JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 3, August 2014
http://dx.doi.org/10.14403/jcms.2014.27.3.469

ON PROPERTIES OF DERIVATIVES WITH RESPECT TO FUZZY MEASURES

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Abstract. Fuzzy measures in this paper are a class of distorted Lebesgue measures (see [5]). We investigate some properties of the derivatives with respect to distorted Lebesgue measures and give some important examples.

1. Introduction

Let $\mathcal{B}$ be the smallest σ-algebra including all the closed intervals in $\mathbb{R}^+ = [0, \infty)$. We consider a measurable space $(\mathbb{R}^+, \mathcal{B})$. Let μ be a fuzzy measure on $(\mathbb{R}^+, \mathcal{B})$. A fuzzy measure is defined as a set function $\mu : \mathcal{B} \to \mathbb{R}^+$ such that

(1) $\mu(\phi)=0$,
(2) $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$,
(3) $(A_n \uparrow A \Longrightarrow \mu(A_n) \uparrow \mu(A))$ and $(A_n \downarrow A \Longrightarrow \mu(A_n) \downarrow \mu(A))$.

The Choquet integral is a non-additive integral of a function with respect to a fuzzy measure (see [1,4]). In [5], Sugeno discussed Choquet integral equations of a measurable, non-negative, continuous and increasing function g on the non-negative real line. The Choquet integral of g with respect to a fuzzy measure μ on a set A was defined as

$$f(t) = (C) \int_A g(t) d\mu = \int_0^\infty \mu \left(\{t | g(t) \geq r\} \cap A \right) dr. \tag{1.1}$$

Let $\mathcal{F}^+$ be a class of measurable, non-negative, continuous and increasing functions such that $g : \mathbb{R}^+ \to \mathbb{R}^+$. Let λ be a Lebesgue measure such that $\lambda([a, b]) = b - a$ for $[a, b] \subset [0, \infty)$.

Now we quote two definitons and one theorem from Sugeno (see [5]) to understand our problems.

Received May 19, 2014; Accepted June 30, 2014.
2010 Mathematics Subject Classification: Primary 28E10, 46A55.
Key words and phrases: Choquet integral, fuzzy measures.

Definition 1.1. Let $m : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and increasing function $m(0) = 0$. A fuzzy measure μ_m, a distorted Lebesgue measure, is defined by

$$\mu_m(\cdot) = m(\lambda(\cdot)). \tag{1.2}$$

Theorem 1.2. *Let $g \in \mathcal{F}^+$, then the Choquet integral of g with respect to μ on $[0, t]$ is represented as*

$$\int_0^\infty \mu(\{t|g(t) \geq r\} \cap [0, t])dr = -\int_0^t \mu'([\tau, t])g(\tau)d\tau. \tag{1.3}$$

In particular, for $\mu = \mu_m$

$$\int_0^\infty \mu(\{t|g(t) \geq r\} \cap [0, t])dr = \int_0^t m'(t - \tau)g(\tau)d\tau. \tag{1.4}$$

Definition 1.3. For a continuous and increasing function $f(t)$ with $f(0) = 0$, the derivatives of a function f with respect to a fuzzy measure μ_m is defined as the inverse operation of the Choquet integral based on (1.1) and (1.4) by

$$\frac{df(t)}{d\mu_m(t)} = g(t), \tag{1.5}$$

if $g(t)$ is found to be an element of $\mathcal{F}^+$.

In section 2, we investigate properties of derivatives with respect to fuzzy measures. Moreover some important and easy examples are given.

2. Properties of derivatives

From Definition 1.3, we need to consider a class of $f(t)$'s for a given $m(t)$ such that

$$\mathcal{I}_m(\mathcal{F}^+) = \{f|f(t) = \int_0^t m'(t - \tau)g(\tau)d\tau, \quad g(t) \in \mathcal{F}^+\}. \tag{1.6}$$

Note that $f(t) \in \mathcal{I}_m(\mathcal{F}^+)$ is a continous and increasing function with $f(0) = 0$. From the condition of $g(t)$, we show that (1.5) is linear only for non-negative constants.

