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SEPARATION AXIOMS ON BI-GENERALIZED TOPOLOGICAL SPACES

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Abstract. In this paper, introducing various separation axioms on a bi-GTS, it has been observed that such separation axioms actually unify the well-known separation axioms on topological spaces. Several characterizations of such separation properties of a bi-GTS are established in terms of γ_{μ_i,μ_j}-closure operator, generalized cluster sets of functions and graph of functions.

1. Introduction and preliminaries

The concept of bi-Generalized topology (in short, bi-GTS) was introduced by Á. Császár and and E.Makai Jr. in [5]. We study certain separation axioms on bi-GTS and find their characterizations in terms of γ_{μ_i,μ_j}-closure operator [5], graph of a function and generalized cluster sets [2] of a function. It is worth noting that the well-known separation axioms of bi-topological and hence topological spaces, follow as special cases for suitable choices of the bi-GTs.
In the next section, we investigate the behaviour of a bi-GTS obeying separation properties, in terms of a generalized closure operator called γ_{μ_i,μ_j}-closure operator [5]; while in the last section, a bi-GTS under separation properties are discussed in the light of graph of a function and generalized cluster sets [2] of a function.
We now state certain useful definitions and quote several existing results that we require in the next two sections.

Definition 1.1. ([4]) Let X be a nonempty set and μ be a collection of subsets of X (i.e. $\mu \subseteq \mathcal{P}(X)$). μ is called a generalized topology

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(briefly GT) on X iff $\emptyset \in \mu$ and $G_\lambda \in \mu$ for $\lambda \in \Lambda(\neq \emptyset)$ implies $\cup_{\lambda\in\Lambda} G_\lambda \in \mu$. The pair (X, μ) is called a generalized topological space (briefly GTS). The elements of μ are called μ-open sets and their complements are called μ-closed sets. The generalized closure of a subset S of X, denoted by $c_\mu(S)$, is the intersection of all μ-closed sets containing S. The set of all μ-open sets containing an element $x \in X$ is denoted by $\mu(x)$.

For a topological space (X, τ), set of all open, δ-open [18], semi open [10] and pre open [11] subsets of X are denoted respectively by $\tau(X)$, $\Delta(X)$, $SO(X)$ and $PO(X)$.
Let μ_1, μ_2 be two GTs on a non-empty set X. Then (X, μ_1, μ_2) is called bi-Generalized topological space (briefly bi-GTS).

Definition 1.2. ([5]) On a bi-GTS (X, μ_1, μ_2), $\gamma_{\mu_i,\mu_j} : P(X) \to P(X)$ is defined by

$$\gamma_{\mu_i,\mu_j}(A) = \{x \in X : c_{\mu_j} M \cap A \neq \phi \ for\ all\ M \in \mu_i(x)\},$$

for each $A \subseteq X, i, j = 1, 2 (i \neq j)$. $\theta(\mu_i, \mu_j)$, $\delta(\mu_i, \mu_j) \subseteq P(X)$, defined respectively by

$$\theta(\mu_i, \mu_j) = \{A \subset X : for\ each\ x \in A\ there\ exists\ M \in \mu_i(x)\ such\ that\ c_{\mu_j} M \subset A\}, i, j = 1, 2(i \neq j),$$

and
$\delta(\mu_i, \mu_j) = \{A \subseteq X :\ for\ each\ x \in A\ \exists\ \mu_j - closed\ set\ Q\ with\ x \in i_{\mu_i} Q \subseteq A\}$, $i, j = 1, 2(i \neq j)$,
also form GTs on X. The elements of $\theta(\mu_i, \mu_j)$(resp. $\delta(\mu_i, \mu_j)$) are called $\theta(\mu_i, \mu_j)$(resp. $\delta(\mu_i, \mu_j)$)-open and the complements are called $\theta(\mu_i, \mu_j)$(resp. $\delta(\mu_i, \mu_j)$)-closed.

Theorem 1.3. ([5]) *Let (X, μ_1, μ_2) be a bi-GTS and $A \subseteq X$. Then the following hold:*

(1) $\theta(\mu_i, \mu_j) \subseteq \delta(\mu_i, \mu_j) \subseteq \mu_i$.
(2) $A \subseteq \gamma_{\mu_i,\mu_j}(A) \subseteq c_{\theta(\mu_i,\mu_j)}(A)$.
(3) *A is $\theta(\mu_i, \mu_j)$-closed iff $A = \gamma_{\mu_i,\mu_j}(A)$.*

Theorem 1.4. ([14]) *Let (X, μ_1, μ_2) be a bi-GTS. Then for any μ_j-open set A we have $\gamma_{\mu_i,\mu_j}(A) = c_{\mu_i}(A)$.*

Theorem 1.5. *For any subset A in a bi-GTS (X, μ_1, μ_2), $\gamma_{\mu_i,\mu_j}(A) = \cap\{c_{\mu_i} V : A \subseteq V \in \mu_j\}$.*

Let μ_1, μ_2 be two GTs on a non-empty set X and $A \subseteq X$. A is said to be $r(\mu_i, \mu_j)$-open (resp. $r(\mu_i, \mu_j)$-closed) [5] if $A = i_{\mu_i}(c_{\mu_j}(A))$(resp. $A = c_{\mu_i}(i_{\mu_j}(A))$).

Theorem 1.6. ([5]) $x \in c_{\delta(\mu_i,\mu_j)}A$ *iff* $A \cap R \neq \phi$ *for every* $r(\mu_i,\mu_j)$-*open set* R *containing* x.

Let (X, μ_1, μ_2) and (Y, η_1, η_2) be two bi-GTS. The GT $\nu_i(i = 1, 2)$ on the cartesian product $X \times Y$ is defined by $\nu_i = \mu_i \times \eta_j$ for $i, j = 1, 2(i \neq j)$; Then $(X \times Y, \nu_1, \nu_2)$ is again a bi-GTS. Also, for the bi-GTS (X, μ_1, μ_2), $(X \times X, \nu_1, \nu_2)$ is a bi-GTS where $\nu_i = \mu_i \times \mu_j$ for $i, j = 1, 2(i \neq j)$.

2. Separation axioms in terms of γ_{μ_i,μ_j}-closure operator

In this section, we introduce different separation axioms on a bi-GTS and establish their interrelationships. Also, such separation axioms are characterized here using generalized closure operator, called γ_{μ_i,μ_j}-closure operator.

Definition 2.1. Let μ be a GT on a non-empty set X. Then for any $A \subseteq X$, $k_\mu A = \cap\{U \in \mu : A \subseteq U\}$.

Definition 2.2. Let (X, μ_1, μ_2) be a bi-GTS. Then for any point $x \in X$ we define $ij\text{-}ck(A) = (c_{\mu_i}A) \cap (k_{\mu_j}A)$; for $i, j = 1, 2$ $(i \neq j)$.

If $A = \{x\}$, we will write $ij\text{-}ck\{x\}$ for $ij\text{-}ck(\{x\})$.

Lemma 2.3. *Let* x *be an arbitrary point in a bi-GTS* (X, μ_1, μ_2). *Then*

(1) $y \in ji\text{-}ck\{x\}$ *iff* $ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$.
(2) $c_{\mu_i}(ij\text{-}ck\{x\}) = c_{\mu_i}\{x\}$.
(3) $k_{\mu_j}(ij\text{-}ck\{x\}) = k_{\mu_j}\{x\}$.
(4) $\gamma_{\mu_i,\mu_j}(ij\text{-}ck\{x\}) = \gamma_{\mu_i,\mu_j}\{x\}$.
(5) *for any* μ_j-*open set* U *containing* x, $k_{\mu_j}\{x\} \subseteq U$.
(6) *for any* μ_i-*closed set* F *containing* x, $c_{\mu_i}(ij\text{-}ck\{x\}) \subseteq F$.
(7) $k_{\mu_i}(k_{\mu_i}A) = k_{\mu_i}A$ *for* $A \subseteq X$.
(8) $\gamma_{\mu_i,\mu_j}(k_{\mu_j}A) = \gamma_{\mu_i,\mu_j}A$ *for* $A \subseteq X$; $i, j = 1, 2$ $(i \neq j)$.

