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# A SEXTIC-ORDER VARIANT OF DOUBLE-NEWTON METHODS WITH A SIMPLE BIVARIATE WEIGHTING FUNCTION

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Abstract. Via extension of the classical double-Newton method, we propose high-order family of two-point methods in this paper. Theoretical and computational properties of the proposed methods are fully investigated along with a main theorem describing methodology and convergence analysis. Typical numerical examples are thoroughly treated to verify the underlying theory.

## 1. Introduction

Since the development of multipoint iterative methods of Traub[9] in the 1960s, numerous high-order multipoint methods for solving a nonlinear equation in the form of $f(x) = 0$ have been investigated. Some of them can be found in [2, 3, 4, 5, 8]. Displayed below in (1.1) is the well-known two-point fourth-order double-Newton method[9], which is a two-step classical Newton method:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \end{cases} \tag{1.1}$$

Such higher-order methods requiring only two derivatives and two functions include three-point sextic-order methods [3, 7], being respectively shown below in (1.2) and (1.3).

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$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}, \\ x_{n+1} = z_n - \frac{f'(x_n)+f'(y_n)}{3f'(y_n)-f'(x_n)} \cdot \frac{f(z_n)}{f'(x_n)}. \end{cases} \tag{1.2}$$

$$\begin{cases} y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n - J_f(x_n) \cdot \frac{f(x_n)}{f'(x_n)}, J_f(x_n) = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{a(z_n-x_n)(z_n-y_n)+\frac{3}{2}J_f(x_n)f'(y_n)+(1-\frac{3}{2}J_f(x_n))f'(x_n)}, a \in \mathbb{R}. \end{cases} \tag{1.3}$$

DEFINITION 1.1. (*Error equation, asymptotic error constant, order of convergence*) Let $x_0, x_1, \cdots, x_n, \cdots$ be a sequence of numbers converging to $\alpha$. Let $e_n = x_n - \alpha$ for $n = 0, 1, 2, \cdots$. If constants $p \geq 1,\ c \neq 0$ exist in such a way that $e_{n+1} = c\, {e_n}^p + O(e_n^{p+1})$ called the *error equation*, then $p$ and $\eta = |c|$ are said to be the *order of convergence* and the *asymptotic error constant*, respectively.

Three-point methods (1.2)–(1.3) possess rather more complicated structures than two-point methods like (1.1). The main aim of this paper is to develop a general class of two-point higher-order extended double-Newton methods. To this end, by introducing a weighting function in the second step of (1.1), we propose a higher-order family of two-point methods in the following form:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - [G(s) + u(\frac{a+bs}{1+rs})] \cdot \frac{f(y_n)}{f'(y_n)},\ s = \frac{f'(y_n)}{f'(x_n)},\ u = \frac{f(y_n)}{f(x_n)}, \end{cases} \tag{1.4}$$

where the function $G : \mathbb{C} \to \mathbb{C}$ is analytic[1] in a neighborhood of 1 with $s = \frac{f'(y_n)}{f'(x_n)} = 1 + O(e_n)$ and $u = \frac{f(y_n)}{f(x_n)} = O(e_n)$. In view of the fact that $s - 1 = O(e_n), u = O(e_n)$, Taylor series expansion of $G(s)$ about 1 up to terms of several order as well as determination of parameters $r, a, b$ will play an essential role in designing two-point sextic-order methods costing two derivatives and two functions.

In Section 2, methodology and analysis is described for a new family of sextic-order methods with appropriate forms of $G$. Section 3 investigates some special cases of $G(s)$, while Section 4 presents numerical experiments and concluding remarks.

## 2. Methodology and analysis

This section describes a main theorem and its proof covering the methodology and convergence behavior on iterative scheme (1.4).