Theorem 2.1. *If $f_1(t), f_2(t) \in \mathcal{I}_m(\mathcal{F}^+)$ and α is a non-negative real number, then $\frac{d}{d\mu_m}(f_1(t) + f_2(t))$ and $\frac{d}{d\mu_m}(\alpha f_1(t))$ exist and satisfy the followings:*

(1) $\dfrac{d}{d\mu_m}\left(f_1(t)+f_2(t)\right)=\dfrac{df_1(t)}{d\mu_m}+\dfrac{df_2(t)}{d\mu_m}$,

(2) $\dfrac{d}{d\mu_m}\left(\alpha f_1(t)\right)=\alpha\dfrac{df_1(t)}{d\mu_m}$.

Proof. From the condition of $f_1(t)$ and $f_2(t)$, we see that

$$f_1(t)=\int_0^t m'(t-\tau)\frac{df_1(\tau)}{d\mu_m}d\tau \tag{1.7}$$

and

$$f_2(t)=\int_0^t m'(t-\tau)\frac{df_2(\tau)}{d\mu_m}d\tau. \tag{1.8}$$

Adding (1.7) and (1.8), we obtain that

$$f_1(t)+f_2(t)=\int_0^t m'(t-\tau)\left(\frac{df_1(\tau)}{d\mu_m}+\frac{df_2(\tau)}{d\mu_m}\right)d\tau.$$

From the definition of $\mathcal{F}^+$ and $\mathcal{I}_m(\mathcal{F}^+)$, we know that

$$\frac{df_1(t)}{d\mu_m}+\frac{df_2(t)}{d\mu_m}\in\mathcal{F}^+ \quad \text{and} \quad f_1(t)+f_2(t)\in\mathcal{I}_m(\mathcal{F}^+).$$

Hence (1) is proved.

From (1.7), we see that

$$\alpha f_1(t)=\int_0^t m'(t-\tau)\alpha\frac{df_1(\tau)}{d\mu_m}d\tau.$$

Since $\alpha\dfrac{df_1(t)}{d\mu_m}\in\mathcal{F}^+$, we obtain that $\alpha f_1(t)\in\mathcal{I}_m(\mathcal{F}^+)$. Hence (2) is proved. □

Note that the definition of Choquet integral is convolution. If we use Laplace transformation, then we can calculate derivatives of some functions. Of course, those can be also proved by direct calculations.

Theorem 2.2. *Let $M(s)$ be the Laplace transform of $m(t)$. We have the followings:*

(1) $\dfrac{dm(t)}{d\mu_m}=1$,

(2) $\dfrac{d}{d\mu_m}\left(n\displaystyle\int_0^t m(\tau)(t-\tau)^{n-1}d\tau\right)=t^n, \quad n=1,2,\ldots.$

Proof. From $1 \in \mathcal{F}^+$, we obtain that

$$\mathcal{L}\left\{\int_0^t m'(t-\tau)d\tau\right\} = sM(s)\frac{1}{s} = M(s)$$

and

$$\int_0^t m'(t-\tau)d\tau = \mathcal{L}^{-1}\{M(s)\} = m(t).$$

Hence (1) is proved.

Since $t^n \in \mathcal{F}^+$, we have

$$\mathcal{L}\left\{\int_0^t m'(t-\tau)\tau^n d\tau\right\} = sM(s)\frac{n!}{s^{n+1}} = n!\frac{M(s)}{s^n}$$

and

$$\begin{aligned}\int_0^t m'(t-\tau)\tau^n d\tau = \mathcal{L}^{-1}\left\{n!\frac{M(s)}{s^n}\right\} &= n!\int_0^t m(\tau)\frac{(t-\tau)^{n-1}}{(n-1)!}d\tau \\ &= n\int_0^t m(\tau)(t-\tau)^{n-1}d\tau.\end{aligned}$$

Hence (2) is proved. □

By the almost same method, we can answer what are anti-derivatives of some transcendental functions with respect to distorted fuzzy measures.