Proof.
(1) Let, $y \in ji\text{-}ck\{x\}$. Suppose $z \in ij\text{-}ck\{x\}$. Now $y \in ji\text{-}ck\{x\}$ implies $y \in c_{\mu_j}\{x\}$, $y \in k_{\mu_i}\{x\}$ and $z \in ij\text{-}ck\{x\}$ implies $z \in c_{\mu_i}\{x\}$, $z \in k_{\mu_j}\{x\}$. Again $z \in c_{\mu_i}\{x\}$ and $y \in k_{\mu_i}\{x\}$ together imply $z \in c_{\mu_i}\{y\}$. Also $y \in c_{\mu_j}\{x\}$ and $z \in k_{\mu_j}\{x\}$ together imply $z \in k_{\mu_j}\{y\}$. So, $z \in c_{\mu_i}\{y\} \cap k_{\mu_j}\{y\} = ij\text{-}ck\{y\}$. Hence $ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$.
Conversely, let $ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$. Since, $x \in ij\text{-}ck\{x\} \subseteq ij\text{-}ck\{y\}$, So

$x \in c_{\mu_i}\{y\}$ and $x \in k_{\mu_j}\{y\}$. Now $x \in c_{\mu_i}\{y\}$ implies $y \in k_{\mu_i}\{x\}$. Also $x \in k_{\mu_j}\{y\}$ implies $y \in c_{\mu_j}\{x\}$. So, $y \in c_{\mu_j}\{x\} \cap k_{\mu_i}\{x\} = ji\text{-}ck\{x\}$.

(2) Let $z \in c_{\mu_i}(ij\text{-}ck\{x\})$. Therefore for all $U \in \mu_i(z)$, $U \cap (ij\text{-}ck\{x\}) \neq \phi$ and so $U \cap (c_{\mu_i}\{x\}) \neq \phi$ i.e. $z \in c_{\mu_i}(c_{\mu_i}\{x\}) = c_{\mu_i}\{x\}$. Hence $c_{\mu_i}(ij\text{-}ck\{x\}) \subseteq c_{\mu_i}\{x\}$.
Conversely, $\{x\} \subseteq ij\text{-}ck\{x\}$ implies $c_{\mu_i}\{x\} \subseteq c_{\mu_i}(ij\text{-}ck\{x\})$. Thus $c_{\mu_i}(ij\text{-}ck\{x\}) = c_{\mu_i}\{x\}$.

(3) Let $y \in k_{\mu_j}(ij\text{-}ck\{x\})$ but $y \notin k_{\mu_j}\{x\}$. Then there exists $U \in \mu_j(x)$ such that $y \notin U$. Also, $y \in k_{\mu_j}(ij\text{-}ck\{x\}) \Rightarrow ij\text{-}ck\{x\} \cap c_{\mu_j}\{y\} \neq \phi \Rightarrow c_{\mu_j}\{y\} \cap k_{\mu_j}\{x\} \neq \phi$. Hence there exists $z \in c_{\mu_j}\{y\} \cap k_{\mu_j}\{x\}$. Then every μ_j-open neighbourhood of x contains y, a contradiction.

(4) Let, $y \in \gamma_{\mu_i,\mu_j}(ij\text{-}ck\{x\})$ and if possible let $y \notin \gamma_{\mu_i,\mu_j}\{x\}$. Then there exists $U \in \mu_i(y)$ such that $x \notin c_{\mu_j}U$. Again $y \in \gamma_{\mu_i,\mu_j}(ij\text{-}ck\{x\})$ implies $c_{\mu_j}U \cap ij\text{-}ck\{x\} \neq \phi$ i.e. $c_{\mu_j}U \cap k_{\mu_j}\{x\} \neq \phi$ and so there exists $z \in c_{\mu_j}U \cap k_{\mu_j}\{x\}$. Again since, $x \in X \backslash c_{\mu_j}U \in \mu_j$ and $z \in k_{\mu_j}\{x\}$, so , $z \in X \backslash c_{\mu_j}U$, which is not possible. Hence, $\gamma_{\mu_i,\mu_j}(ij\text{-}ck\{x\}) \subseteq \gamma_{\mu_i,\mu_j}\{x\}$.
Conversely, $x \in ij\text{-}ck\{x\}$ implies $\gamma_{\mu_i,\mu_j}\{x\} \subseteq \gamma_{\mu_i,\mu_j}(ij\text{-}ck\{x\})$.

(5) Let, $z \in k_{\mu_j}\{x\}$ and $U \in \mu_j(x)$. Clearly $z \in U$. Thus $k_{\mu_j}\{x\} \subseteq U$.

(6) By (2) $c_{\mu_i}(ij\text{-}ck\{x\}) = c_{\mu_i}\{x\}$ and cosequently $c_{\mu_i}(ij\text{-}ck\{x\}) \subseteq F$.

(7) R.H.S $\subseteq$ L.H.S. We now show that L.H.S $\subseteq$ R.H.S. Let $y \notin R.H.S.$ Then there exists a μ_i open set containing A s.t $y \notin U$. Again $A \subseteq U$ and $U \in \mu_i$ implies that $k_{\mu_i}\{A\} \subseteq U$ and consequently $y \notin R.H.S.$

(8) $R.H.S \subseteq L.H.S.$ We now show that L.H.S $\subseteq$ R.H.S. Let $y \notin R.H.S.$ Then there exists a μ_i open set U containing y s.t. $c_{\mu_j}U \cap A = \phi$, Consequently $c_{\mu_j}U \cap k_{\mu_j}\{A\} = \phi$ (Since, $k_{\mu_j}\{A\}$ is the intesection of all μ_j open set containing A). Hence $y \notin L.H.S.$ □

Corollary 2.4. *For any point x in a bi-GTS (X, μ_1, μ_2) the following hold :*

(1) *For any μ_j-open set U containing x, $ij\text{-}ck\{x\} \subseteq U$.*
(2) *For any μ_i-closed set F containing x, $ij{-}ck\{x\} \subseteq F$.*
(3) *For any point x, $ij\text{-}ck(\{ij\text{-}ck\{x\}) = ij\text{-}ck\{x\}$.*

Proof.

(1) Follows from (5) of Lemma 2.3 and definition of $ij\text{-}ck\{x\}$.
(2) Follows from (6) of Lemma 2.3 and definition of $c_{\mu_i}\{x\}$.
(3) Follows from (2) and (3) of Lemma 2.3. □

Definition 2.5. A bi-GTS (X, μ_1, μ_2) is said to be pairwise R_0-space if for each μ_i-open set G and for each $x \in G$, $c_{\mu_j}\{x\} \subseteq G$; for $i, j = 1, 2$ $(i \neq j)$.

μ_1	μ_2	pairwise R_0
τ	τ	R_0 [8]
$SO(X)$	$SO(X)$	semi R_0 [7]
$PO(X)$	$PO(X)$	pre R_0 [3]

THEOREM 2.6. *If (X, μ_1, μ_2) is pairwise R_0, then for each $x \in X$, $\gamma_{\mu_j,\mu_i}\{x\}\backslash c_{\mu_i}\{x\}$ is a union of μ_j-closed sets; for $i, j = 1, 2$ $(i \neq j)$.*

Proof. Let, $y \in \gamma_{\mu_j,\mu_i}\{x\}\backslash c_{\mu_i}\{x\}$. Then $y \in X\backslash c_{\mu_i}\{x\}$. Since X is pairwise R_0, $c_{\mu_i}\{x\} \cap c_{\mu_j}\{y\} = \phi$. Now $y \in \gamma_{\mu_j,\mu_i}\{x\}$ implies $c_{\mu_j}\{y\} \subseteq \gamma_{\mu_j,\mu_i}\{x\}$. Thus $c_{\mu_j}\{y\} \subseteq \gamma_{\mu_j,\mu_i}\{x\}\backslash c_{\mu_i}\{x\}$. Consequently $\gamma_{\mu_j,\mu_i}\{x\}\backslash c_{\mu_i}\{x\}$ is a union of c_{μ_j}-closed sets. □

THEOREM 2.7. *If for every pair of distinct point x, y in a bi-GTS (X, μ_1, μ_2), either $c_{\mu_i}\{x\} = c_{\mu_j}\{y\}$ or $c_{\mu_i}\{x\} \cap c_{\mu_j}\{y\} = \phi$, for $i, j = 1, 2$ $(i \neq j)$, then (X, μ_1, μ_2) is pairwise R_0.*

Proof. Let G be a μ_i-open set containing $y \in X$. For any $x \in X\backslash G$ as $y \notin c_{\mu_i}\{x\}$, $c_{\mu_i}\{x\} \neq c_{\mu_j}\{y\}$. By the hypothesis, $c_{\mu_i}\{x\} \cap c_{\mu_j}\{y\} = \phi$ which gives $x \notin c_{\mu_j}\{y\}$; i.e. there exists $V_x \in \mu_j(x)$ such that $y \notin V_x$. Let $A = \cup\{V_x : x \in X\backslash G\}$. Then $y \notin A$ and $A \in \mu_j$. So $X\backslash A$ is a μ_j-closed set containing y. Also $X\backslash G \subseteq A$ i.e. $X\backslash A \subseteq G$. Therefore $c_{\mu_j}\{y\} \subseteq G$ and hence (X, μ_1, μ_2) is pairwise R_0. □

DEFINITION 2.8. A bi-GTS (X, μ_1, μ_2) is said to be pairwise R_1 if for any two points $x, y \in X$ such that $x \notin c_{\mu_i}\{y\}$, there are μ_i-open set U containing x and μ_j-open set V containing y such that $U \cap V = \phi$; where $i, j = 1, 2$ $(i \neq j)$.

