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A TOPOLOGICAL CHARACTERIZATION OF Ω-LIMIT SETS ON DYNAMICAL SYSTEMS

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Abstract. In this article, we deal with the notion of Ω-limit sets in dynamical systems. We show that the Ω-limit set of a compact subset of a phase space is quasi-attracting.

1. Introduction

The theory for the notion of attractors is important for the classical theory of dynamical systems. Conley [7] introduced a topological definition of attractors for a flow on a compact metric space. Hurley [9, 10] obtained results which is related to the correspondence between attractors and Lyapunov functions on noncompact spaces. Akin [1] and McGehee [12] obtained many properties of attractors in set-valued dynamics.

The concept of omega-limit set, arising from their ubiquitous applications in dynamical systems, is also an extremely used tool in the abstract theory of dynamical systems. Especially, the notion of omega-limit sets is much related to the notion of attractors. These notions are used to describe eventually the positive time behavior for dynamical systems. In recent years, Choy and Chu [4] described the characterizations of omega-limit sets for analytic flows on $\mathbb{R}^2$.

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A pseudo-orbit(chain) was firstly used by Bowen [3] and Conley [7]. The notion is a very strong tool to understand important theories in several fields of Mathematics and generates many results about the induced concepts, for example, chain transitive, chain recurrence, shadowing property and so on. See [2, 9, 10, 11, 13, 14].

In [8], Ding introduced the concept of chain prolongation and investigated the notion of chain stability which takes an intermediate concept between absolute stability and asymptotic stability. Especially he proved that the chain prolongation set of closed set is quasi-attracting. Recently, Chu et al [5] discussed the notion of chain prolongations in locally compact spaces.

In this paper, we focus on the relationship between the notion of Ω-limit sets and the notion of quasi-attracting sets. The quasi-attracting sets are considered as a general version of the notion of attracting sets. More precisely speaking, we show that the Ω-limit set of a compact set is quasi-attracting.

The paper is organized as follows.

In section 2, we explain the elementary definitions for the proof of the main theorems.

In section 3, we briefly sketch for the theories of attracting sets and quasi-attracting sets. Next, we also prove that the Ω-limit set of a compact subset of X becomes an quasi-attracting set.

2. Ω-limit set in topological dynamics

Let (X, d) be a locally compact metric space. A *flow* on X is the continuous map $\pi : X \times \mathbb{R} \to X$ that satisfies the following group laws; for every $x \in X$, $\pi(x, 0) = x$ and for every $t, s \in \mathbb{R}$, $x \in X$, $\pi(\pi(x, t), s) = \pi(x, t + s)$. For a convenience, we briefly write $x \cdot t = \pi(x, t)$. For any $x \in X$, we define an *orbit* of x to be the subset $\{x \cdot t \mid t \in \mathbb{R}\}$ of X which is denoted by $O(x)$. We say that a subset Y of X is *positively invariant* (*invariant*) under π if $Y \cdot \mathbb{R}^+ = Y (Y \cdot \mathbb{R} = Y)$.

Next, we introduce the subsets of X which are related to the eventual orbit of a point under the flow π. For $x \in X$, the *limit set* of x, denoted by $\Lambda^+(x)$, is defined by

$$\Lambda^+(x) := \bigcap_{t \geq 0} \overline{x \cdot [t, \infty)}.$$

The limit set of x has a major role in Conley's theory, and for its basic properties we refer to [7, 8]. For $x \in X$, we also call the *first prolongational limit set* and *first prolongational set* of x as defined, respectively, by

$$J^+(x) := \bigcap_{U \in N(x), t \geq 0} \overline{U \cdot [t, \infty)},$$
$$D^+(x) := \bigcap_{U \in N(x)} \overline{U \cdot \mathbb{R}^+},$$

where $N(x)$ is the set of all neighborhoods of x.

In [2], Bae, Choi and Park studied limit sets and prolongational sets in topological dynamics. Next remark is immediately proved from the definitions. See [6].

Remark 2.1. ([6]) For $x \in X$, the following equivalences are well-known.

(1) $y \in \Lambda^+(x)$ if and only if there is a sequence $\{t_n\}$ in $\mathbb{R}^+$ with $t_n \to \infty$ such that $x \cdot t_n \to y$.
(2) $y \in J^+(x)$ if and only if there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \to x$, $t_n \to \infty$ and $x_n \cdot t_n \to y$.
(3) $y \in D^+(x)$ if and only if there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \to x$ and $x_n t_n \to y$.

