = 0, 1, 2, \cdots, p$, integrating on the both sides of the above equation from 0 to t, one obtains

$$u'(t) = u'(0) - \sum_{t_k < t} b_k - \int_0^t e(s) ds.$$

Using the boundary condition $u'(1) = a$, we have $u'(0) = \sum_{t_k < 1} b_k + \int_0^1 e(s) ds + a$, and then

(2.3)
$$u'(t) = \sum_{t \le t_k} b_k + \int_t^1 e(s) ds + a.$$

Integrate on the both sides of (2.3) from 0 to t, and one obtains

(2.4)
$$u(t) = u(0) + \sum_{t_k < t} a_k + \sum_{t_k < t} b_k t_k + \sum_{t \le t_k} b_k t + \int_0^t s e(s) ds + \int_t^1 t e(s) ds + at,$$

Multiplying (2.4) with $g(t)$ and integrating it from 0 to 1, we have

(2.5)
$$u(0) = \frac{1}{1 - \sigma} \left[\int_0^1 g(t) \sum_{t_k < t} a_k dt + \int_0^1 g(t) \sum_{t_k < t} b_k t_k dt + \int_0^1 g(t) \sum_{t \le t_k} b_k t dt \right.$$
$$\left. + \int_0^1 g(t) \left(\int_0^t s e(s) ds + \int_t^1 t e(s) ds \right) dt \right] + \frac{\int_0^1 t g(t) dt}{1 - \sigma} a,$$

and substituting (2.5) into (2.4) yields

$$u(t) = \frac{1}{1-\sigma}\left[\int_0^1 g(t)\sum_{t_k<t} a_k dt + \int_0^1 g(t)\sum_{t_k<t} b_k t_k dt + \int_0^1 g(t)\sum_{t\leq t_k} b_k t dt\right.$$

$$+ \int_0^1 g(t)(\int_0^t se(s)ds + \int_t^1 te(s)ds)dt\bigg] + \sum_{t_k<t} a_k + \sum_{t_k<t} b_k t_k + \sum_{t\leq t_k} b_k t$$

$$+ \int_0^t se(s)ds + \int_t^1 te(s)ds + \frac{\int_0^1 tg(t)dt}{1-\sigma}a,$$

that is,

$$u(t) = \sum_{t_k<t} a_k + \sum_{k=1}^p G(t,t_k)b_k + \int_0^1 G(t,s)e(s)ds + \frac{1}{1-\sigma}\left[\int_0^1 g(t)\sum_{t_k<t} a_k dt\right.$$

$$+ \int_0^1 g(t)\sum_{k=1}^p G(t,t_k)b_k dt + \int_0^1 g(t)\int_0^1 G(t,s)e(s)dsdt\bigg] + \frac{\int_0^1 tg(t)dt}{1-\sigma}a, \ t \in \mathbb{J}.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In order to solve (1.1), we consider the following BVP

(2.6)
$$\begin{cases} u''(t) + q(t)F(t,u(t),u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = \bar{I}_k(u(t_k),u'(t_k)), k = 1,2,\cdots,p, \\ \Delta u'(t_k) = -\bar{L}_k(u(t_k),u'(t_k)), k = 1,2,\cdots,p, \\ u(0) = \int_0^1 g(t)u(t)dt, \ u'(1) = a, \end{cases}$$

where $F : \mathbb{J} \times \mathbb{R}^2 \to (0,\infty)$ is continuous, $\bar{I}_k$, $\bar{L}_k : \mathbb{R}^2 \to [0,\infty)(k = 1,2,\cdots,p)$ are continuous, q, g are the same as in (H_1), and $a \geq 0$ is a constant.

Let $u \in PC^1[\mathbb{J}]$. We define an operator $T : PC^1[\mathbb{J}] \to PC^1[\mathbb{J}]$ by

(2.7)
$$(Tu)(t) = \sum_{t_k<t} \bar{I}_k(u(t_k),u'(t_k)) + \sum_{k=1}^p G(t,t_k)\bar{L}_k(u(t_k),u'(t_k))$$

$$+ \int_0^1 G(t,s)q(s)F(s,u(s),u'(s))ds + \frac{1}{1-\sigma}\left[\int_0^1 g(t)\sum_{t_k<t} \bar{I}_k(u(t_k),u'(t_k))dt\right.$$

$$+ \int_0^1 g(t)\sum_{k=1}^p G(t,t_k)\bar{L}_k(u(t_k),u'(t_k))dt$$

$$+ \int_0^1 g(t)\int_0^1 G(t,s)q(s)F(s,u(s),u'(s))dsdt\bigg] + \frac{\int_0^1 tg(t)dt}{1-\sigma}a.$$

We have the following result:

Lemma 2.5. $T : PC^1[\mathbb{J}] \to PC^1[\mathbb{J}]$ *is completely continuous.*

Proof. It is easy to prove that $T : PC^1[\mathbb{J}] \to PC^1[\mathbb{J}]$ is well defined.