μ_1	μ_2	pairwise R_1
τ	τ	R_1 [8]
$SO(X)$	$SO(X)$	semi R_1 [7]
$PO(X)$	$PO(X)$	pre R_1 [3]

REMARK 2.9. Every pairwise R_1 space is pairwise R_0.

Proof. Let (X, μ_1, μ_2) be pairwise R_1. Let G be a μ_i open set and $x \in G$. If $X\backslash G = \phi$ then the proof is obvious. So let us consider the case $X\backslash G \neq \phi$ and $y \notin G$. Consequently $x \notin c_{\mu_i}\{y\}$. Since X is pairwise R_1 there exist $U_y \in \mu_i(x)$ and $V_y \in \mu_j(y)$ s.t. $U_y \cap V_y = \phi$. Let $V = \cup_{y \notin G} V_y$ and $F = X\backslash V$. Then F is a μ_j closed set containing x s.t. $F \subseteq G$ i.e. $c_{\mu_j}\{x\} \subseteq G$. Hence X is pairwise R_0. □

But the converse is not true. This follows from the following example.

EXAMPLE 2.10. *Let us consider the set* $X = \{a, b, c\}$. *Let* $\mu_1 = \mu_2 = \mu = \{\phi, \{a, b\}, \{b, c\}, \{c, a\}, X\}$. *Then* $a \notin c_{\mu_i}\{b\} = \{b\}$ *but every* μ-*open set containig them intersect each other. i.e.* X *is not pairwise* R_1. *Again for every* μ_i-*open set* G *and for each* $x \in G$, $c_{\mu_j}\{x\} \subseteq G$, *for* $i, j = 1, 2$. *i.e* X *is pairwise* R_0.

THEOREM 2.11. *Let* (X, μ_1, μ_2) *be a bi-GTS. Then the following are equivalent:*

(a) X *is pairwise* R_1.
(b) $ij\text{-}ck\{x\} = \gamma_{\mu_i,\mu_j}\{x\}$, *for each* $x \in X$.
(c) $ij\text{-}ck\{x\}$ *is* $\theta(\mu_i, \mu_j)$-*closed set, for each* $x \in X$.
(d) $\gamma_{\mu_i,\mu_j}\{x\} = c_{\mu_i}\{x\}$, *for each* $x \in X$.
(e) $\gamma_{\mu_i,\mu_j}\{x\} = k_{\mu_j}\{x\}$, *for each* $x \in X$.
(f) $c_{\mu_i}\{x\}$ *is* $\theta(\mu_i, \mu_j)$-*closed, for each* $x \in X$.
(g) $k_{\mu_j}\{x\}$ *is* $\theta(\mu_i, \mu_j)$-*closed, for each* $x \in X$.
(h) *If* F *is* μ_i-*closed set containing* x, *then* $\gamma_{\mu_i,\mu_j}\{x\} \subseteq F$, *for each* $x \in X$.
(i) *If* U *is a* μ_j-*open set containing* x, *then for each* $x \in X$, $\gamma_{\mu_i,\mu_j}\{x\} \subseteq U$; $i, j = 1, 2$ $(i \neq j)$.

Proof.
$(a) \Rightarrow (b)$: Let $x \in X$. Also let $y \in X$ be such that $y \notin ij\text{-}ck\{x\}$, then $y \notin c_{\mu_i}\{x\} \cap k_{\mu_j}\{x\}$. Now if $y \notin k_{\mu_j}\{x\}$ then $x \notin c_{\mu_j}\{y\}$. since X is pairwise R_1, there exist $U \in \mu_j(x)$ and $V \in \mu_i(y)$ such that $U \cap V = \phi$. Then $y \notin c_{\mu_i}\{x\}$. Thus $y \notin k_{\mu_j}\{x\}$ implies $y \notin c_{\mu_i}\{x\}$. If possible let $y \in \gamma_{\mu_i,\mu_j}\{x\}$, then for all μ_i-open set W containing y, $x \in c_{\mu_j}W$. Since $y \notin c_{\mu_i}\{x\}$ and X is pairwise R_1 there exist $W_1 \in \mu_i(y)$ and $W_2 \in \mu_j(x)$ such that $W_1 \cap W_2 = \phi$ i.e. $x \notin c_{\mu_j}W_1$, a contradiction. So $y \notin \gamma_{\mu_i,\mu_j}\{x\}$ and hence $\gamma_{\mu_i,\mu_j}\{x\} \subseteq ij\text{-}ck\{x\}$.
On the other hand if $y \in ij\text{-}ck\{x\}$, then $y \in c_{\mu_i}\{x\} \subseteq \gamma_{\mu_i,\mu_j}\{x\}$ so that $ij\text{-}ck\{x\} \subseteq \gamma_{\mu_i,\mu_j}\{x\}$.
$(b) \Leftrightarrow (c)$: follows from lemma 2.3.
$(b) \Rightarrow (d)$: This is evident from the fact that $ij\text{-}ck\{x\} \subseteq c_{\mu_i}\{x\} \subseteq \gamma_{\mu_i,\mu_j}\{x\}$, for each $x \in X$.
$(d) \Rightarrow (a)$: Let $x, y \in X$ with $y \notin c_{\mu_i}\{x\} = \gamma_{\mu_i,\mu_j}\{x\}$. Then there exists $U \in \mu_i(y)$ such that $x \notin c_{\mu_j}U$. Hence $X \backslash c_{\mu_j}U$ is a μ_j-open set containing x such that $(X \backslash c_{\mu_j}U) \cap U = \phi$, proving that (X, μ_1, μ_2) is pairwise R_1.
$(d) \Rightarrow (e)$: Let $y \in \gamma_{\mu_i,\mu_j}\{x\} = c_{\mu_i}\{x\}$. If possible let $y \notin k_{\mu_j}\{x\}$. Then $x \notin c_{\mu_j}\{y\} = \gamma_{\mu_j,\mu_i}\{x\}$, a contradiction. Thus $\gamma_{\mu_i,\mu_j}\{x\} \subseteq k_{\mu_j}\{x\}$.
Conversely, if $y \notin \gamma_{\mu_i,\mu_j}\{x\}$, then there exists $W \in \mu_i(y)$ such that

$x \notin c_{\mu_j} W \supseteq c_{\mu_j}\{y\}$ and so $y \notin k_{\mu_j}\{x\}$. Thus $k_{\mu_j}\{x\} \subseteq \gamma_{\mu_i,\mu_j}\{x\}$ and hence $\gamma_{\mu_i,\mu_j}\{x\} = k_{\mu_j}\{x\}$.

$(e) \Rightarrow (d)$: Let, $y \in \gamma_{\mu_i,\mu_j}\{x\} = k_{\mu_j}\{x\}$, then $x \in c_{\mu_j}\{y\}$. Now $y \notin c_{\mu_i}\{x\}$ implies $x \notin k_{\mu_i}\{y\} = \gamma_{\mu_j,\mu_i}\{y\} \supseteq c_{\mu_j}\{y\}$ which is a contradiction. Consequently, $\gamma_{\mu_i,\mu_j}\{x\} \subseteq c_{\mu_i}\{x\}$. The other part, i.e. $c_{\mu_i}\{x\} \subseteq \gamma_{\mu_i,\mu_j}\{x\}$ is obvious.

$(a) \Rightarrow (f)$: Follows from $(b), (c)$ and (d).

