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ON THE STABILITY OF AN ADDITIVE SET-VALUED FUNCTIONAL EQUATION

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Abstract. In this paper, we consider the additive set-valued functional equation $nf(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i) \oplus \sum_{1 \leq i < j \leq n} f(x_i + x_j)$ where $n \geq 2$ is an integer, and prove the Hyers-Ulam stability of the functional equation.

1. Introduction

The stability problem of functional equations is originated from the question of S. M. Ulam [16] concerning the stability of group homomorphisms. Let G_1 be a group and G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, D. H. Hyers [9] considered the case of approximately additive mappings $f : E_1 \to E_2$ where E_1 and E_2 are Banach spaces and f satisfies inequality $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E_1$. He proved that the function $T : E_1 \to E_2$ which is given by $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ for all $x \in E_1$ is the unique additive mapping satisfying $\|f(x) - T(x)\| < \varepsilon$.

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Hyers' theorem has been generalized by Aoki [1] for additive mapping. In 1978, Th. M. Rassias [15] proved the following theorem.

THEOREM 1.1. *Let $f : E_1 \to E_2$ be a mapping from a normed vector space E_1 into a Banach space E_2 subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E_1$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$, then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Moreover, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E_2 is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear.

For the case $p \geq 1$ related to the theorem, in 1991, Z. Gajda [7] proved the question for the case $p > 1$. Recently, P. Nakmahachalasint [14] proved the Hyers-Ulam-Rassias stability of the following n-dimensional additive functional equation

$$nf(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j). \tag{1.3}$$

In this paper, we improve to establish the generalized Hyers-Ulam-Rassias stability for the set-valued functional equation which is closely related by the functional equation (1.3) and prove the Hyers-Ulam-Rassias stability problem for the set-valued functional equation. The study for set-valued functional equations in Banach spaces has been developed in the last decades. The papers by G. Debreu [6] and R.J. Aumann [3] were inspired by problems arising in control theory and mathematical economics. The stability problems of several functional equation have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2], [5], [4], [8], [11], [12]).

Throughout this paper, let X be a real vector space and Y be a Banach space.

Now, we will introduce the properties for set-valued functional equations which goes into a Banace space. We define $C_b(Y)$ the set of all closed bounded subset of Y and $C_c(Y)$ the set of all closed convex subset of Y. We denote $C_{cb}(Y)$ the set of all closed convex bounded subsets of Y.

Let $A, A' \in C_c(Y)$ and let α, β be positive real numbers. Then we denote $A \oplus A' := \overline{A + A'}$. So it is easy to prove that $\alpha A + \alpha A' = \alpha(A + A')$ and $(\alpha + \beta)A \subseteq \alpha A + \beta A$ for all $\alpha, \beta \in \mathbb{R}^+$. Moreover, we obtain that for every positive real number α and β, $(\alpha + \beta)A = \alpha A + \beta A$.

For a subset $A \subset Y$, the distance function $d(\cdot, A)$ and the support function $s(\cdot, A)$ are defined by $d(x, A) := inf\{\|x - y\|: y \in A\}$ for $x \in Y$ and $s(x^*, A) := sup\{< x^*, x > \mid x \in A\}$ for $x^* \in Y^*$, respectively.

For $A, A' \in C_b(Y)$, the Hausdorff distance $h(A, A')$ is defined by

$$h(A, A') := inf\{\alpha \geq 0 | \ A \subseteq A' + \alpha B_Y, A' \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y. In [4], it was proved that $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup. G. Debreu [6] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space. The following remark is easily proved from the definition of the Hausdorff distance.

Remark 1.2. For $A, A', B, B' \in C_{cb}(Y)$ and $\alpha > 0$, the followings hold :

(a) $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$;
(b) $h(\alpha A, \alpha B) = \alpha h(A, B)$.

Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\| \cdot \|_u$. We define a function j from $(C_{cb}(Y), \oplus, h)$ to $C(B_{Y^*})$ which is induced from s given by $j(A) := s(\cdot, A)$ for each $A \in (C_{cb}(Y), \oplus, h)$.

Then the following properties also hold. (See [6].)

(a) $j(A \oplus B) = j(A) + j(B)$
(b) $j(\alpha A) = \alpha j(A)$
(c) $h(A, B) = \| j(A) - j(B) \|_u$
(d) $j(C_{cb}(Y)$ is closed in $C(B_{Y^*})$

for each $A, B \in C_{cb}(Y)$ and $\alpha \geq 0$.