By the continuity of $\bar{I}_k$, $\bar{L}_k(k = 1, 2, \cdots, p)$ and F, we have T is continuous.

Next we shall show that T is compact. Suppose $B = \{u \in PC^1[\mathbb{J}]|\ ||u||_{PC^1} \leq r\} \subset PC^1[\mathbb{J}]$ is a bounded set. For any $u \in B$, which implies $||u|| \leq r$, $||u'|| \leq r$, we have

$$||Tu|| \leq \frac{2 - \sigma}{1 - \sigma}\Bigg[\sum_{k=1}^{p} \max_{-r \leq x \leq r, -r \leq y \leq r} \bar{I}_k(x, y) + \sum_{k=1}^{p} \max_{-r \leq x \leq r, -r \leq y \leq r} \bar{L}_k(x, y)$$

$$+ \max_{0 \leq s \leq 1, -r \leq x \leq r, -r \leq y \leq r} F(s, x, y) \int_0^1 q(s)ds\Bigg] + \frac{\int_0^1 tg(t)dt}{1 - \sigma}a.$$

In addition,

$$||(Tu)'|| = \max_{t \in \mathbb{J}}\Bigg[\sum_{t \leq t_k} \bar{L}_k(u(t_k), u'(t_k)) + \int_t^1 q(s)F(s, u(s), u'(s))ds + a\Bigg]$$

$$\leq \sum_{k=1}^{p} \max_{-r \leq x \leq r, -r \leq y \leq r} \bar{L}_k(x, y) + \max_{0 \leq s \leq 1, -r \leq x \leq r, -r \leq y \leq r} F(s, x, y)\int_0^1 q(s)ds + a$$

$$:= r_0.$$

This implies that $T(B)$ is uniformly bounded.

For any given $\varepsilon > 0$, t, $s \in \mathbb{J}_k(k = 0, 1, \cdots, p)$ (without loss of generality, let $s < t$), when $t \to s$, we obtain

$$|(Tu)(t) - (Tu)(s)| = \left|\int_s^t (Tu)'(\tau)d\tau\right| \leq r_0|t - s| \to 0.$$

Additional,

$$|(Tu)'(t) - (Tu)'(s)| = \left|\sum_{s \leq t_k \leq t} \bar{L}_k(u(t_k), u'(t_k)) + \int_s^t q(s)F(s, u(s), u'(s))ds\right|$$

$$\leq \sum_{s \leq t_k \leq t} \max_{-r \leq x \leq r, -r \leq y \leq r} \bar{L}_k(x, y)$$

$$+ \max_{0 \leq \tau \leq 1, -r \leq x \leq r, -r \leq y \leq r} F(\tau, x, y)\int_s^t q(s)ds$$

$$\to 0.$$

This implies that $T(B)$ is quasiequicontinuous. By Lemma 2.3, $T(B)$ is relatively compact. Therefore, T is completely continuous. $\qquad\square$

Now we state a existence principle, which plays an important role in our proof of main results.

Lemma 2.6 (Existence Principle). *Assume that there exists a constant*

$$R > \frac{\int_0^1 tg(t)dt}{1 - \sigma} a \geq 0$$

independent of λ, such that for $\lambda \in (0,1)$, $||u||_{PC^1} \neq R$, where $u(t)$ satisfies

$$(2.8)_\lambda \quad \begin{cases} u''(t) + \lambda q(t)F(t, u(t), u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = \lambda \bar{I}_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -\lambda \bar{L}_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ u(0) = \int_0^1 g(t)u(t)dt, \ u'(1) = a. \end{cases}$$

Then $(2.8)_1$ has at least one solution $u(t)$ such that $||u||_{PC^1} \leq R$.

Proof. For any $\lambda \in \mathbb{J}$, $u \in PC^1[\mathbb{J}]$, define one operator

$$\begin{aligned}
(2.9) \quad (N_\lambda u)(t) = {}& \lambda \sum_{t_k < t} \bar{I}_k(u(t_k), u'(t_k)) + \lambda \sum_{k=1}^p G(t, t_k)\bar{L}_k(u(t_k), u'(t_k)) \\
& + \lambda \int_0^1 G(t, s)q(s)F(s, u(s), u'(s))ds \\
& + \frac{\lambda}{1 - \sigma} \Bigg[\int_0^1 g(t) \sum_{t_k < t} \bar{I}_k(u(t_k), u'(t_k))dt \\
& + \int_0^1 g(t) \sum_{k=1}^p G(t, t_k)\bar{L}_k(u(t_k), u'(t_k))dt \\
& + \int_0^1 g(t) \int_0^1 G(t, s)q(s)F(s, u(s), u'(s))dsdt \Bigg] + \frac{\int_0^1 tg(t)dt}{1 - \sigma} a.
\end{aligned}$$