$(f) \Rightarrow (d)$: $\{x\} \subseteq c_{\mu_i}\{x\}$ gives $\gamma_{\mu_i,\mu_j}\{x\} \subseteq \gamma_{\mu_i,\mu_j}(c_{\mu_i}\{x\}) = c_{\mu_i}\{x\}$ and $c_{\mu_i}\{x\} \subseteq \gamma_{\mu_i,\mu_j}\{x\}$ is obvious. Hence $\gamma_{\mu_i,\mu_j}\{x\} = c_{\mu_i}\{x\}$ for each $x \in X$.

$(a) \Rightarrow (g)$: It follows from $(b), (c)$ and (e).

$(g) \Rightarrow (e)$: Since $k_{\mu_j}\{x\}$ is $\theta(\mu_i,\mu_j)$-closed for each $x \in X$, $\gamma_{\mu_i,\mu_j}(k_{\mu_j}\{x\}) = k_{\mu_j}\{x\}$ and so (e) follows from Corollary 2.4.

$(h) \Rightarrow (d)$: For each $x \in X$, $c_{\mu_i}\{x\}$ is a μ_i-closed set containing x and hence by (h), $\gamma_{\mu_i,\mu_j}\{x\} \subseteq c_{\mu_i}\{x\}$. Again since $c_{\mu_i}\{x\} \subseteq \gamma_{\mu_i,\mu_j}\{x\}$ is obvious, we have $c_{\mu_i}\{x\} = \gamma_{\mu_i,\mu_j}\{x\}$. The implications "$(b) \Rightarrow (h)$", "$(b) \Rightarrow (i)$" and "$(i) \Rightarrow (e)$" follow respectively from (4.), (3.) and (2.) of corollary 2.4. □

Corollary 2.12. *If (X, μ_1, μ_2) is pairwise R_1-space, then $\gamma_{\mu_i,\mu_j}\{x\}$ is $\theta(\mu_i,\mu_j)$-closed for each $x \in X$.*

Definition 2.13. For any subset A of a bi-GTS (X, μ_1, μ_2) we define $\gamma'_{\mu_i,\mu_j}(A) = \{x \in X : \gamma_{\mu_i,\mu_j}\{x\} \cap A \neq \phi\}$; $i, j = 1, 2$ $(i \neq j)$.

It is easy to observe from the above definition that for $A, B \subseteq X$, (i) $A \subseteq k_{\mu_i} A \subseteq \gamma'_{\mu_i,\mu_j} A$, (ii) $A \subseteq B \Rightarrow \gamma'_{\mu_i,\mu_j} A \subseteq \gamma'_{\mu_i,\mu_j} B$ and (iii) $\gamma'_{\mu_i,\mu_j}(A \cup B) = \gamma'_{\mu_i,\mu_j} A \cup \gamma'_{\mu_i,\mu_j} B$.

Definition 2.14. ([2]) Let (X, μ_1, μ_2) be two bi-GTS. Then X is said to be pairwise-Hausdorff if for $x \neq y$ in X, there exist $U \in \mu_i(x)$, $V \in \mu_j(y)$ such that $U \cap V = \emptyset$. $i, j = 1, 2$ $(i \neq j)$.

μ_1	μ_2	pairwise-Hausdorff
τ	τ	T_2
$SO(X)$	$SO(X)$	semi T_2 [12]
$PO(X)$	$PO(X)$	pre T_2 [9]

Every pairwise Hausdorff space is also a pairwise R_1 space. The example below shows that the converse is not neccessarily true.

Example 2.15. *Let us consider the set $X = \{a, b, c\}$. Let $\mu_1 = \mu_2 = \mu = \{\phi, \{a\}, \{b, c\}, X\}$. Then for the point b and c there exist no pair of disjoint μ-open containing them. i.e. X is not pairwise Hausdorff. But* *for any two points $x, y \in X$ s.t. $x \in c_{\mu_1}\{y\}$, there are μ_1-open set U containing x and μ_2-open set V containinig y s.t. $U \cap V = \phi$. i.e. X is pairwise R_1.*

THEOREM 2.16. *For any bi-GTS (X, μ_1, μ_2) the following are equivalent :*

(a) *X is pairwise Hausdorff.*
(b) *For each $x \in X$, $\{x\} = \gamma_{\mu_i,\mu_j}\{x\} \cup \gamma_{\mu_j,\mu_i}\{x\}$.*
(c) *For any two distinct points x, y of X, $\gamma_{\mu_i,\mu_j}\{x\} \cap \gamma_{\mu_j,\mu_i}\{y\} = \phi$.*
(d) *For any subset A of X, $A = \gamma'_{\mu_i,\mu_j}(A)$; $i, j = 1, 2$ $(i \neq j)$.*

Proof.
$(a) \Rightarrow (b)$: Let $y \in X$ such that $y \neq x$. Then there exist $U \in \mu_i(x)$ and $V \in \mu_j(y)$ such that $U \cap V = \phi$. Thus $x \notin c_{\mu_i}V$ and hence $y \notin \gamma_{\mu_j,\mu_i}\{x\}$. Similarly $y \notin \gamma_{\mu_i,\mu_j}\{x\}$. Consequently, $\{x\} = \gamma_{\mu_i,\mu_j}\{x\} \cup \gamma_{\mu_j,\mu_i}\{x\}$.
$(b) \Rightarrow (c)$: straightforward.
$(c) \Rightarrow (d)$: $A \subseteq \gamma'_{\mu_i,\mu_j}(A)$ is evident. Now let $x \in \gamma'_{\mu_i,\mu_j}(A)$ so that $\gamma_{\mu_i,\mu_j}\{x\} \cap A \neq \phi$. let $y \in X$ such that $y \neq x$. Then $\gamma_{\mu_i,\mu_j}\{y\} \cap \gamma_{\mu_i,\mu_j}\{x\} = \phi$ and consequently, $y \notin \gamma_{\mu_i,\mu_j}\{x\}$. Thus $x \in A$ and hence $\gamma'_{\mu_i,\mu_j}(A) \subseteq A$.
$(d) \Rightarrow (a)$: Let x and y be any two distinct points of X. Now, $\{x\} = \gamma'_{\mu_i,\mu_j}\{x\}$ implies $y \notin \gamma'_{\mu_i,\mu_j}\{x\}$ and hence $x \notin \gamma_{\mu_i,\mu_j}\{y\}$. So there exists a $U \in \mu_i(x)$ such that $y \notin c_{\mu_j}U$, i.e. $y \in (X \backslash c_{\mu_j}U)(= V$, say $) \in \mu_j$ and $U \cap V = \phi$. Hence the bi-GTS is pairwise Hausdorff. □

COROLLARY 2.17. *The following statements are equivalent for a bi-GTS (X, μ_1, μ_2):*

(a) *X is pairwise Hausdorff.*
(b) *For each $x \in X$, $\{x\} = \gamma_{\mu_1,\mu_2}\{x\}$, i.e. every singleton of X is $\theta(\mu_1, \mu_2)$-closed.*
(c) *For each $x \in X$, $\{x\} = \gamma_{\mu_2,\mu_1}\{x\}$, i.e. every singleton of X is $\theta(\mu_2, \mu_1)$-closed.*

DEFINITION 2.18. A bi-GTS (X, μ_1, μ_2) is said to be pairwise Urysohn if for any two distinct point x, y of X, there exist $U \in \mu_1(x)$ and $V \in \mu_2(x)$ such that $c_{\mu_2}U \cap c_{\mu_1}V = \phi$.

μ_1	μ_2	Pairwise Urysohn
τ	τ	Urysohn [19]
$SO(X)$	$SO(X)$	semi-Urysohn [1]
$PO(X)$	$PO(X)$	pre-Urysohn [16]

Every pairwise Urysohn space is also a pairwise Hausdorff space. The converse does not always hold. This follows from the following example.

EXAMPLE 2.19. *Let* $X = \{a, b, c, d, e\}$. *Let us consider* $\mu = \mu_1 = \mu_2 = \{\phi, \{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, c, e\}, \{b, d, e\}, \{b, c, d, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{a, b, c, d\}, X\}$. *Then for every pair of distinct* x, y *there exist disjoint* μ*-open set* U, V *containing* x, y *respectively. i.e.* X *is pairwise Hausdorff. But if we take* a *and* c *then there exist no pair of* μ*-open set* U, V *containing* x, y *respectively s.t.* $c_{\mu_2}U \cap c_{\mu_1}V = \phi$. *i.e.* X *is not pairwise Urysohn.*

DEFINITION 2.20. Let (X, μ_1, μ_2) be a bi-GTS. Then for any subset A of X we define,
$P(A) = (\cap\{\gamma_{\mu_1,\mu_2}(\gamma_{\mu_1,\mu_2}U) : A \subseteq U \in \mu_2\}) \cup (\cap\{\gamma_{\mu_2,\mu_1}(\gamma_{\mu_2,\mu_1}U) : A \subseteq U \in \mu_1\})$.