2. Stability of the set-valued functional equation

Let $f : X \to C_{cb}(Y)$ be a function. The *additive set-valued functional equation* is defined by

$$nf(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i) \oplus \sum_{1 \leq i < j \leq n} f(x_i + x_j) \tag{2.1}$$

for all $x_1, \cdots, x_n \in X$, where $n \geq 2$ is an integer. Every solution of the additive set-valued functional equation is called an *additive set-valued mapping.*

THEOREM 2.1. *Let $n \geq 2$ be an integer and let $\phi : X^n \to [0, \infty)$ be a function satisfying the following properties*

(2.2)
$$\sum_{k=0}^{\infty} \frac{1}{2^k}\phi(2^k x, 2^k x, 0, \cdots, 0) < \infty, \lim_{k\to\infty} \frac{1}{2^k}\phi(2^k x_1, 2^k x_2, \cdots, 2^k x_n) = 0$$

for all $x_1, \cdots, x_n \in X$ and $x \in X$.

Suppose that $f : X \longrightarrow (C_{cb}(Y), h)$ is a set-valued mapping with $f(0) = \{0\}$ and

$$h(nf(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1\leq i<j\leq n} f(x_i + x_j)) \leq \phi(x_1, \cdots, x_n) \tag{2.3}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique additive set-valued mapping $\mathcal{T} : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), \mathcal{T}(x)) \leq \frac{1}{2n-2}\sum_{k=0}^{\infty} \frac{1}{2^k}\phi(2^k x, 2^k x, 0, \cdots, 0) \tag{2.4}$$

for all $x \in X$.

Proof. Set $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_n = 0$ in (2.3). Since the range of f is convex, we have

$$h(nf(2x), f(2x) \oplus (2n-2)f(x)) \leq \phi(x, x, 0, \cdots, 0)$$

for all $x \in X$. By Remark 1.2, we get

$$h((n-1)f(2x), (2n-2)f(x)) \leq \phi(x, x, 0, \cdots, 0) \tag{2.5}$$

for all $x \in X$. Dividing both sides of (2.5) by $2n-2$, we get

$$h(\frac{f(2x)}{2}, f(x)) \leq \frac{1}{2n-2}\phi(x, x, 0, \cdots, 0) \tag{2.6}$$

for all $x \in X$. Replacing x by $2^k x$ and deviding both sides of (2.6) by 2^k, we obtain

$$h(\frac{f(2^{k+1}x)}{2^{k+1}}, \frac{f(2^k x)}{2^k}) \leq \frac{1}{2n-2}\frac{1}{2^k}\phi(2^k x, 2^k x, 0, \cdots, 0) \tag{2.7}$$

for all $x \in X$. Let k, m be integers with $k > m \geq 0$. So we have

$$(2.8)\qquad \begin{aligned} h(\frac{f(2^k x)}{2^k}, \frac{f(2^m x)}{2^m}) &\leq \sum_{j=m}^{k-1} h(\frac{1}{2^j} f(2^j x), \frac{1}{2^{j+1}} f(2^{j+1} x)) \\ &\leq \frac{1}{2n-2} \sum_{j=m}^{k-1} \frac{1}{2^j} \phi(2^j x, 2^j x, 0, \cdots, 0) \end{aligned}$$

for all $x \in X$. Therefore, we obtain from (2.2) and (2.8) that the sequence $\{\frac{1}{2^k} f(2^k x)\}$ is a Cauchy sequence for every $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^k} f(2^k x)\}$ converges in Y. Therefore, we can define a mapping $T : X \to (C_{cb}(Y), h)$ as $T(x) := \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$. Putting $m = 0$ and taking the limit as $k \to \infty$ in (2.8), we get the following inequality

$$h(T(x), f(x)) \leq \frac{1}{2n-2} \sum_{k=0}^{\infty} \frac{1}{2^k} \phi(2^k x, 2^k x, 0, \cdots, 0)$$

for all $x \in X$. It follows from (2.3) and (2.2) that

$$\begin{aligned} h(nT(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} T(x_i) \oplus \sum_{1\leq i<j\leq n} T(x_i + x_j)) \\ &\leq \lim_{k\to\infty} \frac{1}{2^k} \phi(2^k x_1, 2^k x_2, \cdots, 2^k x_n) \\ &= 0 \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in X$. Hence we have that the mapping T is an additive set-valued mapping.