Let $\Gamma : X \to 2^X$ be a function and $A \subseteq X$, then we canonically define $\Gamma(A) := \cup_{x \in A} \Gamma(x)$. We define the composition $\Gamma^2 = \Gamma \circ \Gamma$ given by $\Gamma^2(x) = \Gamma(\Gamma(x)) = \cup_{y \in \Gamma(x)} \Gamma(y)$. According to the above inductive definition for the set-valued function Γ, we can consider naturally the iteration $\Gamma^n : X \to 2^X$ inductively by $\Gamma^1(x) = \Gamma(x)$ and $\Gamma^n(x) = \Gamma(\Gamma^{n-1}(x))$. To unfold the applications for the properties of the set-valued function Γ, we need to express the trajectory for the function by the union of the iteration Γ^n. For a family of functions $\Gamma_i : X \to 2^X (i \in I)$, we give the new map $\cup_{i \in I} \Gamma_i : X \to 2^X$ defined by $(\cup_{i \in I} \Gamma_i)(x) = \cup_{i \in I} \Gamma_i(x)$.

Let $\mathcal{P}$ be the set of all functions from X to its power set 2^X and let $\Gamma \in \mathcal{P}$. We define the mappings D and S from the set $\mathcal{P}$ to itself given by

$$(D\Gamma)(x) = \cap_{U \in N(x)} \overline{\Gamma(U)} \quad \text{and} \quad (S\Gamma)(x) = \cup_{n=1}^{\infty} \Gamma^n(x),$$

for every $x \in X$. We call D a *closure funtion* for Γ and S an *orbital function* for Γ defined on $\mathcal{P}$.

Remark 2.2. ([5]) Let $\Gamma : X \to 2^X$ be a mapping and $x \in X$. Then $D\Gamma(x)$ is the set of all points $y \in X$ with the property that there exist sequences (x_n) and (y_n) in X with $y_n \in \Gamma(x_n)$ such that $x_n \to x$, $y_n \to y$. Furthermore, $S\Gamma(x)$ is the set of all points $y \in X$ such that there is a finite subset $\{x_1, \cdots, x_k\}$ of X with the properties that $x_1 = x, x_k = y$ and $x_{i+1} \in \Gamma(x_i)$, $i = 1, \cdots, k-1$.

REMARK 2.3. ([6]) The new mappings D and S have interesting properties, especially the iterations of the mappings are just the original mappings. More precisely speaking, a closure function D is idempotent and so is an orbital function S, that is, $D^2(x) = D(x)$ and $S^2(x) = S(x)$ for every $x \in X$.

We recall the notions of chains and Ω-limit sets in [7] for details. Let x, y be elements of X and ϵ, t positive real numbers. An (ϵ, t)-*chain* from x to y means a pair of finite sequences $x = x_1, x_2, \cdots, x_n, x_{n+1} = y$ in X and $t_1, t_2, \cdots, t_n$ in $\mathbb{R}^+$ such that $t_i \geq t$ and $d(x_i \cdot t_i, x_{i+1}) \leq \epsilon$ for all $i = 1, 2, \cdots, n$. Define a relation R in $X \times X$ as follows. By $(x, y) \in R$, we mean that for every $\varepsilon > 0$ and $t > 0$, there exists (ϵ, t)-chain from x to y.

We define the set-valued map $\Omega : X \to 2^X$ given by x to $\Omega(x)$, where $\Omega(x) := \{y \in X : (x, y) \in R\}$. $\Omega(x)$ is called the Ω-*limit set* of x.

In [7], Conley investigated the several notions of topological dynamics in a compact metric space. He proved that the chain relation R is closed and transitive on X and also showed that if $(x, y) \in R$ and $(s_1, s_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, then $(x \cdot s_1, y \cdot s_2) \in R$. We observe that $\Omega(x)$ is a closed invariant subset of a compact metric space X and $J^+(x) \subseteq \Omega(x)$ (see [7, p.36] and [8, p.2721]). Note that $\Omega(M) = \cup_{x \in M} \Omega(x)$ for every subset M of X.

3. Attracting sets and quasi-attracting sets

In this section, we investigate the notions of Ω-limit sets, attracting sets and quasi-attracting sets in a locally compact space X.