By Lemma 2.5, $N_\lambda : PC^1[\mathbb{J}] \to PC^1[\mathbb{J}]$ is completely continuous. It can be verified that a solution of BVP $(2.8)_\lambda$ equivalent to a fixed point of N_λ in $PC^1[\mathbb{J}]$. Let $\Omega = \{u \in PC^1[\mathbb{J}] | \ ||u||_{PC^1} < R\}$, then Ω is an open set in $PC^1[\mathbb{J}]$. If there exists $u \in \partial\Omega$ such that $N_1 u = u$, then $u(t)$ is a solution of $(2.8)_1$ with $||u||_{PC^1} \leq R$. Thus the conclusion is true. Otherwise, for any $u \in \partial\Omega$, $N_1 u \neq u$. If $\lambda = 0$, for $u \in \partial\Omega$, $(I - N_0)u(t) = u(t) - N_0 u(t) = u(t) - \frac{\int_0^1 tg(t)dt}{1-\sigma} a \not\equiv 0$ since $||u||_{PC^1} = R > \frac{\int_0^1 tg(t)dt}{1-\sigma} a$, so $N_0 u \neq u$ for any $u \in \partial\Omega$. For $\lambda \in (0,1)$, if there is a solution $u(t)$ to BVP $(2.8)_\lambda$, by the assumption, one gets $||u||_{PC^1} \neq R$, which is a contradiction to $u \in \partial\Omega$.

In a word, for any $u \in \partial\Omega$ and $\lambda \in \mathbb{J}$, $N_\lambda u \neq u$. Homotopy invariance of Leray-Schauder degree deduce that

$$\text{Deg}\{I - N_1, \Omega, 0\} = \text{Deg}\{I - N_0, \Omega, 0\} = 1.$$

Hence, N_1 has a fixed point u in Ω. That is, BVP $(2.8)_1$ has a solution $u(t)$ with $||u||_{PC^1} \leq R$. The proof is completed. $\square$

Lemma 2.7. *Suppose* (H_1) *holds. If u is a solution to problem* (2.6), *then*
  (i) $u(t)$ *is concave on* $\mathbb{J}_k(k = 0, 1, \cdots, p)$;
  (ii) $u'(t) \geq a,\ t \in \mathbb{J}',\ u'(t_k - 0) \geq u'(t_k + 0) \geq a,\ \Delta u(t_k) \geq 0,\ k = 1, 2, \cdots, p$;
  (iii) $u(t) \geq \frac{\int_0^1 tg(t)dt}{1-\sigma}a$ *and* $u(t) \geq t||u||,\ t \in \mathbb{J}$.

Proof. (i) Because $u(t)$ is a solution of problem (2.6), we have

$$u''(t) = -q(t)F(t, u(t), u'(t)) < 0,\ t \in \mathbb{J}'.$$

Therefore u' is nonincreasing on $\mathbb{J}'$, which implies $u(t)$ is concave on $\mathbb{J}_k(k = 0, 1, \cdots, p)$.

(ii) Since u' is nonincreasing on $\mathbb{J}'$, and $u'(1) = a$, therefore $u'(t) \geq a,\ t \in \mathbb{J}',\ u'(t_k - 0) \geq u'(t_k + 0) \geq a,\ \Delta u(t_k) \geq 0,\ k = 1, 2, \cdots, p$.

(iii) From Lemma 2.4, we have for $t \in \mathbb{J}$,

$$u(t) = \sum_{t_k < t} \bar{I}_k + \sum_{k=1}^{p} G(t, t_k)\bar{L}_k + \int_0^1 G(t, s)F(s, u(s), u'(s))ds$$

$$+ \frac{1}{1-\sigma}\left[\int_0^1 g(t)\sum_{t_k < t}\bar{I}_k dt + \int_0^1 g(t)\sum_{k=1}^{p}G(t, t_k)\bar{L}_k dt\right.$$

$$\left.+ \int_0^1 g(t)\int_0^1 G(t, s)F(s, u(s), u'(s))dsdt\right] + \frac{\int_0^1 tg(t)dt}{1-\sigma}a$$

$$\geq \frac{\int_0^1 tg(t)dt}{1-\sigma}a.$$

Because $u(t)$ is concave, we have

$$\frac{u(t)}{t} \geq \frac{u(1)}{1},\ t \in \mathbb{J},$$

thus $u(t) \geq t||u||,\ t \in \mathbb{J}$. $\square$

3. Existence Results

In this section, we give the main results for BVP (1.1) in this paper.