THEOREM 2.1. *Assume that* $f : \mathbb{C} \to \mathbb{C}$ *has a simple root* $\alpha$ *and is analytic* [1] *in a region containing* $\alpha$. *Let* $\Delta = f'(\alpha)$ *and* $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ *for* $j = 2, 3, \cdots$. *Let* $x_0$ *be an initial guess chosen in a sufficiently small neighborhood of* $\alpha$. *Let* $G : \mathbb{C} \to \mathbb{C}$ *be analytic in a neighborhood of* 1. *Let* $G_j = \frac{1}{j!}\frac{d^j}{ds^j}G(s)\big|_{s=1}$ *for* $0 \le j \le 4$. *If* $G_0 = 1, G_1 = 0$, $G_2 = \frac{3}{4}, G_3 = -\frac{1+2r}{2(1+r)}, a = -(1+r)$ *and* $b = 1 + r$ *are satisfied with* $r$ *free, then iterative scheme* (1.4) *defines a family of two-point sextic-order methods satisfying the error equation below: for* $n = 0, 1, 2, \cdots$,

$$e_{n+1} = \Big\{ -\frac{1}{4}c_2(-88c_2^4 + 12c_2^2c_3 + 3c_3^2 - 4c_2c_4 + 64c_2^4G_4) \\ -2c_2^3 \cdot \frac{c_3(1+r) + 4c_2^2(1+2r)}{(1+r)^2}\Big\}\, e_n^6 + O(e_n^7). \quad (2.1)$$

*Proof.* Taylor series expansion of $f(x_n)$ about $\alpha$ up to 6th-order terms with $f(\alpha) = 0$ leads us to:

$$f(x_n) = \Delta\{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\}. \quad (2.2)$$

It follows that

$$f'(x_n) = \Delta\{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + O(e_n^7)\}. \quad (2.3)$$

For simplicity, we will denote $e_n$ by $e$ from now on. With the aid of symbolic computation of Mathematica[10], we have:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e^2 - 2(c_2^2 - c_3)e^3 + Y_4e^4 + Y_5e^5 + Y_6e_n^6 + O(e^7), \quad (2.4)$$

where $Y_4 = (4c_2^3 - 7c_2c_3 + 3c_4), Y_5 = -2(4c_2^4 - 10c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)$ and $Y_6 = (16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 + 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)$. In view of the fact that $f'(y_n) = f'(x_n)|_{e_n \to (y_n - \alpha)}$, we get:

$$f'(y_n) = \Delta[1 + 2c_2^2e^2 - 4c_2(c_2^2 - c_3)e^3 + D_4e^4 + \Sigma_{i=4}^6\, D_i\, e^i + O(e^7)], \quad (2.5)$$

where $D_4 = c_2(8c_2^3 - 11c_2c_3 + 6c_4), D_i = D_i(c_2, c_3, \cdots, c_6)$ for $5 \le i \le 6$. Hence we have:

$$s = \frac{f'(y_n)}{f'(x_n)} = 1 - 2c_2e + 3(2c_2^2 - c_3)e^2 - 4(4c_2^3 - 4c_2c_3 + c_4)e^3 + \Sigma_{i=4}^6\, E_i\, e^i + O(e^7), \quad (2.6)$$

where $E_i = E_i(c_2, c_3, \cdots, c_6)$ for $4 \le i \le 6$. In view of the fact that $f(y_n) = f(x_n)|_{e_n \to (y_n - \alpha)}$, we get:

$$f(y_n) = \Delta[c_2e^2 - 2(c_2^2 - c_3)e^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e^4 + \Sigma_{i=5}^6\, F_i\, e^i + O(e^7)], \quad (2.7)$$

where $F_i = F_i(c_2, c_3, \cdots, c_6)$ for $5 \le i \le 6$. Hence we have:

$$u = \frac{f(y_n)}{f(x_n)} = c_2e - (3c_2^2 - 2c_3)e^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e^3 + \Sigma_{i=4}^6\, L_i\, e^i + O(e^7), \quad (2.8)$$

where $L_i = L_i(c_2, c_3, \cdots, c_6)$ for $4 \le i \le 6$. Noting that $O(f(x_n)) = O(s-1) = O(u) = O(e)$ and $O(f(y_n)) = O(e^2)$, Taylor expansion of $G(s)$ about $s = 1$ up to fourth-order yields after retaining up to fourth-order terms:

$$G(s) = G_0 + G_1(s-1) + G_2(s-1)^2 + G_3(s-1)^3 + G_4(s-1)^4 + O(e^5). \quad (2.9)$$

By direct substitution of $z_n, f(x_n), f(y_n), f'(x_n), f'(y_n), s, u$ and $G(s)$ in (1.4), we find

$$x_{n+1} = y_n - [G(s) + u(\frac{a+bs}{1+rs})] \cdot \frac{f(y_n)}{f'(y_n)} = \alpha + (1-G_0)c_2 e^2 + \Sigma_{i=3}^{6} \Gamma_i \, e^i + O(e^7), \quad (2.10)$$

where $\Gamma_i = \Gamma_i(c_2, c_3, \cdots, c_6, r, a, b, G_j)$, for $3 \le i \le 6$, $0 \le j \le 4$.