For a subset Y of X, we define the *limit set* of Y by

$$\omega(Y) = \bigcap_{t \geq 0} \overline{Y \cdot [t, \infty)}.$$

Note that $\omega(Y)$ is a maximal invariant subset in $Y \cdot [0, \infty)$ and is generally larger than $\Lambda^+(Y) = \bigcup_{x \in Y} \Lambda^+(x)$.

In [6], it is proved that the connectedness is invariant under the notion of limit. More precisely speaking, assume that the limit set $\omega(Y)$ of a connected subset Y of X is compact, then the limit set is connected.

A positively invariant closed subset A of X is called an *attracting set* if A admits a neighborhood U such that $\omega(U) \subseteq A$. A closed set is a *quasi-attracting set* if it is an intersection of attracting sets.

It is easy to see that if A is a (quasi-)attracting set, so is $A \cdot t$, for $t \in \mathbb{R}$. Let $\{B_i\}_{i\in I}$ (here, I is some index set) be the family of quasi-attracting sets, then $\bigcap_{i\in I} B_i$ is also a quasi-attracting set. We note that a quasi-attracting set is just positively invariant. In general, actually (quasi-)attracting set need not invariant. If an (quasi-)attracting set is invariant, the set is called an (quasi-)attractor in the sense of Conley(see [7]); that is, an invariant attracting set A is an attractor.

For positive real numbers ε and t, we define the set $P_t(M, \varepsilon)$ by

$$P_t(M, \varepsilon) := \{y \in X \mid \text{there is an } (\varepsilon, t)\text{−chain from } x \text{ to } y \text{ for some } x \in M\}.$$

In the following lemma, we show that the Ω-limit set $\Omega(M)$ of compact set M is represented by the intersection of the above subsets. This representation plays an important role in the proof of Theorem 3.4.

Lemma 3.1. *Let M be a compact subset of X. Then*

$$\Omega(M) = \bigcap_{\varepsilon,t>0} P_t(M, \varepsilon).$$

Proof. Firstly we recall the equality $\Omega(M) = \cup_{x\in M}\Omega(x)$. From the equality, we can easily prove the inclusion $\Omega(M) \subseteq \bigcap_{\varepsilon,t>0} P_t(M, \varepsilon)$.

Conversely, we let $y \in \bigcap_{\varepsilon,t>0} P_t(M, \varepsilon)$. Then, for each positive integer n, since y is an element of $P_n(M, \frac{1}{n})$, there exist an element x_n of M and a $(\frac{1}{n}, n)$-chain from x_n to y

$$\{x_n = x_1^n, x_2^n, \cdots, x_{m_n}^n, x_{m_n+1}^n = y; t_1^n, t_2^n, \cdots, t_{m_n}^n\}.$$

Since M is compact, the sequence $\{x_n\}$ in M has a convergent subsequence. Without loss of generality, we can assume that the original sequence $\{x_n\}$ converges to some point x in M. For any $\varepsilon > 0$ and $t > 0$, there exists a positive real number δ such that if $d(x, z) < \delta$, then $d(x \cdot t, z \cdot t) < \varepsilon$. We can take a positive integer n such that

$$d(x, x_n) < \delta, n > 2t \text{ and } \frac{1}{n} < \varepsilon.$$

Since $d(x, x_n) < \delta$, we have $d(x\cdot t, x_n\cdot t) < \varepsilon$. Thus the following sequence

$$\{x, x_n \cdot t, x_2^n, \cdots, x_{m_n}^n, x_{m_n+1}^n = y; t, t_1^n - t, t_2^n, \cdots, t_{m_n}^n\}$$

is (ε, t)-chain from x to y. Hence y is an element of $\Omega(M)$, which completes the proof. □

In [8, p.2724], Ding showed that the set $P_t(M,\varepsilon)$ is open and proved the inclusion $\overline{P_t(M,\varepsilon)\cdot[t,\infty)} \subseteq P_t(M,\varepsilon)$. Put $A_{\varepsilon,t} := \overline{P_t(M,\varepsilon)\cdot[t,\infty)}$. Then he obtins that the set $P_t(M,\varepsilon)$ is an open neighborhood of $A_{\varepsilon,t}$ and that furthermore, $A_{\varepsilon,t}$ is a positively invariant closed attracting set.