Theorem 3.1. *Suppose* (H_1)-(H_5) *hold, then BVP* (1.1) *has at least one positive solution.*

Proof. Step 1. From (H_5), we choose $M > 0$ and $0 < \frac{\int_0^1 tg(t)dt}{1-\sigma}\varepsilon < M$ such that

(3.1)

$$\frac{(1-\sigma)M}{\sum\limits_{k=1}^{p}\max\limits_{u,v\in[0,M]}I_k(u,v)+\Gamma^{-1}(\int_0^\varepsilon \frac{dz}{f_1(z)+f_2(z)}+\frac{\sum\limits_{k=1}^{p}\max\limits_{u,v\in[0,M]}L_k(u,v)}{f_1(M)}+\max\limits_{u\in[0,M]}h(u)\int_0^1 q(t)dt)}>1.$$

Furthermore, we have

(3.2) $$\frac{M}{\Gamma^{-1}(\int_0^\varepsilon \frac{dz}{f_1(z)+f_2(z)}+\frac{\sum\limits_{k=1}^{p}\max\limits_{u,v\in[0,M]}L_k(u,v)}{f_1(M)}+\max\limits_{u\in[0,M]}h(u)\int_0^1 q(t)dt)}>1.$$

Let $n_0 \in \{1,2,\cdots\}$ satisfy that $\dfrac{1}{n_0} < \varepsilon$, and set $\mathbb{N}_0 = \{n_0, n_0+1, n_0+2, \cdots\}$.

In what follows, we show that the following BVP

(3.3)
$$\begin{cases} u''(t) + q(t)f(t, u(t), u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = I_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -L_k(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ u(0) = \displaystyle\int_0^1 g(t)u(t)dt, \ u'(1) = \frac{1}{m} \end{cases}$$

has a positive solution for each $m \in \mathbb{N}_0$.

To this end, we consider the following BVP

(3.4)
$$\begin{cases} u''(t) + q(t)f^*(t, u(t), u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = I_k^*(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -L_k^*(u(t_k)u'(t_k)), k = 1, 2, \cdots, p, \\ u(0) = \displaystyle\int_0^1 g(t)u(t)dt, \ u'(1) = \frac{1}{m}, \end{cases}$$

where

$$f^*(t, u, v) = \begin{cases} f(t, u, v), & u \geq 0, \ v \geq \frac{1}{m}, \\ f(t, u, \frac{1}{m}), & u \geq 0, \ v < \frac{1}{m}, \end{cases}$$

$$I_k^*(u, v) = \begin{cases} I_k(u, v), & u \geq 0, \ v \geq \frac{1}{m}, \\ I_k(u, \frac{1}{m}), & u \geq 0, \ v < \frac{1}{m}, \end{cases}$$

$$L_k^*(u, v) = \begin{cases} L_k(u, v), & u \geq 0, \ v \geq \frac{1}{m}, \\ L_k(u, \frac{1}{m}), & u \geq 0, \ v < \frac{1}{m}, \end{cases}$$

then $f^* : \mathbb{J} \times [0, \infty) \times \mathbb{R} \to (0, \infty)$, $I_k^*, \ L_k^* : [0, \infty) \times \mathbb{R} \to [0, \infty)$, $(k = 1, 2, \cdots, p)$.

To obtain a solution of BVP (3.4) for each $m \in \mathbb{N}_0$, by applying Lemma 2.6, we consider the family of BVPs

$$(3.5) \quad \begin{cases} u''(t) + \lambda q(t) f^*(t, u(t), u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = \lambda I_k^*(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -\lambda L_k^*(u(t_k), u'(t_k)), k = 1, 2, \cdots, p, \\ u(0) = \int_0^1 g(t) u(t) dt, \ u'(1) = \frac{1}{m}, \end{cases}$$

where $\lambda \in \mathbb{J}$. Let $u(t)$ be a solution of (3.5). From Lemma 2.7, we observe that $u(t)$ is concave on $\mathbb{J}_k (k = 0, 1, \cdots, p)$, $u'(t) \geq \frac{1}{m}$, $t \in \mathbb{J}'$, $u'(t_k - 0) \geq u'(t_k + 0) \geq \frac{1}{m}$, $k = 1, 2, \cdots, p$ and $u(t) \geq \frac{\int_0^1 t g(t) dt}{1-\sigma} \frac{1}{m}$, $u(t) \geq t \|u\|_{PC^1}$, $t \in \mathbb{J}$.