By setting $G_0 = 1$ from (2.10) along with $\Gamma_3 = 0$, we immediately solve for $K_{01}$ as

$$G_1 = \frac{a+b}{2(1+r)}. \quad (2.11)$$

By substituting $G_0 = 1, G_1$ into $\Gamma_4 = 0$, we find two independent relations below:

$$a + b = 0, \; 1 + 2b + 2r - 2ar + r^2 - 4G_2(1+r)^2 = 0. \quad (2.12)$$

As a result, we find

$$a = -b, G_2 = \frac{1+2b+r}{4(1+r)}. \quad (2.13)$$

By substituting $G_0 = 1, G_1 = 0, G_2, a$ into $\Gamma_5 = 0$, we find two independent relations below:

$$1 - b + r = 0, \; 1 + (2+b)r + r^2 + 2G_3(1+r)^2 = 0. \quad (2.14)$$

Solving (2.14) for $b, G_3$ yields

$$b = 1 + r, \; G_3 = -\frac{(1+2r)}{2(1+r)}. \quad (2.15)$$

By substituting $G_0 = 1, G_1 = 0, G_2 = \frac{3}{4}, G_3, a, b$ into $\Gamma_6$, we find with $r$ as a free parameter to be chosen:

$$\Gamma_6 = -\frac{1}{4}c_2(-88c_2^4 + 12c_2^2 c_3 + 3c_3^2 - 4c_2 c_4 + 64c_2^4 G_4)$$

$$-2c_2^3 \cdot \frac{c_3(1+r) + 4c_2^2(1+2r)}{(1+r)^2} \quad (2.16)$$

as desired in (2.1). This completes the proof. □

## 3. Selection of $G(s)$ and $r$

We first put $K_f(s,u) = [G(s) + u(\frac{a+bs}{1+rs})]$ for notational convenience. Then using relations (2.11), (2.13) and (2.15), the Taylor-form bivariate polynomial $K_f(s,u)$ is given by

$$K_f(s,u) = 1 + \frac{3}{4}(s-1)^2 - \frac{(1+2r)(s-1)^3}{2(1+r)} + G_4(s-1)^4 + u\frac{(1+r)(s-1)}{(1+rs)}, \quad (3.1)$$

where notations $s = f'(y_n)/f'(x_n), u = f(y_n)/f(x_n)$ are introduced for simplicity. Special cases of (3.1) are considered here with appropriate determination of $G_4$ and free selection of $r$. In each case, relevant coefficients are determined based on relations (2.11)–(2.15).

**Case 1:** $G = 1 + \frac{3}{4}(s-1)^2 - \frac{(1+2r)(s-1)^3}{2(1+r)} + G_4(s-1)^4$, $G_4$: selected

$$K_f(s,u) = 1 + \tfrac{3}{4}(s-1)^2 - \tfrac{(1+2r)(s-1)^3}{2(1+r)} + G_4(s-1)^4 + u\tfrac{(1+r)(s-1)}{(1+rs)}. \quad (3.2)$$

In what follows, we consider five interesting weighting functions with some values of $r, G_4$.

| SN | $r$ | $G_4$ | $K_f(s,u)$ |
|---|---|---|---|
| **1A** | 0 | 0 | $-\frac{1}{4}(s-3)(2s^2-3s+3)+(s-1)u$ |
| **1B** | $-\frac{1}{2}$ | 0 | $1+\frac{3}{4}(s-1)^2-\frac{(s-1)}{(s-2)}u$ |
| **1C** | $-\frac{2}{3}$ | 0 | $\frac{1}{4}(s+1)(2s^2-5s+5)-\frac{(s-1)}{(2s-3)}u$ |
| **1D** | $-\frac{5}{3}$ | 0 | $-\frac{1}{4}(s-2)(7s^2-10s+7)+\frac{2(s-1)}{(5s-3)}u$ |
| **1E** | $-\frac{1}{2}$ | $\frac{9}{64}$ | $\left(\frac{3s^2-6s+11}{8}\right)^2-\frac{(s-1)}{(s-2)}u$ |