LEMMA 2.21. *For any point* x *in a bi-GTS* (X, μ_1, μ_2), $p_1[(X \times \{x\}) \cap \gamma_{\nu_i,\nu_j}\Delta] = p_2[(\{x\} \times X) \cap \gamma_{\nu_j,\nu_i}\Delta]$; $i, j = 1, 2$ $(i \neq j)$.

Proof. Let $y \notin L.H.S.$ This implies that $(y, x) \notin \gamma_{\nu_i,\nu_j}\Delta$. So there exists $V \in \mu_i(y)$ and $U \in \mu_j(x)$ such that $c_{\nu_j}(V \times U) \cap \Delta = \phi$ i.e. $(c_{\mu_j}V \times c_{\mu_i}U) \cap \Delta = \phi$ which gives $c_{\mu_j}V \cap c_{\mu_i}U = \phi$. Thus we have $(c_{\mu_i}U \times c_{\mu_j}V) \cap \Delta = \phi$ i.e $c_{\nu_i}(U \times V) \cap \Delta = \phi$ which gives $(x, y) \notin \gamma_{\mu_j,\mu_i}\Delta$ i.e $y \notin R.H.S.$
By reversing the above arguments we can similarly show that $y \notin R.H.S.$ implies $y \notin L.H.S.$ □

LEMMA 2.22. *If* (X, μ_1, μ_2) *is a bi-GTS and* $x \in X$, *then*

$$\begin{aligned} P(\{x\}) &= p_1[(X \times \{x\}) \cap \gamma_{\nu_i,\nu_j}\Delta] \cup p_2[(\{x\} \times X) \cap \gamma_{\nu_i,\nu_j}\Delta] \\ &= p_1[(X \times \{x\}) \cap \gamma_{\nu_1,\nu_2}\Delta] \cup p_1[(X \times \{x\}) \cap \gamma_{\nu_2,\nu_1}\Delta] \\ &= p_2[(\{x\} \times X) \cap \gamma_{\nu_1,\nu_2}\Delta] \cup p_2[(\{x\} \times X) \cap \gamma_{\nu_2,\nu_1}\Delta] \end{aligned}$$

Proof. In view of Lemma 2.21 it is sufficies to show that $P(\{x\}) = p_1[(X \times \{x\}) \cap \gamma_{\nu_1,\nu_2}\Delta] \cup p_1[(X \times \{x\}) \cap \gamma_{\nu_2,\nu_1}\Delta]$. Now, if $y \notin P(\{x\})$ then there exist $U_2 \in \mu_2(x)$ and $U_1 \in \mu_1(x)$ such that $y \notin \gamma_{\mu_1,\mu_2}(\gamma_{\mu_1,\mu_2}U_2)$ and $y \notin \gamma_{\mu_2,\mu_1}(\gamma_{\mu_2,\mu_1}U_1)$. Consequently we have $V_1 \in \mu_1(y)$ and $V_2 \in \mu_2(y)$ such that $c_{\mu_2}V_1 \cap \gamma_{\mu_1,\mu_2}U_2 = \phi = c_{\mu_1}V_2 \cap \gamma_{\mu_2,\mu_1}U_1$ i.e. $c_{\mu_2}V_1 \cap c_{\mu_1}U_2 = \phi = c_{\mu_1}V_2 \cap c_{\mu_2}U_1$ (by Theorem 1.4). Then $(c_{\mu_2}V_1 \times c_{\mu_1}U_2) \cap \Delta = \phi = (c_{\mu_1}V_2 \times c_{\mu_2}U_1) \cap \Delta$ i.e. $c_{\nu_2}(V_1 \times U_2) \cap \Delta = \phi = c_{\nu_1}(V_2 \times U_1) \cap \Delta$ which gives $(y, x) \notin \gamma_{\nu_1,\nu_2}\Delta$ and $(y, x) \notin \gamma_{\nu_2,\nu_1}\Delta$. Hence $y \notin p_1[(X \times \{x\}) \cap \gamma_{\nu_1,\nu_2}\Delta]$ and $y \notin p_1[(X \times \{x\}) \cap \gamma_{\nu_2,\nu_1}\Delta]$ so that $y \notin R.H.S.$ By reversing the above argument we can similarly show that $y \notin R.H.S.$ implies $y \notin L.H.S.$ □

Theorem 2.23. *For a bi-GTS (X, μ_1, μ_2) the following are equivalent:*

(1) *X is pairwise Urysohn.*
(2) *For each $x \in X, \{x\} = P(\{x\})$.*
(3) *$\Delta = (\gamma_{\nu_1,\nu_2}\Delta) \cup (\gamma_{\nu_2,\nu_1}\Delta)$.*

Proof.
(1) ⇒ (2): Let $x \in X$ and y be any point of X with $y \neq x$. Then there exist $U \in \mu_1(x)$ and $V \in \mu_2(y)$ such that $c_{\mu_2}U \cap c_{\mu_1}V = \phi$ and so we have $\gamma_{\mu_2,\mu_1}U \cap c_{\mu_1}V = \phi$ (by Theorem 1.4) so that $y \notin \gamma_{\mu_2,\mu_1}(\gamma_{\mu_2,\mu_1}U)$. Similarly we can find $W \in \mu_2(x)$ such that $y \notin \gamma_{\mu_1,\mu_2}(\gamma_{\mu_1,\mu_2}W)$. Hence we get (2).
(2) ⇒ (3) : We have,

$$\begin{aligned}(x,y) \notin \Delta &\Leftrightarrow y \notin P(\{x\})\\ &\Leftrightarrow y \notin p_2[(\{x\} \times X) \cap \gamma_{\nu_1,\nu_2}\Delta] \cup p_2[(\{x\} \times X) \cap \gamma_{\nu_2,\nu_1}\Delta]\\ &\Leftrightarrow (x,y) \notin \gamma_{\nu_1,\nu_2}\Delta \text{ and } (x,y) \notin \gamma_{\nu_2,\nu_1}\Delta\end{aligned}$$

Thus (3) follows.
(3) ⇒ (1): Let $x, y \in X$ such that $x \neq y$. Since $(x, y) \notin \Delta, (x, y) \notin \gamma_{\nu_1,\nu_2}\Delta$ so that $c_{\nu_2}(U_1 \times V_2) \cap \Delta = \phi$ for some $U_1 \in \mu_1(x)$ and $V_2 \in \mu_2(y)$. Then $(c_{\mu_2}U_1 \times c_{\mu_1}V_2) \cap \Delta = \phi$ i.e. $c_{\mu_2}U_1 \cap c_{\mu_1}V_2 = \phi$, proving that X is pairwise Urysohn. □

Corollary 2.24. *A bi-GTS (X, μ_1, μ_2) is pairwise Urysohn iff any one of the following conditions holds:*

(1) *For each $x \in X, \{x\} = \cap\{\gamma_{\mu_1,\mu_2}(\gamma_{\mu_1,\mu_2}U) : U \in \mu_2(x)\}$.*
(2) *For each $x \in X, \{x\} = \cap\{\gamma_{\mu_2,\mu_1}(\gamma_{\mu_2,\mu_1}U) : U \in \mu_1(x)\}$.*
(3) *$\Delta = \gamma_{\mu_1,\mu_2}\Delta$.*
(4) *$\Delta = \gamma_{\mu_2,\mu_1}\Delta$.*

Definition 2.25. ([14]) Let (X, μ_1, μ_2) be a bi-GTS. Then X is said to be (μ_i, μ_j)-regular if for any $x \in X$ and any μ_i-closed set F not containing x, there exist $U \in \mu_i$ and $V \in \mu_j$ with $x \in U, F \subseteq V$ such that $U \cap V = \emptyset$; $i, j = 1, 2$ $(i \neq j)$.

If X is (μ_1, μ_2) and (μ_2, μ_1) regular then X is called pairwise regular.