Now we prove the uniqueness for the additive set-valued mapping satisfying the inequality (2.4). To prove the uniqueness for the mapping, let $T' : X \to C_{cb}(Y)$ be another additive set-valued mapping satisfying (2.3) and (2.4) . Then

$$(2.9)\qquad \begin{aligned} h(T(x), T'(x)) &\leq h(T(x), f(x)) + h(f(x), T'(x)) \\ &\leq \frac{1}{n-1} \sum_{j=0}^{k-1} \frac{1}{2^j} \phi(2^j x, 2^j x, 0, \cdots, 0) \end{aligned}$$

for all $x \in X$. Taking the limit as $k \to \infty$ in (2.9), we have $T(x) = T'(x)$ for all $x \in X$. This completes the proof. □

Corollary 2.2. *Let $n \geq 2$ be an integer and let $\theta \geq 0$, $0 < p < 1$. Suppose that $f : X \to C_{cb}(Y)$ is a set-valued mapping satisfying*

$$h(nf(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1\leq i<j\leq n} f(x_i + x_j)) \leq \theta \sum_{i=1}^{n} \| x_i \|^p \tag{2.10}$$

for all $x_1, x_2, \cdots, x_n \in X$. Then there exists a unique additive set-valued mapping $T : X \to C_{cb}(Y)$ satisfying the functional equation

$$nT(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} T(x_i) \oplus \sum_{1\leq i<j\leq n} T(x_i + x_j)$$

and

$$h(f(x), T(x)) \leq \frac{2\theta}{(n-1)(2-2^p)} \| x \|^p$$

for all $x_1, x_2, \cdots, x_n \in X$ and $x \in X$.

Proof. Putting $x_1 = x_2 = \cdots = x_n = 0$ in (2.10), we have

$$h(nf(0), nf(0) \oplus {}_nC_2 f(0)) \leq \theta \cdot 0 = 0,$$

which yields $f(0) = \{0\}$. So we let $\phi(x_1, x_2, \cdots, x_n) = \theta \sum_{i=1}^{n} \| x_i \|^p$ in Theorem 2.1 and obtain the desired results. □

Theorem 2.3. *Let $n \geq 2$ be an integer and let $\phi : X^n \to [0, \infty)$ be a function satisfying the following properties*

$$\sum_{k=0}^{\infty} 2^k \phi(\frac{x}{2^k}, \frac{x}{2^k}, 0, \cdots, 0) < \infty \quad \text{and} \quad \lim_{k\to\infty} 2^k \phi(\frac{x_1}{2^k}, \frac{x_2}{2^k}, \cdots, \frac{x_n}{2^k}) = 0 \tag{2.11}$$

for all $x_1, \cdots, x_n \in X$ and $x \in X$.

Suppose that $f : X \longrightarrow (C_{cb}(Y), h)$ is a set-valued mapping with $f(0) = \{0\}$ and

$$h(nf(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1\leq i<j\leq n} f(x_i + x_j)) \leq \phi(x_1, \cdots, x_n) \tag{2.12}$$

for all $x_1, \cdots, x_n \in X$. Then there exists a unique additive set-valued mapping $T : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{2n-2} \sum_{k=0}^{\infty} 2^k \phi(\frac{x}{2^k}, \frac{x}{2^k}, 0, \cdots, 0) \tag{2.13}$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2^k}$ and multiplying by 2^k in (2.6), we have the following inequality

$$h(2^{k-1}f(\frac{x}{2^{k-1}}), 2^k f(\frac{x}{2^k})) \leq \frac{2^k}{2n-2}\phi(\frac{x}{2^k}, \frac{x}{2^k}, 0, \cdots, 0)$$

for all $x \in X$. The rest of this proof is similar to the proof of Theorem 2.1. □

COROLLARY 2.4. *Let $n \geq 2$ be an integer and let $\theta \geq 0$, $p > 1$. Suppose that $f : X \to C_{cb}(Y)$ is a set-valued mapping satisfying the following property*

$$h(nf(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1\leq i<j\leq n} f(x_i + x_j)) \leq \theta \sum_{i=1}^{n} \| x_i \|^p \tag{2.14}$$

for all $x_1, x_2, \cdots, x_n \in X$. Then there exists a unique additive set-valued mapping $T : X \to C_{cb}(Y)$ satisfying the functional equation

$$nT(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} T(x_i) \oplus \sum_{1\leq i<j\leq n} T(x_i + x_j)$$

and

$$h(f(x), T(x)) \leq \frac{2\theta}{(n-1)(2^p - 2)} \| x \|^p$$

for all $x_1, x_2, \cdots, x_n \in X$ and $x \in X$.