THEOREM 2.3. *If $m(t)$ has the Laplace transform, then we have the followings:*

(1) $$\frac{d}{d\mu_m}\left(m(t) + a\int_0^t m(\tau)e^{a(t-\tau)}d\tau\right) = e^{at}, \quad a \geq 0,$$

(2) $$\frac{d}{d\mu_m}\left(\int_0^t \frac{m(t-\tau)}{\tau+1}d\tau\right) = \ln(t+1).$$

Proof. Since $e^{at} \in \mathcal{F}^+$, we have

$$\begin{aligned}\mathcal{L}\left\{\int_0^t m'(t-\tau)e^{a\tau}d\tau\right\} &= sM(s)\frac{1}{s-a} = \left(1+\frac{a}{s-a}\right)M(s) \\ &= M(s) + aM(s)\frac{1}{s-a}\end{aligned}$$

and

$$\begin{aligned}\int_0^t m'(t-\tau)e^{a\tau}d\tau &= \mathcal{L}^{-1}\left\{M(s) + aM(s)\frac{1}{s-a}\right\} \\ &= m(t) + a\int_0^t m(\tau)e^{a(t-\tau)}d\tau.\end{aligned}$$

Hence (1) is proved.

From $\ln(t+1) \in \mathcal{F}^+$ and integration by parts, we have

$$\int_0^t m'(t-\tau)\ln(\tau+1)d\tau = -m(t-\tau)\ln(\tau+1)|_0^t + \int_0^t \frac{m(t-\tau)}{\tau+1}d\tau$$
$$= \int_0^t \frac{m(t-\tau)}{\tau+1}d\tau.$$

□

For a given $f(t)$ and a fuzzy measure μ_m, to find $g(t)$ is to obtain the solution of a Volterra integral equation of the first kind (see [3]). Moreover, we point out that the existence of a solution $g(t)$ is based on the Radon-Nikodym Theorem (see [2]). The existence of derivatives depends on a fuzzy measure μ_m. Now we give some examples.

EXAMPLE 2.4. *$t \notin \mathcal{I}_m(\mathcal{F}^+)$, that is $\nexists \frac{dt}{d\mu_m}$ for $m(t) = e^t - 1$.*

Proof. Suppose that $t \in \mathcal{I}_m(\mathcal{F}^+)$. From the definition of $\mathcal{I}_m(\mathcal{F}^+)$, we have

$$t = \int_0^t m'(t-\tau)x_1(\tau)d\tau = \int_0^t e^{t-\tau}x_1(\tau)d\tau \tag{1.9}$$

where $\frac{dt}{d\mu_m} = x_1(t) \in \mathcal{F}^+$.

But differentiating (1.9), we have

$$1 = e^t \int_0^t e^{-\tau}x_1(\tau)d\tau + x_1(t) = t + x_1(t).$$

From the definition of $\mathcal{F}^+$, we obtain

$$x_1(t) = 1 - t \notin \mathcal{F}^+.$$

It is proved by this controdiction. □

EXAMPLE 2.5. *$t \in \mathcal{I}_m(\mathcal{F}^+)$, that is $\frac{dt}{d\mu_m} \in \mathcal{F}^+$ for $m(t) = t$.*

Proof.

$$t = \int_0^t m'(t-\tau)x_1(\tau)d\tau = \int_0^t x_1(\tau)d\tau \tag{1.10}$$

where $\frac{dt}{d\mu_m} = x_1(t) \in \mathcal{F}^+$.

Differentialing (1.10), we have

$$x_1(t) = 1 \in \mathcal{F}^+.$$

□

Acknowledgment.

The author wishes to thank the referees for careful reading this paper.

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