TABLE 1. Typical $K_f(s,u)$ of Case 1 with values of $r, G_4$

**Case 2:** $G = 1 + (s-1)^2\frac{(b_0+b_1 s)}{(1+a_1 s)}$, $a_1$: selected, $b_0, b_1, G_4$: found

$$K_f(s,u) = 1 + (s-1)^2\frac{(b_0+b_1 s)}{(1+a_1 s)} + u\frac{(1+r)(s-1)}{(1+rs)}, r = free. \quad (3.3)$$

Using $G_0 = 1, G_1 = 0, G_2 = \frac{3}{4}, G_3 = -\frac{(1+2r)}{2(1+r)}$, we find that

$$b_0 = \frac{1-2a_1-(1+4a_1)r}{4(1+r)}, \; b_1 = \frac{2+5a_1+(4+7a_1)r}{4(1+r)}, \; a_1, r : free. \quad (3.4)$$

In what follows, we consider seven weighting functions with values of $a_1, b_0, b_1, r, G_4$.

| SN | $a_1$ | $b_0$ | $b_1$ | $r$ | $G_4$ | $K_f(s,u)$ |
|---|---|---|---|---|---|---|
| **2A** | 2 | $\frac{9}{4}$ | 0 | 0 | $\frac{1}{3}$ | $1+\frac{9(s-1)^2}{4(2s+1)}+(s-1)u$ |
| **2B** | $-\frac{1}{4}$ | 0 | $\frac{9}{16}$ | $-\frac{3}{4}$ | $\frac{1}{3}$ | $1-\frac{9(s-1)^2 s}{4(s-4)}-\frac{(s-1)}{(3s-4)}u$ |
| **2C** | 1 | $\frac{3}{2}$ | 0 | $-\frac{1}{5}$ | $\frac{3}{16}$ | $1+\frac{3(s-1)^2}{2(s+1)}-\frac{4(s-1)}{(s-5)}u$ |
| **2D** | $-\frac{4}{7}$ | $\frac{9}{14}$ | $-\frac{9}{28}$ | $-\frac{3}{5}$ | $\frac{1}{3}$ | $1+\frac{9(s-1)^2(s-2)}{4(4s-7)}-\frac{2(s-1)}{(3s-5)}u$ |
| **2E** | $-\frac{1}{3}$ | $\frac{1}{2}$ | 0 | $-\frac{7}{11}$ | $\frac{3}{16}$ | $1-\frac{3(s-1)^2}{2(s-3)}-\frac{4(s-1)}{(7s-11)}u$ |
| **2F** | -$\frac{1}{4}$ | $\frac{15}{32}$ | $\frac{3}{32}$ | $-\frac{7}{11}$ | $\frac{1}{8}$ | $1-\frac{3(s-1)^2(s+5)}{8(s-4)}-\frac{4(s-1)}{(7s-11)}u$ |
| **2G** | -$\frac{1}{4}$ | $-\frac{9}{16}$ | $\frac{9}{8}$ | $-\frac{9}{11}$ | $\frac{7}{12}$ | $1-\frac{9(s-1)^2(2s-1)}{4(s-4)}-\frac{2(s-1)}{(9s-11)}u$ |

TABLE 2. Typical $K_f(s,u)$ of Case 1 with values of $a_1, b_0, b_1, r, G_4$

## 4. Numerical experiments and concluding remarks

Root-finding problems under normal circumstances frequently display the relevant numerical results of approximately 6 or 7 significant decimal digits with second-order Newton-like methods using common programming languages *Fortran* or *C*. In such programming languages, empirically 15 or 16 decimal working-precision digits are adopted for numerical results with 6 or 7 significant decimal digits. Likewise, about 48 decimal working-precision digits would be reasonable for approximately 21 significant decimal digits with sextic-order numerical methods. Computing asymptotic error constants $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ require a sufficient number of significant digits due to the indeterminate form of a small-number division near the root $\alpha$. Consequently, increased working-precision digits are needed for reliable numerical results having a moderate number of significant digits.