Proof. From the proof of the Corollary 2.2, we get $f(0) = \{0\}$. Applying $\phi(x_1, x_2, \cdots, x_n) = \theta \sum_{i=1}^{n} \| x_i\|^p$ in Theorem 2.3, we can obtain the desired results. □

3. Stability of the additive set-valued functional equation by fixed point method

In this section, we will prove the stability of the additive set-valued functional equation using the fixed point method. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following properties:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the following theorem by Margolis and Diaz[13].

THEOREM 3.1. *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \ \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set* $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

Next, using the fixed point method, we prove the stability of the additive set-valued functional equation.

THEOREM 3.2. *Let $n \geq 2$ be an integer. Suppose that a set-valued mapping $f : X \to (C_{cb}(Y), h)$ with $f(0) = \{0\}$ satisfies the functional inequality*

$$h(nf(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1\leq i<j\leq n} f(x_i + x_j)) \leq \phi(x_1, \cdots, x_n) \tag{3.1}$$

for all $x_1, \cdots, x_n \in X$ and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0,\infty)$ satisfies

$$\phi(x, x, 0, \cdots, 0) \leq 2L\phi(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0) \tag{3.2}$$

for all $x \in X$. Then there exists a unique additive set-valued mapping $T : X \to (C_{cb}(Y), h)$ given by $T(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$ such that

$$h(f(x), T(x)) \leq \frac{L}{(n-1)(1-L)} \phi(x, x, 0, \cdots, 0) \tag{3.3}$$

for all $x \in X$.

Proof. Put $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_n = 0$ in (3.1). Since the range of f is convex, we have

$$h(nf(2x), f(2x) \oplus (2n-2)f(x)) \leq \phi(x, x, 0, \cdots, 0)$$

for all $x \in X$. By Remark 1.2, we get

$$h((n-1)f(2x), (2n-2)f(x)) \leq \phi(x, x, 0, \cdots, 0) \tag{3.4}$$

for all $x \in X$. Dividing both sides of (3.4) by $2n-2$, we get

$$h(\frac{1}{2}f(2x), f(x)) \leq \frac{1}{2n-2}\phi(x, x, 0, \cdots, 0) \leq \frac{L}{n-1}\phi(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0) \tag{3.5}$$

for all $x \in X$. Let $S := \{g \mid g : X \to C_{cb}(Y),\ g(0) = \{0\}\}$. For $g_1, g_2 \in S$, we consider the generalized metric $d(g_1, g_2)$ on S defined by

$$\inf\{\mu \in (0, \infty) \mid h(g_1(x), g_2(x)) \leq \mu\phi(x, x, 0, \cdots, 0),\ \forall x \in X\}$$

and $\inf\{\emptyset\} = \infty$. It is easy to prove that (S, d) is complete(see [10]). Now, we define the linear mapping J: $S \to S$ given by $Jg(x) := \frac{1}{2}g(2x)$ for all $x \in X$.

For $g_1, g_2 \in S$, let $d(g_1, g_2) < \mu$, we get

$$\begin{aligned} h(Jg_1(x), Jg_2(x)) &= h(\frac{1}{2}g_1(2x), \frac{1}{2}g_2(2x)) \\ &\leq \frac{\mu}{2}\phi(2x, 2x, 0, \cdots, 0) \\ &\leq \mu L\phi(x, x, 0, \cdots, 0) \end{aligned} \tag{3.6}$$

for all $x \in X$. The above inequality show that $d(Jg_1, Jg_2) \leq Ld(g_1, g_2)$ for all $g_1, g_2 \in S$. Hence J is a strictly contractive mapping with Lipschitz constant L. So we obtain $d(Jf, f) \leq \frac{L}{n-1} < \infty$ in (3.5). By Theorem 3.1, we get that the mapping $T : X \to C_{cb}(Y)$ satisfies the following properties:

(1) T has a fixed point of J, that is, $T(2x) = 2T(x)$ fo all $x \in X$. The mapping T has a fixed point of J in the set $M = \{g \in S : d(f, g) < \infty\}$. This implies that T is a unique mapping such that there exists a $\mu \in (0, \infty)$ satisfying $h(f(x), T(x)) \leq \mu\phi(x, x, 0, \cdots, 0)$ for all $x \in X$.

(2) T is defined by the limit mapping as following

$$T(x) := lim_{k\to\infty}\frac{f(2^k x)}{2^k} = lim_{k\to\infty}J^k f(x) \tag{3.7}$$

for all $x \in X$.