For any $x \in \mathbb{J}'$, by (H$_2$), we have

$$-u''(x) = \lambda q(x) f^*(x, u(x), u'(x)) = \lambda q(x) f(x, u(x), u'(x))$$
$$\leq q(x) h(u(x)) [f_1(u'(x)) + f_2(u'(x))].$$

Multiply the above inequality by $\frac{1}{f_1(u'(x)) + f_2(u'(x))}$ and integrate it from $t(t \in \mathbb{J})$ to 1 yield that

$$\int_0^{u'(t)} \frac{dz}{f_1(z) + f_2(z)} \leq \int_0^{\frac{1}{m}} \frac{dz}{f_1(z) + f_2(z)} + \frac{\sum_{k=1}^p \max_{u \in [0, u(1)], v \in [0, u'(0)]} L_k(u, v)}{f_1(u'(0))}$$
$$+ \max_{u \in [0, u(1)]} h(u) \int_0^1 q(x) dx,$$

For any $t \in \mathbb{J}$, we have

$$(3.6) \quad u'(t) \leq \Gamma^{-1} \Bigg(\int_0^{\frac{1}{m}} \frac{dz}{f_1(z) + f_2(z)} + \frac{\sum_{k=1}^p \max_{u \in [0, u(1)], v \in [0, u'(0)]} L_k(u, v)}{f_1(u'(0))}$$
$$+ \max_{u \in [0, u(1)]} h(u) \int_0^1 q(x) dx \Bigg)$$

and

$$(3.7) \quad u'(0) \leq \Gamma^{-1} \Bigg(\int_0^{\varepsilon} \frac{dz}{f_1(z) + f_2(z)} + \frac{\sum_{k=1}^p \max_{u \in [0, u(1)], v \in [0, u'(0)]} L_k(u, v)}{f_1(u'(0))}$$
$$+ \max_{u \in [0, u(1)]} h(u) \int_0^1 q(x) dx \Bigg).$$

Integrate (3.6) from 0 to 1, and one obtains

(3.8)
$$(1-\sigma)u(1) \leq \sum_{k=1}^{p} \max_{u\in[0,u(1)],v\in[0,u'(0)]} I_k(u,v) + \Gamma^{-1}\bigg(\int_0^{\varepsilon} \frac{dz}{f_1(z)+f_2(z)}$$

$$+ \frac{\sum_{k=1}^{p} \max_{u\in[0,u(1)],v\in[0,u'(0)]} L_k(u,v)}{f_1(u'(0))} + \max_{u\in[0,u(1)]} h(u) \int_0^1 q(x)dx \bigg).$$

If $u'(0) \geq u(1)$, then $||u||_{PC^1} = \max\{u(1),u'(0)\} = u'(0)$. By (3.7) one obtains

$$\frac{u'(0)}{\Gamma^{-1}(\int_0^{\varepsilon} \frac{dz}{f_1(z)+f_2(z)} + \frac{\sum_{k=1}^{p} \max_{u,v\in[0,u'(0)]} L_k(u,v)}{f_1(u'(0))} + \max_{u\in[0,u'(0)]} h(u)\int_0^1 q(t)dt)} \leq 1,$$

which together with (3.2) implies

$$||u||_{PC^1} = u'(0) \neq M.$$

If $u'(0) < u(1)$, then $||u||_{PC^1} = \max\{u(1),u'(0)\} = u(1)$. By (3.8), we obtain

$$\frac{(1-\sigma)u(1)}{\sum_{k=1}^{p} \max_{u,v\in[0,u(1)]} L_k(u,v) + \Gamma^{-1}(\int_0^{\varepsilon} \frac{dz}{f_1(z)+f_2(z)} + \frac{\sum_{k=1}^{p} \max_{u,v\in[0,u(1)]} L_k(u,v)}{f_1(u(1))} + \max_{u\in[0,u(1)]} h(u)\int_0^1 q(t)dt)}$$
$$\leq 1,$$

which together with (3.1) implies

$$||u||_{PC^1} = u(1) \neq M.$$

By Lemma 2.6, for any fixed $m \in \mathbb{N}_0$, BVP (3.4) has at last one positive solution, denoted by $u_m(t)$, and $||u_m||_{PC^1} \leq M$. From Lemma 2.7, we note that $u_m(t) \geq \frac{\int_0^1 tg(t)dt}{1-\sigma}\frac{1}{m}, t \in \mathbb{J}, u'_m(t) \geq \frac{1}{m}, \ t \in \mathbb{J}', u'(t_k-0) \geq u'(t_k+0) \geq \frac{1}{m}$. So $f^*(t,u_m(t),u'_m(t)) = f(t,u_m(t),u'_m(t))$, $I_k^*(u_m(t),u'_m(t)) = I_k(u_m(t),u'_m(t))$, $L_k^*(u_m(t),u'_m(t)) = L_k(u_m(t),u'_m(t))(k = 1,2,\cdots,p)$. Therefore, $u_m(t)$ is the solution to BVP (3.3).