μ_1	μ_2	(μ_1, μ_2)-regular
τ	τ	regular
Δ	τ	almost regular [17]
$SO(X)$	$SO(X)$	semi regular [6]
$PO(X)$	$PO(X)$	strong regular [13]

Theorem 2.26. [14] *A bi-GTS* (X, μ_1, μ_2) *is* (μ_i, μ_j)*-regular iff for each point* $x \in X$ *and each* μ_i*-open set* G *containing* x*, there is a* μ_i*-open set* H *containing* x *such that* $c_{\mu_j}H \subseteq G$*;* $i, j = 1, 2$ $(i \neq j)$*.*

Theorem 2.27. *A bi-GTS* (X, μ_1, μ_2) *is* (μ_i, μ_j)*-regular iff for any set* A *in* X*,* $c_{\mu_i}A = \gamma_{\mu_i,\mu_j}A$*;* $i, j = 1, 2$ $(i \neq j)$*.*

Proof. First suppose that X is (μ_i, μ_j)-regular. Obviously $c_{\mu_i}A \subseteq \gamma_{\mu_i,\mu_j}A$ for $A \subseteq X$. Now let $x \in \gamma_{\mu_i,\mu_j}A$ and U be any μ_i-open set containing x, then by Theorem 2.23 there exists a μ_i-open set V containing x such that $c_{\mu_j}V \subseteq U$. Now since $x \in \gamma_{\mu_i,\mu_j}A$, we get $c_{\mu_j}V \cap A \neq \phi$ and hence $U \cap A \neq \phi$. Thus $x \in c_{\mu_i}A$ and consequently, $\gamma_{\mu_i,\mu_j}A = c_{\mu_i}A$.
Conversly, let $x \in X$ and U be a μ_i-open set containing x. Then $x \notin X\backslash U = c_{\mu_i}(X\backslash U) = \gamma_{\mu_i,\mu_j}(X\backslash U)$. Thus there exists a μ_i-open set V containing x such that $c_{\mu_j}V \cap (X\backslash U) = \phi$ i.e. $c_{\mu_j}V \subseteq U$ and hence X is (μ_i, μ_j)-regular. □

Corollary 2.28. *A Bi-BTS* (X, μ_i, μ_j) *is pairwise regular iff every* μ_i*-closed set is* $\theta(\mu_i, \mu_j)$*-closed;* $i, j = 1, 2$ $(i \neq j)$*.*

Definition 2.29. ([15]) A bi-GTS (X, μ_1, μ_2) is said to be (μ_i, μ_j)-almost regular if for each $x \in X$ and $r(\mu_i, \mu_j)$-closed set F with $x \notin F$, there exist $U \in \mu_i$ and $V \in \mu_j$ such that $x \in U, F \subseteq V$ and $U \cap V = \phi$; $i, j = 1, 2$ $(i \neq j)$.
X is called pairwise almost regular if it is both (μ_1, μ_2)-almost regular and (μ_2, μ_1)-almost regular.

It is easy to check that every pairwise regular space is also a pairwise almost regular space. But the converse is not so. This follows from the following example.

Example 2.30. *Let us consider the set* $X = \{a, b, c, d\}$*. Let* $\mu_1 = \mu_2 = \mu = \{\phi, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$*. Then for the point* a *and the* μ*-closed set* $F = \{b, c\}$ *there exist no pair of* μ*-open sets* U *and* V *s.t.* $a \in U$*,* $F \subseteq V$ *and* $U \cap V = \phi$*. i.e.* X *is not pairwise regular. But the* $r(\mu_i, \mu_j)$*-closed set in* X *are* $\phi, \{a, b\}, \{c, d\}, \{b, d\}, \{a, c\}$*. So for each* $x \in X$ *and* $r(\mu_i, \mu_j)$*-closed set* F *with* $x \notin F$*, there exist* $U \in \mu_i$ *and* $V \in \mu_j$ *such that* $x \in U, F \subseteq V$ *and* $U \cap V = \phi$*. Hence* X *is pairwise almost regular.*

Theorem 2.31. ([15]) A bi-GTS (X, μ_1, μ_2) is (μ_i, μ_j)-almost regular iff for $x \in X$ and $r(\mu_i, \mu_j)$-open set U containing x, there exists μ_i-open set V containing x such that $c_{\mu_j}V \subseteq U$; $i, j = 1, 2$ $(i \neq j)$.

Theorem 2.32. *For a bi-GTS* (X, μ_1, μ_2) *the following are equivalent:*

(*a*) *X is* (μ_i, μ_j)*-almost regular.*
(*b*) *For any set* $A \subseteq X$, $\gamma_{\mu_i,\mu_j} A = c_{\delta(\mu_i,\mu_j)} A$.
(*c*) *For any set* $A \subseteq X$, $\gamma_{\mu_i,\mu_j}(\gamma_{\mu_i,\mu_j} A) = \gamma_{\mu_i,\mu_j} A$.
(*d*) *For any* μ_j*-open set* A, $\gamma_{\mu_i,\mu_j}(\gamma_{\mu_i,\mu_j} A) = \gamma_{\mu_i,\mu_j} A$; $i, j = 1, 2$ $(i \neq j)$.

Proof.
$(a) \Rightarrow (b)$: It is always true that $c_{\delta(\mu_i,\mu_j)} A \subseteq \gamma_{\mu_i,\mu_j} A$. Let, $x \in \gamma_{\mu_i,\mu_j} A$ and $U \in \mu_i(x)$. Then by (a), there exists $V \in \mu_i(x)$ such that $c_{\mu_j} V \subseteq i_{\mu_i} c_{\mu_j} U$. Since $c_{\mu_j} V \cap A \neq \phi$, we have $(i_{\mu_i} c_{\mu_j} U) \cap A \neq \phi$ and thus $x \in c_{\delta(\mu_i,\mu_j)} A$.
$(b) \Rightarrow (c)$: We have,

$$\begin{aligned}\gamma_{\mu_i,\mu_j}(\gamma_{\mu_i,\mu_j} A) &= \gamma_{\mu_i,\mu_j}(c_{\delta(\mu_i,\mu_j)} A) = c_{\delta(\mu_i,\mu_j)}(c_{\delta(\mu_i,\mu_j)} A) \\ &= c_{\delta(\mu_i,\mu_j)} A = \gamma_{\mu_i,\mu_j} A.\end{aligned}$$

$(c) \Rightarrow (d)$: Straightforward.
$(d) \Rightarrow (a)$: Let F be any $r(\mu_i, \mu_j)$-open set in X and $p \in F$. Now $A = X \backslash F$ is an $r(\mu_i, \mu_j)$-closed set and then $A = c_{\mu_i}(i_{\mu_j} A)$. Put $B = i_{\mu_j} A$. Then $\gamma_{\mu_i,\mu_j} A = \gamma_{\mu_i,\mu_j}(c_{\mu_i} B) = \gamma_{\mu_i,\mu_j}(\gamma_{\mu_i,\mu_j} B) = \gamma_{\mu_i,\mu_j} B = c_{\mu_i} B = A$. Then $p \notin \gamma_{\mu_i,\mu_j} A$ and hence there exist $G \in \mu_i(p)$ such that $c_{\mu_j} G \cap A = \phi$ i.e. $c_{\mu_j} G \subseteq X \backslash A = F$. Hence X is (μ_i, μ_j)-almost regular. □

3. Separation axioms via generalized cluster sets and graph of a function

This section is devoted to establish necessary and sufficient conditions for separation properties of a bi-GTS via generalized cluster sets and graph of a function. We begin with a few useful lemmas and already known definitions.

Lemma 3.1. *Let* f *be a function from a set* X *to a set* Y*. Then for any* $A \subseteq X$ *and any* $B \subseteq Y$, $f(A) \cap B = \{y \in Y : (x, y) \in ((A \times B) \cap G(f)),\ for\ some\ x \in X\}$.

Lemma 3.2. *Let* (X, μ_1, μ_2) *and* (Y, η_1, η_2) *be bi-GTS. Then* $\gamma_{\nu_i,\nu_j}\{(x, y)\} = \gamma_{\mu_i,\mu_j}\{x\} \times \gamma_{\eta_j,\eta_i}\{y\}$, *for any* $(x, y) \in X \times Y$.