LEMMA 3.2. *Let $P_t(M,\varepsilon)$ and $A_{\varepsilon,t}$ be the same as notations in the above statements. Then we have*

$$\bigcap_{\varepsilon,t>0} P_t(M,\varepsilon) = \bigcap_{\varepsilon,t>0} A_{\varepsilon,t}.$$

Proof. From Ding's results in the above, it is obvious that the set $\bigcap_{\varepsilon,t>0} A_{\varepsilon,t}$ is contained in the intersection $\bigcap_{\varepsilon,t>0} P_t(M,\varepsilon)$.

To show the converse, firstly, let y be an element of $\bigcap_{\varepsilon,t>0} P_t(M,\varepsilon)$. For an arbitrary positive real numbers ε and t, we can choose a positive real number $t^{'}$ larger than $2t$. We note that the action $y\cdot(-s)(s>0)$ is continuous. Let δ be an arbitrary positive real number. Using the continuity of the action, we can choose a positive real number $\delta^{'}$ such that if $d(y,y^{'})<\delta^{'}$ then

$$d(y\cdot(-t), y^{'}\cdot(-t)) < \delta. \tag{1}$$

Put $\varepsilon^{'} := min(\delta^{'},\varepsilon)$. Since y is an element of $P_{t^{'}}(M,\varepsilon^{'})$, there exists $(\varepsilon^{'},t^{'})$-chain from x_0 to y, say $\{x_0,x_1,\cdots,x_m,x_{m+1}=y;t_0,t_1,\cdots,t_m\}$, for some x_0 in M. Since $t^{'}$ is larger than $2t$, we can construct the new (ε,t)-chain from x_0 to $x_m\cdot(t_m-t)$ as follows,

$$\{x_0,x_1,\cdots,x_m,x_m\cdot(t_m-t);t_0,t_1,\cdots,t_m-t\}.$$

Thus we have that $x_m\cdot(t_m-t)$ is an element of $P_t(M,\varepsilon)$. Note that the inequalities $d(y,x_m\cdot t_m)<\varepsilon^{'}\leq\delta^{'}$. By (1), we obtain that $d(y\cdot(-t),x_m\cdot(t_m-t))<\delta$. Since δ is arbitrary, it yields that $y\cdot(-t)\in\overline{P_t(M,\varepsilon)}$. Then we obtain the following inclusions

$$\begin{aligned} y=(y\cdot(-t))\cdot t &\in \overline{P_t(M,\varepsilon)}\cdot t \\ &\subseteq \overline{P_t(M,\varepsilon)\cdot t} \\ &\subseteq \overline{P_t(M,\varepsilon)\cdot[t,\infty)}, \end{aligned}$$

for arbitrary positive real numbers ε,t. Therefore y is an element of $\bigcap_{\varepsilon,t>0} A_{\varepsilon,t}$. □

To prove the theorem 3.4, we also need a basic property for Ω-limit sets as follows.

LEMMA 3.3. ([5]) If $p_n\to p, q_n\to q$ and $q_n\in\Omega(p_n)$, then $q\in\Omega(p)$.

We now complete the consequence for the notion of quasi-attracting sets. The following theorem describes that the Ω-limit set on X is quasi-attracting for the compact case.

THEOREM 3.4. *For a compact subset M of X, $\Omega(M)$ is quasi-attracting.*

Proof. First of all, we show that $\Omega(M)$ is a closed subset. Let y be an element of $\overline{\Omega(M)}$. Then there exists a sequence $\{y_n\}$ in $\Omega(M)$ such that $\{y_n\}$ converges to y. Thus, for every n, there exists a point x_n in M such that $y_n \in \Omega(x_n)$. By the compactness of M, the sequence $\{x_n\}$ has a convergent subsequence. Without loss of generality, we can assume that the original sequence $\{x_n\}$ converges to some point x in M. By Lemma 3.3, y is an element of $\Omega(x)$ and thus, $\Omega(M)$ is closed.

By combining Lemma 3.1 and 3.2, we get the following equalities

$$\Omega(M) = \bigcap_{\varepsilon,t>0} P_t(M,\varepsilon) = \bigcap_{\varepsilon,t>0} A_{\varepsilon,t}.$$

As a consequence, the Ω-limit set is the above the intersection of the attracting sets, that is, quasi-attracting set. This completes the proof. □

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