Current numerical experiments with *Mathematica*(Version 7) have been carried out with 112 working-precision digits, which minimize round-off errors and accurately compute the asymptotic error constants requiring small-number divisions. The error bound $\epsilon = \frac{1}{2} \times 10^{-80}$ was assigned. The initial guesses $x_0$ were selected close to $\alpha$ to guarantee the convergence of the iterative methods. Only 15 significant digits of approximated roots $x_n$ are displayed in Tables 3–4 due to the limited paper space, although 80 significant digits are available. Numerical experiments have been performed on a personal computer equipped with an AMD 3.1 Ghz dual-core processor and 64-bit Windows 7 operating system.

| $\binom{\mathbf{MT}}{F_i}$ | $n$ | $x_n$ | $\lvert F(x_n)\rvert$ | $\lvert e_n\rvert$ | $\frac{e_n}{{e_{n-1}}^6}$ | $\eta$ | $p_n$ |
|---|---|---|---|---|---|---|---|
| $\binom{\mathbf{T1E}}{F_1}$ | 0 | 0.91 | 0.219354 | 0.0237731 | | | |
| | 1 | 0.886226925225390 | $2.18\times10^{-9}$ | $2.27\times10^{-10}$ | 1.259554099 | 20.85601714 | 6.75066 |
| | 2 | 0.886226925452758 | $2.76\times10^{-56}$ | $2.88\times10^{-57}$ | 20.85601741 | | 6.00000 |
| | 3 | 0.886226925452758 | $0.0\times10^{-111}$ | $0.0\times10^{112}$ | | | |
| $\binom{\mathbf{T2A}}{F_2}$ | 0 | $\binom{1.54}{-0.98}^*$ | 0.0406260 | 0.0367208 | | | |
| | 1 | $\binom{1.57079632084450}{-0.999999998324032}$ | $6.76\times10^{-9}$ | $6.18\times10^{-9}$ | 2.521470323 | 2.615238385 | 6.01105 |
| | 2 | $\binom{1.57079632679490}{-1.00000000000000}$ | $1.59\times10^{-49}$ | $1.45\times10^{-49}$ | 2.615238349 | | 6.00000 |
| | 3 | $\binom{1.57079632679490}{-1.00000000000000}$ | $0.0\times10^{-111}$ | $0.0\times10^{-111}$ | | | |

$p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$, $^*\binom{1.54}{-0.98} = 1.54 - 0.98i$

TABLE 3. Convergence for sample test functions $F_1(x)-F_2(x)$ with methods **T1E, T2A**

Iterative methods (1.4) with all sub-cases of both Case 1 and Case 2 were respectively identified by **T1A, T1B, T1C, T1D, T1E** and **T2A, T2B, T2C, T2D, T2E, T2F, T2G**, being **T**-prefixed. Among them, two typical methods **T1E** and **T2A** have been successfully applied to two test functions shown below:

$$\begin{cases} \mathbf{T1E}: F_1(x) = 2\cos(x^2) - \log(1+4x^2-\pi) - \sqrt{2},\ \alpha = \frac{\sqrt{\pi}}{2}, \\ \mathbf{T2A}: F_2(x) = 2x + 2i - \pi + \cos(x+i)\cdot\log(x^2+1),\ \alpha = -i + \frac{\pi}{2},\ i = \sqrt{-1}, \\ \text{where } \log \mathrm{z}\ (\mathrm{z} \in \mathbb{C}) \text{ represents a principal branch such that } -\pi \le \mathrm{Im}(\log \mathrm{z}) < \pi. \end{cases}$$

DEFINITION 4.1. (*Asymptotic Convergence Order*) Assume that the asymptotic error constant $\eta = \lim_{n\to\infty} \frac{|e_n|}{|e_{n-1}|^p}$ is known. Then we can define the *asymptotic convergence order* $p_a = \lim_{n\to\infty} \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$, being abbreviated by ACO.

Methods **T1E, T2A** in Table 3 clearly confirmed sextic-order convergence. Table 3 lists iteration indexes $n$, approximate zeros $x_n$, residual errors $|f(x_n)|$, errors $|e_n| = |x_n - \alpha|$ and computational asymptotic error constants $\eta_n = |\frac{e_n}{{e_{n-1}}^6}|$ as well as the theoretical asymptotic error constant $\eta$ and computational asymptotic convergence order $p_n = \frac{\log|e_n/\eta|}{\log|e_{n-1}|}$. The values of $\eta_n$ agree up to 8 significant digits with $\eta$. The computational asymptotic order of convergence undoubtedly approaches 6.