(3) $d(f, T) \leq \frac{1}{1-L}d(f, Jf)$ implies the inequality $d(f, T) \leq \frac{L}{(n-1)(1-L)}$ and also implies that the inequality (3.3) holds. From (3.1) and (3.7), we have that

$$
\begin{aligned}
& h(nT(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} T(x_i) \oplus \sum_{1 \le i < j \le n} T(x_i + x_j)) \\
& = \lim_{k \to \infty} \frac{1}{2^k} h(nT(\sum_{i=1}^{n} 2^k x_i), \sum_{i=1}^{n} T(2^k x_i) \\
(3.8) \qquad & \oplus \sum_{1 \le i < j \le n} T(2^k x_i + 2^k x_j)) \\
& \le \lim_{k \to \infty} \frac{1}{2^k} \phi(\frac{x}{2^k}, \frac{x}{2^k}, 0, \cdots, 0) \\
& = 0
\end{aligned}
$$

Therefore, T is a unique additive set-valued mapping satisfying the inequality (3.3), as desired.

□

COROLLARY 3.3. *Let $\theta \ge 0$, $0 < p < 1$ be real numbers and X be a real normed space. Suppose that $f : X \to C_{cb}(Y)$ is a set-valued mapping satisfying*

$$
h(nf(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1 \le i < j \le n} f(x_i + x_j)) \le \theta \sum_{i=1}^{n} \| x_i \|^p \tag{3.9}
$$

for all $x_1, x_2, \cdots, x_n \in X$, then there exists a unique additive set-valued mapping $T : X \to C_{cb}(Y)$ satisfying

$$
h(f(x), T(x)) \le \frac{\theta \ 2^p}{(n-1)(2-2^p)} \| x \|^p
$$

for all $x_1, x_2, \cdots, x_n \in X$ and $x \in X$.

Proof. We first take the function ϕ in Theorem 3.2 given by

$$
\phi(x_1, x_2, \cdots, x_n) := \theta \sum_{k=1}^{n} \| x_k \|^p .
$$

Then by choosing $L = 2^{p-1}$, we get the desired result. □

In the following theorem, we focus on changes of the condition for the control function ϕ on the inequality (3.1).

THEOREM 3.4. *Let $n \geq 2$ be an integer. Suppose that a set-valued mapping $f : X \to (C_{cb}(Y), h)$ with $f(0) = \{0\}$ satisfies the functional inequality*

$$h(nf(\sum_{i=1}^{n} x_i), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1\leq i<j\leq n} f(x_i + x_j)) \leq \phi(x_1, \cdots, x_n) \tag{3.10}$$

for all $x_1, \cdots, x_n \in X$ and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(x, x, 0, \cdots, 0) \leq \frac{L}{2}\phi(2x, 2x, 0, \cdots, 0) \tag{3.11}$$

for all $x \in X$. Then there exists a unique additive set-valued mapping $T : X \to (C_{cb}(Y), h)$ given by $T(x) = \lim_{k\to\infty} 2^k f(\frac{x}{2^k})$ such that

$$h(f(x), T(x)) \leq \frac{L}{(2n-2)(1-L)}\phi(x, x, 0, \cdots, 0) \tag{3.12}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$h\Big(\frac{f(2x)}{2}, f(x)\Big) \leq \frac{1}{2n-2}\phi(x, x, 0, \cdots, 0) \tag{3.13}$$

for all $x \in X$. Then we obtain the linear mapping J from S to itself with satisfying $Jg(x) = 2f(\frac{x}{2})$ for all $x \in X$. The rest of this proof is similar to the proof of Theorem 3.2. □

COROLLARY 3.5. *Let $\theta \geq 0$, $p > 1$ be real numbers and X be a real normed space. Suppose that $f : X \to C_{cb}(Y)$ is a set-valued mapping satisfying*

$$h\Big(nf\Big(\sum_{i=1}^{n} x_i\Big), \sum_{i=1}^{n} f(x_i) \oplus \sum_{1\leq i<j\leq n} f(x_i + x_j)\Big) \leq \theta \sum_{i=1}^{n} \| x_i \|^p \tag{3.14}$$

for all $x_1, x_2, \cdots, x_n \in X$, then there exists a unique additive set-valued mapping $T : X \to C_{cb}(Y)$ satisfying

$$h(f(x), T(x)) \leq \frac{\theta}{(n-1)(2^{p-1}-1)} \| x \|^p$$

for all $x_1, x_2, \cdots, x_n \in X$ and $x \in X$.

Proof. We first take the function ϕ in Theorem 3.4 given by

$$\phi(x_1, x_2, \cdots, x_n) := \theta \sum_{k=1}^{n} \| x_k \|^p .$$

Then by choosing $L = 2^{1-p}$, we get the desired result. □

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