Step 2. By

(3.9)
$$0 < \frac{\int_0^1 tg(t)dt}{1-\sigma}\frac{1}{m} \leq u_m(t) \leq M, \ t \in \mathbb{J},$$

$$0 < \frac{1}{m} \leq u'_m(t) \leq M, \ t \in \mathbb{J}',$$

$$0 < \frac{1}{m} \leq u'_m(t_k+0) \leq u'_m(t_k-0) \leq M, \ k = 1,2,\cdots,p,$$

we conclude that

(3.10)
$$u'_m(t) \geq L^{\frac{1}{1-\gamma}} \int_t^1 s^\gamma q(s)\psi_{M,M}(s)ds := \varphi(t), \quad t \in J^0,$$
$$u'_m(t_k - 0) \geq u'_m(t_k + 0) \geq \varphi(t_k), \quad k = 1, 2, \cdots, p.$$

where $L > 0$ is a constant.

In fact, (H$_3$) guarantees the existence of a function $\psi_{M,M}$ which is continuous on $\mathbb{J}$ and positive on $(0,1)$ with

$$f(t, u_m(t), u'_m(t)) \geq \psi_{M,M}(t)u^\gamma, \quad t \in \mathbb{J}, \ \gamma \in [0,1).$$

By Lemma 2.4 and Lemma 2.7, one has

(3.11)

$$u_m(t) = \sum_{t_k < t} I_k(u_m(t_k), u'_m(t_k)) + \sum_{k=1}^p G(t, t_k)L_k(u_m(t_k), u'_m(t_k))$$

$$+ \int_0^1 G(t, s)q(s)f(s, u_m(s), u'_m(s))ds + \frac{1}{1-\sigma}\left[\int_0^1 g(t) \sum_{t_k < t} I_k(u_m(t_k), u'_m(t_k))dt\right.$$

$$+ \int_0^1 g(t) \sum_{k=1}^p G(t, t_k)L_k(u_m(t_k), u'_m(t_k))dt$$

$$\left. + \int_0^1 g(t) \int_0^1 G(t, s)q(s)f(s, u_m(s), u'_m(s))dsdt\right] + \frac{\int_0^1 tg(t)dt}{1-\sigma}\frac{1}{m}$$

$$\geq \int_0^1 G(t, s)q(s)f(s, u_m(s), u'_m(s))ds$$

$$+ \frac{1}{1-\sigma}\int_0^1 g(t) \int_0^1 G(t, s)q(s)f(s, u_m(s), u'_m(s))dsdt$$

$$\geq t \int_0^1 sq(s)\psi_{M,M}(s)(u_m(s))^\gamma ds + \frac{\int_0^1 tg(t)dt}{1-\sigma}\int_0^1 sq(s)\psi_{M,M}(s)(u_m(s))^\gamma ds$$

$$\geq (u_m(1))^\gamma[t \int_0^1 s^{\gamma+1}q(s)\psi_{M,M}(s)ds + \frac{\int_0^1 tg(t)dt}{1-\sigma}\int_0^1 s^{\gamma+1}q(s)\psi_{M,M}(s)ds],$$

$$= (u_m(1))^\gamma(tL_1 + L_0), \ t \in \mathbb{J},$$

where $L_1 := \int_0^1 s^{\gamma+1}q(s)\psi_{M,M}(s)ds$, $L_0 := \frac{\int_0^1 tg(t)dt}{1-\sigma}\int_0^1 s^{\gamma+1}q(s)\psi_{M,M}(s)ds$. Furthermore, we have

$$u_m(1) \geq (L_1 + L_0)^{\frac{1}{1-\gamma}} := L^{\frac{1}{1-\gamma}},$$
$$u_m(t) \geq L^{\frac{1}{1-\gamma}}(tL_1 + L_0), \ t \in \mathbb{J}.$$

Because $u_m(t)$ is a solution of (3.3), for $s \in \mathbb{J}'$,

$$-u_m''(s) = q(s)f(s, u_m(s), u_m'(s)),$$

and integrate it from $t(t \in \mathbb{J})$ to 1, one obtains

$$u'(t) = \frac{1}{m} + \sum_{t < t_k} L_k(u(t_k, u'(t_k))) + \int_t^1 q(s)f(s, u(s), u'(s))ds$$

$$\geq \int_t^1 s^\gamma q(s)\psi_{M,M}(s)ds(u_m(1))^\gamma$$

$$\geq L^{\frac{1}{1-\gamma}} \int_t^1 s^\gamma q(s)\psi_{M,M}(s)ds = \varphi(t),$$

and then, we have

$$u_m'(t) \geq \varphi(t), \quad t \in J^0,$$
$$u_m'(t_k - 0) \geq u_m'(t_k + 0) \geq \varphi(t_k), \quad k = 1, 2, \cdots, p,$$

Thus, (3.10) holds.