Proof. Let $(a, b) \in \gamma_{\nu_i,\nu_j}\{(x, y)\}$ and $U \in \mu_i(a)$, $V \in \eta_j(b)$. Then $(x, y) \in c_{\nu_j}(U \times V) \Rightarrow (x, y) \in c_{\mu_j} U \times c_{\eta_i} V \Rightarrow x \in c_{\mu_j} U$ and $y \in c_{\eta_i} V$. Hence $a \in \gamma_{\mu_i,\mu_j}\{x\}$ and $b \in \gamma_{\eta_j,\eta_i}\{y\}$. This shows that $(a, b) \in \gamma_{\mu_i,\mu_j}\{x\} \times \gamma_{\eta_j,\eta_i}\{y\}$. Then $\gamma_{\nu_i,\nu_j}\{(x, y)\} \subseteq \gamma_{\mu_i,\mu_j}\{x\} \times \gamma_{\eta_j,\eta_i}\{y\}$. Reversing the argument we get the reverse inclusion. Hence $\gamma_{\nu_i,\nu_j}\{(x, y)\} = \gamma_{\mu_i,\mu_j}\{x\} \times \gamma_{\eta_j,\eta_i}\{y\}$ for any $(x, y) \in X \times Y$. □

DEFINITION 3.3. ([2]) Let $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ be a function. Then for any $x \in X$, then the generalized cluster set of f at any point x is given by $\mathcal{G}_{ij}^{kl}(f, x) = \cap\{\gamma_{\eta_k,\eta_l} f(c_{\mu_j} U) : U \in \mu_i(x)\}$ $i, j, k, l = 1, 2$ $(i \neq j$ and $k \neq l)$.

LEMMA 3.4. *Let $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ be a function and $x \in X$. Then*

(1) $p_2((\{x\} \times Y) \cap \gamma_{\nu_i,\nu_j} G(f)) = \mathcal{G}_{ij}^{ji}(f, x)$.
(2) $p_2((\{x\} \times Y) \cap k_{\nu_i} G(f)) = k_{\eta_j}(f(c_{\mu_i}\{x\}))$.
(3) $p_2((\{x\} \times Y) \cap \gamma'_{\nu_i,\nu_j} G(f)) = \gamma'_{\eta_j,\eta_i}(f(\gamma_{\mu_i,\mu_j}\{x\}))$.

Proof.

(1) Let $y \in \mathcal{G}_{ij}^{ji}(f, x)$ and $U \in \mu_i(x), V \in \eta_j(y)$. Then $y \in \gamma_{\eta_j,\eta_i} f(c_{\mu_j} U)$ and so $c_{\eta_i} V \cap f(c_{\mu_j} U) \neq \phi$ i.e. $(c_{\mu_j} U \times c_{\eta_i} V) \cap G(f) \neq \phi$ i.e. $c_{\nu_j}(U \times V) \cap G(f) \neq \phi$. This shows that $(x, y) \in \gamma_{\nu_i,\nu_j} G(f)$; so that $y \in p_2((\{x\} \times Y) \cap \gamma_{\nu_i,\nu_j} G(f))$. Reversing the step we get the reverse inclusion. Hence $p_2((\{x\} \times Y) \cap \gamma_{\nu_i,\nu_j} G(f)) = \mathcal{G}_{ij}^{ji}(f, x)$.

(2) Let $y \in L.H.S.$ Then $(x, y) \in k_{\nu_i} G(f)$ i.e. $c_{\nu_i}\{(x, y)\} \cap G(f) \neq \phi$, which gives $(c_{\mu_i}\{x\} \times c_{\eta_j}\{y\}) \cap G(f) \neq \phi$ so $f(c_{\mu_i}\{x\}) \cap c_{\eta_j}\{y\} \neq \phi$ and hence $y \in k_{\eta_j}(f(c_{\mu_i}\{x\}))$. i.e. $y \in R.H.S.$ Then $L.H.S. \subseteq R.H.S.$

(3) Let $y \in p_2((\{x\} \times Y) \cap \gamma'_{\nu_i,\nu_j} G(f))$. Then $(x, y) \in \gamma'_{\nu_i,\nu_j} G(f)$ i.e. $\gamma_{\nu_i,\nu_j}\{(x, y)\} \cap G(f) \neq \phi$. So by lemma 3.2 $(\gamma_{\mu_i,\mu_j}\{x\} \times \gamma_{\eta_j,\eta_i}\{y\}) \cap G(f) \neq \phi$. Then by lemma 3.1 $f(\gamma_{\mu_i,\mu_j}\{x\}) \cap \gamma_{\eta_j,\eta_i}\{y\} \neq \phi$, which gives $y \in \gamma'_{\eta_j,\eta_i}(f(\gamma_{\mu_i,\mu_j}\{x\}))$. Hence $p_2((\{x\} \times Y) \cap \gamma'_{\nu_i,\nu_j} G(f)) \subseteq \gamma'_{\eta_j,\eta_i}(f(\gamma_{\mu_i,\mu_j}\{x\}))$. Reversal of above aruments yields the inclusion the other way round. □

COROLLARY 3.5. *If $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is a function then the following are equivalent:*

(1) *f has a $\theta(\nu_i, \nu_j)$-closed graph.*
(2) *$\{f(x)\} = p_2((\{x\} \times Y) \cap \gamma_{\nu_i,\nu_j} G(f))$, for each $x \in X$.*
(3) *$\mathcal{G}_{ij}^{ji}(f, x) = \{f(x)\}$, for each $x \in X$.*

DEFINITION 3.6. ([2]) Let (X, μ_1, μ_2) and (Y, η_1, η) be two bi-GTS. Then $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is said to be $(\mu_i\mu_j, \eta_k)$-continuous at $x \in X$ if for each $V \in \eta_k(f(x))$, there exists $U \in \mu_i(x)$ such that $f(c_{\mu_j} U) \subseteq V$. $i, j, k = 1, 2$ $(i \neq j)$.
If f is $(\mu_i\mu_j, \eta_k)$-continuous at each $x \in X$ then f is called $(\mu_i\mu_j, \eta_k)$-continuous on X.
If f is both $(\mu_i\mu_j, \eta_1)$nd $(\mu_i\mu_j, \eta_2)$-continuous then f is called pairwise $(\mu_i\mu_j, \eta_k)$-continuous.

Lemma 3.7. *If $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is a function such that f is $(\mu_i\mu_j, \eta_i)$ continuous then, $\mathcal{G}_{ij}^{ji}(f, x) = \gamma_{\eta_j,\eta_i}\{f(x)\}$, for each $x \in X$.*

Proof. Let $V \in \eta_i$ such that $f(x) \in V$. Since f is $(\mu_i\mu_j, \eta_i)$ continuous, there exists $U \in \mu_i(x)$ such that $f(c_{\mu_j}U) \subseteq V$. Then $\mathcal{G}_{ij}^{ji}(f, x) \subseteq \gamma_{\eta_j,\eta_i} f(c_{\mu_j}U) \subseteq \gamma_{\eta_j,\eta_i} V = c_{\eta_j} V$ (by Theorem 1.4). Thus $\mathcal{G}_{ij}^{ji}(f, x) \subseteq c_{\eta_j} V$ for all $V \in \eta_i$ with $f(x) \in V$. By Theorem 1.5 $\mathcal{G}_{ij}^{ji}(f, x) \subseteq \gamma_{\eta_j,\eta_i}\{f(x)\}$. On the other hand, $\gamma_{\eta_j,\eta_i}\{f(x)\} \subseteq \mathcal{G}_{ij}^{ji}(f, x)$ is obvious. Hence the lemma. □

Corollary 3.8. *A function $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ has the property that f is $(\mu_i\mu_j, \eta_i)$ continuous , has a $\theta(\nu_i, \nu_j)$-closed graph iff it has $\theta(\eta_j, \eta_i)$-closed point images.*

Lemma 3.9. *A function $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is $(\mu_i\mu_j, \eta_k)$ continuous iff $f(\gamma_{\mu_i,\mu_j} A) \subseteq c_{\eta_k} f(A); i, j, k = 1, 2$ $(i \neq j)$.*

Proof. Let f be a $(\mu_i\mu_j, \eta_k)$ continuous and $y \in f(\gamma_{\mu_i,\mu_j} A)$. There exists $x \in X$ such that $x \in \gamma_{\mu_i,\mu_j} A$ and $f(x) = y$. Let $V \in \eta_k(f(x))$. Then there exists $U \in \mu_i(x)$ such that $f(c_{\mu_j}U) \subseteq V$. Again since $x \in \gamma_{\mu_i,\mu_j} A$ we have $c_{\mu_j}U \cap A \neq \phi$ and so $f(c_{\mu_j}U) \cap f(A) \neq \phi$ i.e $V \cap f(A) \neq \phi$ i.e $f(x) \in c_{\eta_k} f(A)$ and hence $y \in c_{\eta_k} f(A)$.
Conversely, let $x \in X$ be arbitrary and $V \in \eta_k(f(x))$. Then $f(x) \notin c_{\eta_k}(Y \backslash V)$ and so $f(x) \notin c_{\eta_k}(f f^{-1}(Y \backslash V))$. By the hypothesis $f(x) \notin f(\gamma_{\mu_i,\mu_j}(f^{-1}(Y \backslash V)))$, so that $x \notin \gamma_{\mu_i,\mu_j}(X \backslash (f^{-1}V))$. Thus there exists $U \in \mu_i(x)$ such that $c_{\mu_j}U \subseteq f^{-1}V$ i.e. $f(c_{\mu_j}U) \subseteq V$. Hence f is $(\mu_i\mu_j, \eta_k)$ continuous. □