Following functions are further tested for the convergence behavior of proposed scheme (1.4):

$$\begin{cases} f_1(x) = x^5 + x^2 + xe^{2x} - 7,\ \alpha \approx 0.906962092165271,\ x_0 = 0.85, \\ f_2(x) = \cos(\pi x) + (x-2)^2\sin(\pi x),\ \alpha \approx 1.56068650991399.,\ x_0 = 1.6, \\ f_3(x) = \cos(x^2 - x + \frac{7}{16}) + 4x - 3 - i\sqrt{3},\ \alpha = \frac{1}{2} + i\frac{\sqrt{3}}{4},\ x_0 = 0.45 + 0.5i. \end{cases}$$

For the sake of comparison, we first identify methods (1.1), (1.2), (1.3) by **DBN, PGU, CHU**, respectively. Table 4 displays the values

| $f$ | $x_0$ | $\lvert x_n-\alpha\rvert$ | **DBN** | **PGU** | **CHU** | **T1A** | **T2A** | **T2C** |
|---|---|---|---|---|---|---|---|---|
| $f_1$ | 0.85 | $\lvert x_1-\alpha\rvert$ | 3.38e-5* | 1.79e-7 | 4.11e-8 | 3.59e-6 | 1.60e-6 | 2.53e-6 |
| | | $\lvert x_2-\alpha\rvert$ | 3.76e-18 | 1.35e-40 | *7.03e-45* | 1.34e-31 | 5.36e-34 | 1.23e-32 |
| | | $\lvert x_3-\alpha\rvert$ | 5.71e-70 | 0.0e-112 | 0.0e-112 | 0.0e-112 | 0.0e-112 | 0.0e-112 |
| $f_2$ | 1.6 | $\lvert x_1-\alpha\rvert$ | 2.72e-7 | 9.52e-9 | 8.79e-9 | 9.28e-10 | 1.73e-9 | 6.59e-10 |
| | | $\lvert x_2-\alpha\rvert$ | 2.81e-28 | 1.49e-48 | 7.47e-49 | 5.62e-55 | 2.49e-53 | *6.918e-56* |
| | | $\lvert x_3-\alpha\rvert$ | 3.17e-112 | 0.0e-111 | 0.0e-111 | 0.0e-111 | 0.0e-111 | 0.0e-111 |
| $f_3$ | $-0.45$ | $\lvert x_1-\alpha\rvert$ | 7.41e-8 | 2.80e-9 | 2.87e-9 | 1.31e-9 | 1.26e-9 | 1.32e-9 |
| | $+0.5i$ | $\lvert x_2-\alpha\rvert$ | 2.49e-32 | 2.67e-54 | 3.12e-54 | 1.29e-56 | *1.02e-56* | 1.34e-56 |
| | | $\lvert x_3-\alpha\rvert$ | 0.0e-112 | 0.0e-112 | 0.0e-112 | 0.0e-112 | 0.0e-112 | 0.0e-112 |

* 3.38e-5 denotes 3.38 $\times 10^{-5}$

TABLE 4. Comparison of $|x_n-\alpha|$ for $f_1(x)-f_3(x)$ among listed methods

of $|x_n - \alpha|$ for methods **DBN, PGU, CHU**, **T1A, T2A, T2C**. As Table 4 suggests, proposed methods show favorable or equivalent performance as compared with existing methods **DBN, PGU** and **CHU**. Method **DBN** expectedly displays the largest error of $|x_n - \alpha|$ due to its lower order of four, in comparison with the rest of the listed sextic-order methods. In Table 4, italicized numbers indicate the least errors $|x_n - \alpha|$ within the prescribed error bound. Within the same order of convergence, it is important to be aware that the behavior of local convergence of $|x_n - \alpha|$ is dependent on $c_j$, namely $f(x)$ and $\alpha$ as well as initial guesses $x_0$.

Although limited to the test functions chosen in these numerical experiments, **CHU** has shown best accuracy for $f_1$, while **T2C** for $f_3$ and **T2A** for $f_2$. One should keep in mind that no iterative method always shows better accuracy for all the test functions than the others. The corresponding efficiency index of the proposed family of methods (1.4) is found to be $6^{1/4}$, which is better than $4^{1/4}$ being that of the classical double-Newton method. The current approach will extend to the development of 2-point higher-order family of simple root-finders for a nonlinear equation.

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