Step 3. It remains to show that $\{u_m^{(j)}(t)\}_{m \in \mathbb{N}_0}(j = 0, 1)$ are both uniformly bounded and quasiequicontinuous on $\mathbb{J}$. By (3.9), we have $\{u_m^{(j)}(t)\}_{m \in \mathbb{N}_0}(j = 0, 1)$ are both uniformly bounded on $\mathbb{J}$.

Next we need only to show that $\{u_m^{(j)}(t)\}_{m \in \mathbb{N}_0}(j = 0, 1)$ are quasiequicontinuous on $\mathbb{J}$. By $u_m(t)$ is the solution (3.3), for $s \in \mathbb{J}'$, we have

$$-u_m''(s) = q(s)f(s, u_m(s), u_m'(s)) \leq q(s)h(u_m(s))[f_1(u_m'(s)) + f_2(u_m'(s))],$$

and by integrate it from $t(t \in \mathbb{J})$ to 1, one obtains

$$(3.12) \qquad u_m'(t) \leq \Gamma^{-1}\left(\frac{1}{f_1(M)} + \frac{\sum\limits_{k=1}^p L_k(M, M)}{f_1(M)} + \max_{u \in [0,M]} h(u) \int_0^1 q(x)dx\right).$$

For any $t, s \in \mathbb{J}_k(k = 0, 1, \cdots, p)$,

$$|u_m(t) - u_m(s)| = \left| \int_s^t u_m'(\tau)d\tau \right|$$

$$\leq \Gamma^{-1}\left(\frac{1}{f_1(M)} + \frac{\sum\limits_{k=1}^p L_k(M, M)}{f_1(M)} + \max_{u \in [0,M]} h(u) \int_0^1 q(x)dx\right)|t - s|$$

$$\to 0 \text{ as } t \to s.$$

By the conditions (H$_2$) and (H$_4$), one gets

$$|u'_m(t) - u'_m(s)| = \left| \int_s^t u''_m(\tau)d\tau \right| = \left| \int_s^t q(\tau)f(\tau, u_m(\tau), u'_m(\tau))d\tau \right|$$

$$\leq \max_{u \in [0,M]} h(u) \left[\int_s^t q(\tau)f_1(\varphi(\tau))d\tau + \max_{u \in [0,M]} f_2(u) \int_s^t q(\tau)d\tau \right]$$

$$\to 0 \text{ as } t \to s.$$

Therefore, $\{u^j_m(t)\}_{m \in \mathbb{N}_0}(j = 0, 1)$ are quasiequicontinuous on $\mathbb{J}$.

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}^*$ of $\mathbb{N}_0$ (without loss of generality, we assume $\mathbb{N}^* = \mathbb{N}_0$) and functions $u^{(j)}(t)(j = 0, 1)$ with $u^{(j)}_m(t) \to u^{(j)}(t)(j = 0, 1)$ uniformly on $\mathbb{J}$ as $m \to +\infty$ through $\mathbb{N}^*$. So $u(0) = \int_0^1 g(t)u(t)dt$, $u'(1) = 0$, $||u||_{PC^1} \leq M$, $\lim_{m \to \infty} u_m(t_k + 0) = u(t_k + 0)$, $\lim_{m \to \infty} u'_m(t_k + 0) = u'(t_k + 0)(k = 1, 2, \cdots, p)$, and $u(t) \geq L^{\frac{1}{1-\gamma}}(tL_1 + L_0)$, $u'(t) \geq \phi(t)$, $t \in \mathbb{J}$.

For $t \in (t_p, 1)$, by $u_m(t)(m \in \mathbb{N}^*)$ is the solution of (3.3) and Lemma 2.4, we have

$$(3.13) \qquad u_m(t) = -u_m(1) + u'_m(1)(1 - t) + \int_t^1 xq(x)f(x, u_m(x), u'_m(x))dx.$$

Let $m \to +\infty$ through $\mathbb{N}^*$ in (3.13), one has

$$u(t) = -u(1) + u'(1)(1 - t) + \int_t^1 xq(x)f(x, u(x), u'(x))dx,$$

and furthermore, we have $u''(t) + q(t)f(t, u(t), u'(t)) = 0$, $t \in (t_p, 1)$. Similarly, for any $t \in \mathbb{J}_k(k = 1, \cdots, p - 1)$, $t \in (0, t_1)$, one has $u''(t) + q(t)f(t, u(t), u'(t)) = 0$.