Theorem 3.10. *If $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is pairwise $(\mu_i\mu_j, \eta_k)$ continuous and (Y, η_1, η_2) is pairwise R_1, then $\gamma_{\nu_i,\nu_j} G(f) = k_{\nu_j} G(f)$.*

Proof. Since f is pairwise $(\mu_i\mu_j, \eta_k)$ continuous and (Y, η_1, η_2) is pairwise R_1, by Lemma 3.7, Theorem 2.11 and Lemma 3.9 we have , $\mathcal{G}_{ij}^{ji}(f, x)$ $= \gamma_{\eta_j,\eta_i}\{f(x)\} = k_{\eta_i}\{f(x)\}$ and $f(\gamma_{\mu_i,\mu_j}\{x\}) \subseteq c_{\eta_j}\{f(x)\}$. So $k_{\eta_i}\{f(x)\}$ $\subseteq k_{\eta_i} f(c_{\mu_j}\{x\}) \subseteq k_{\eta_i} f(\gamma_{\mu_j,\mu_i}\{x\}) \subseteq k_{\eta_i}(c_{\eta_j}\{f(x)\})$. Again by theorem 2.11 ji-$ck\{f(x)\} = \gamma_{\eta_j,\eta_i}\{f(x)\} = c_{\eta_j}\{f(x)\}$. So by Lemma 2.3 $k_{\eta_i}\{f(x)\} = k_{\eta_i}(ji\text{-}ck\{f(x)\}) = k_{\eta_i}(c_{\eta_j}\{f(x)\})$. Hence $k_{\eta_i}\{f(x)\} = k_{\eta_i} f(c_{\mu_j}\{x\})$. i.e. $\mathcal{G}_{ij}^{ji}(f, x) = k_{\eta_i} f(c_{\mu_j}\{x\})$. It follows from lemma 3.4 $\gamma_{\nu_i,\nu_j} G(f) = k_{\nu_j} G(f)$. □

Theorem 3.11. *If $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is $(\mu_i\mu_j, \eta_i)$ continuous and (Y, η_1, η_2) is pairwise Hausdorff, then $\gamma_{\nu_i,\nu_j} G(f) = \gamma'_{\nu_i,\nu_j} G(f)$.*

Proof. Since f is $(\mu_i\mu_j, \eta_i)$ continuous and (Y, η_1, η_2) is pairwise Hausdorff, by Lemma 3.7, Corollary 2.17 and Lemma 3.9 we have , $\mathcal{G}^{ji}_{ij}(f, x) = \gamma_{\eta_j,\eta_i}\{f(x)\} = \{f(x)\}$ and $f(\gamma_{\mu_i,\mu_j}\{x\}) \subseteq c_{\eta_i}\{f(x)\}$. Again using Theorem 2.16, Lemma 3.9 and Corollary 2.17 we have

$$\begin{aligned}\{f(x)\} &\subseteq \gamma'_{\eta_j,\eta_i}(f(\gamma_{\mu_i,\mu_j}\{x\})) \\ &= f(\gamma_{\mu_i,\mu_j}\{x\}) \subseteq c_{\eta_i}\{f(x)\} \subseteq \gamma_{\eta_i,\eta_j}\{f(x)\} = \{f(x)\}.\end{aligned}$$

Then $\gamma'_{\eta_j,\eta_i}(f(\gamma_{\mu_i,\mu_j}\{x\})) = \{f(x)\} = \mathcal{G}^{ji}_{ij}(f, x)$. Hence by Lemma 3.4 $\gamma_{\nu_i,\nu_j}G(f) = \gamma'_{\nu_i,\nu_j}G(f)$. □

THEOREM 3.12. *If a bi-GTS (Y, η_1, η_2) is pairwise R_1, then for any bi-GTS (X, μ_1, μ_2) and every $(\mu_i\mu_j, \eta_i)$ continuous function $f : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ with η_j-closed point image, f has a $\theta(\nu_i, \nu_j)$-closed graph.*

Proof. Let f be a $(\mu_i\mu_j, \eta_i)$ continuous function from a bi-GTS (X, μ_1, μ_2) to (Y, η_1, η_2) with η_i closed point image and let Y be a pairwise R_1. Then for each $x \in X$, $\{f(x)\} = c_{\eta_j}\{f(x)\} = \gamma_{\eta_j,\eta_i}\{f(x)\}$(by Theorem 2.11)= $\mathcal{G}^{ji}_{ij}(f, x)$(by Lemma 3.7). Then by the Corollary 3.5 f has a $\theta(\nu_i, \nu_j)$ closed graph. □

THEOREM 3.13. *If a bi-GTS (Y, η_1, η_2) is pairwise Hausdorff then every $(\mu_i\mu_j, \eta_i)$ continuous function f from any (X, μ_1, μ_2) to (Y, η_1, η_2) has an $\theta(\nu_i, \nu_j)$-closed graph.*

Proof. It follows from Lemma 3.7, Corollary 2.17 and Corollary 3.5. □

DEFINITION 3.14. A multifunction $F : (X, \mu_1, \mu_2) \to (Y, \eta_1, \eta_2)$ is called $(\mu_i\mu_j, \eta_k)$ continuous at a point x of X if for each η_k open set W in Y such that $F(x) \subseteq W$, there is a $V \in \mu_i(x)$ satisfying $F(c_{\mu_j}V) \subseteq W$, where $F(V) = \cup\{F(y) : y \in V\}$; F is $(\mu_i\mu_j, \eta_k)$ continuous if F is so at each $x \in X$. $i, j, k = 1, 2$ $(i \neq j)$.

THEOREM 3.15. *If a bi-GTS (Y, η_1, η_2) is pairwise regular. Then for each $x \in X$ and each $(\mu_i\mu_j, \eta_i)$ continuous multifunction F from any bi-GTS (X, μ_1, μ_2) to (Y, η_1, η_2), $\mathcal{G}^{ji}_{ij}(F, x) = c_{\eta_j}F(x)$ for $i, j = 1, 2$ $(i \neq j)$, where $\mathcal{G}^{ji}_{ij}(F, x) = \cap\{\gamma_{\eta_j,\eta_i}F(c_{\mu_j}U) : U \in \mu_i(x)\}$.*

Proof. Let (Y, η_1, η_2) be pairwise regular. Now obviously $c_{\eta_j}F(x) \subseteq \mathcal{G}^{ji}_{ij}(F, x)$. On the other hand, if $x \in X$ and $W \in \eta_i$ such that $F(x) \subseteq W$, then by $(\mu_i\mu_j, \eta_i)$ continuity of F there exists $V \in \mu_i(x)$ such that $F(c_{\mu_j}V) \subseteq W$. So, $\mathcal{G}^{ji}_{ij}(F, x) = \cap\{\gamma_{\eta_j,\eta_i}F(c_{\mu_j}V) : V \in \mu_i(x)\} \subseteq$

$\cap\{\gamma_{\eta_j,\eta_i}W : F(x) \subseteq W \in \eta_i\} = \cap\{c_{\eta_j}W : F(x) \subseteq W \in \eta_i\}$. It sufficies to show that $\cap\{c_{\eta_j}W : F(x) \subseteq W \in \eta_i\} = c_{\eta_j}F(x)$. In fact $c_{\eta_j}F(x) \subseteq \cap\{c_{\eta_j}W : F(x) \subseteq W \in \eta_i\}$ is obvious. Now let $y \in \cap\{c_{\eta_j}W : F(x) \subseteq W \in \eta_i\}$ and $y \notin c_{\eta_j}F(x)$. Since Y is pairwise regular, there exist $U' \in \eta_j$ and $V' \in \eta_i$ with $y \in U'$, $c_{\eta_j}F(x) \subseteq V'$ and $U' \cap V' = \phi$. But since $F(x) \subseteq V' \in \eta_i$ we have $y \in c_{\eta_j}V'$, which contradicts $U' \cap V' = \phi$. Hence $\mathcal{G}^{ji}_{ij}(F,x) = c_{\eta_j}F(x)$. $\square$

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