Thus, we have

$$\begin{cases} u''(t) + q(t)f(t, u(t), u'(t)) = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = I_k(u(t_k), u'(t_k)), \ k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = L_k(u(t_k), u'(t_k)), \ k = 1, 2, \cdots, p, \\ u(0) = \int_0^1 g(t)u(t)dt, \ u'(1) = 0, \end{cases}$$

i.e. $u(t)$ is positive solution of BVP (1.1), and $||u||_{PC^1} \leq M$, $u(t) \geq L^{\frac{1}{1-\gamma}}(tL_1 + L_0)$, $u'(t) \geq \varphi(t)$, $t \in \mathbb{J}$. The proof of Theorem 3.1 is complete.   □

4. AN EXAMPLE

In this section, we give an example to illustrate our results.

Example 4.1. Consider the following BVP

(4.1)
$$\begin{cases} u''(t) + tu^{\frac{1}{2}}(u'(t))^{-\frac{1}{5}} = 0, \ t \in \mathbb{J}', \\ \Delta u(t_k) = c_k(u(t_k) + u'(t_k)), \ k = 1, 2, \cdots, p, \\ \Delta u'(t_k) = -d_k(u(t_k) + u'(t_k)), \ k = 1, 2, \cdots, p, \\ u(0) = \int_0^1 tu(t)dt, \ u'(1) = 0, \end{cases}$$

where $0 < t_1 < t_2 < \cdots < t_p < 1$, c_k, $d_k \geq 0$, $k = 1, 2, \cdots, p$ are constants and $\sum_{k=1}^{p} c_k = \frac{1}{2000}$, $\sum_{k=1}^{p} d_k = \frac{1}{2000}$.

Conclusion. *BVP (4.1) has at least one positive solution.*

Proof. Obviously, $q(t) = t$, $f(t, u, v) = u^{\frac{1}{2}}v^{-\frac{1}{5}}$, $g(t) = t$, $I_k = c_k \geq 0$, $L_k = d_k \geq 0$, $k = 1, 2, \cdots, p$. $\sigma = \int_0^1 tdt = \frac{1}{2} \in [0, 1)$, so (H$_1$) holds.

Let
$$h(u) = u^{\frac{1}{2}}, \ f_1(v) = v^{-\frac{1}{5}}, \ f_2(v) = 0,$$

then (H$_2$) holds. For any K, $N > 0$, choose $\psi_{K,N}(t) = N^{-\frac{1}{5}}$ and $\gamma = \frac{1}{2}$ such that
$$f(t, u, v) = u^{\frac{1}{2}}v^{-\frac{1}{5}} \geq N^{-\frac{1}{5}}u^{\frac{1}{2}}, \ (t, u, v) \in \mathbb{J}' \times [0, K] \times (0, N],$$

thus (H$_3$) holds.

By a direct calculation, we have
$$\rho(t) = N^{-\frac{1}{5}} \int_t^1 s^{\frac{3}{2}}ds = \frac{2}{5}N^{-\frac{1}{5}}(1 - t^{\frac{5}{2}}),$$

and
$$\int_0^1 q(t)f_1(\rho(t))dt = \left(\frac{2}{5}\right)^{-\frac{1}{5}}N^{\frac{1}{25}} \int_0^1 \frac{t}{(1 - t^{\frac{5}{2}})^{\frac{1}{5}}}dt \approx 0.7206746604N^{\frac{1}{25}} < \infty,$$

which implies that condition (H$_4$) holds.

Next, we show that the conditions (H$_5$) holds. In fact, because

$$\sup_{c \in (0,\infty)} \frac{(1 - \sigma)c}{\sum_{k=1}^{p} \max_{u,v \in [0,c]} I_k(u, v) + \Gamma^{-1}\left(\frac{\sum_{k=1}^{p} \max_{u,v \in [0,c]} L_k(u,v)}{f_1(c)} + \max_{u \in [0,c]} h(u) \int_0^1 q(t)dt\right)}$$

$$= \sup_{c \in (0,\infty)} \frac{\frac{1}{2}c}{\frac{1}{1000}c + \Gamma^{-1}\left(\frac{1}{1000}c^{\frac{6}{5}} + \frac{1}{2}c^{\frac{1}{2}}\right)}$$

$$= \sup_{c \in (0,\infty)} \frac{\frac{1}{2}c}{\frac{1}{1000}c + \sqrt[\frac{6}{5}]{\frac{3}{2500}c^{\frac{6}{5}} + \frac{3}{5}c^{\frac{1}{2}}}} > 30 > 1,$$

then (H$_5$) holds. Therefore, by Theorem 3.1, we can obtain that (4.1) has at least one positive solution.   □

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