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Continuous Time Approximations to GARCH(1, 1)-Family Models and Their Limiting Properties

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Abstract

Various modified GARCH(1, 1) models have been found adequate in many applications. We are interested in their continuous time versions and limiting properties. We first define a stochastic integral that includes useful continuous time versions of modified GARCH(1, 1) processes and give sufficient conditions under which the process is exponentially ergodic and β-mixing. The central limit theorem for the process is also obtained.

Keywords: Exponential ergodicity, diffusion limit, Lévy-driven volatility process, modified GARCH (1, 1) process, central limit theorem.

1. Introduction

After seminal work of Engle (1982) and Bollerslev (1986) ARCH/GARCH model and their extensions have been extensively applied in modelling financial time series. Most economic data are available only at discrete time periods. However, continuous time models may be a more realistic description of underlying phenomena. Continuous time models are particularly appropriate for irregularly spaced data or ultra high frequency data as they are more convenient for financial applications such as option pricing (see, *e.g.* Maller *et al.*, 2008; Lindner, 2009).

A common method to construct continuous time processes from discrete ones is to use diffusion limit. An extension to discrete time GARCH models by diffusion approximations had been studied by Nelson (1990) and Duan (1997). Klüppelberg *et al.* (2004) proposed a different approach to construct a continuous time analogue of GARCH(1, 1) model, where increments of any Lévy process replace the noise sequence in the discrete time GARCH model. Different from the diffusion limit of Nelson (1990) which is driven by two independent Brownian motions, the continuous time GARCH model proposed by Klüppelberg *et al.* (2004) is based on a single background driving Lévy process and many stylized facts are captured by a single random process as in discrete time GARCH models (see, *e.g.* Kallsen and Vesenmayer, 2009).

Statistical inference for time series models requires the study of the asymptotics of the process. The goal of this paper is to examine the limiting properties for continuous time versions of GARCH(1, 1)-family models. To obtain the desired results, we first consider the following process $(V_t)_{t\geq 0}$ driven by a Lévy process $(\xi_t)_{t\geq 0}$:

$$V_t = e^{-\xi_t}\left(\int_0^t e^{\xi_{s-}}ds + V_0\right), \quad t \geq 0, \tag{1.1}$$

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where V_0 is a finite random variable, independent of $(\xi_t)_{t\geq 0}$. Stochastic integral in (1.1) is well defined and we can easily show that $(V_t)_{t\geq 0}$ is a time homogeneous Markov process. $(V_t)_{t\geq 0}$ of (1,1) includes the (AP)COGARCH(1, 1) process and diffusion limit of GARCH(1, 1) model studied in Nelson (1990) as well as a version of continuous time modified GARCH(1, 1) process by selecting suitable Lévy processes ξ_t (see Donati-Martin *et al.*, 2001; Fasen, 2010; Lee, 2012a, 2012b *etc.*). We provide sufficient conditions under which the process $(V_t)_{t\geq 0}$ is exponentially ergodic and then examine the exponential ergodicity and mixing properties for diffusion limits of Nelson (1990) and modified COGARCH(1, 1) process. $(V_t)_{t\geq 0}$ in (1.1) is a special case of the generalized Ornstein-Uhlenbeck process. The exponential ergodicity and mixing properties for generalized Ornstein-Uhlenbeck process are studied in Lee (2012b), but the obtained results are new and easy to apply to the continuous time version of GARCH(1, 1)-family process considered in this paper.

A continuous time Markov process $(X_t)_{t\geq 0}$ will be called exponentially ergodic if for some probability measure π and a constant $\alpha > 0$,

$$\left\|P^{(t)}(x,\cdot) - \pi(\cdot)\right\| = O\left(e^{-\alpha t}\right) \quad (t \to \infty),$$

for π-a.a x where $\|\cdot\|$ denotes a total variation norm. It is known that the exponential ergodic process has the exponential β-mixing property.

For more information on Markov chain theory, we refer to Meyn and Tweedie (1993). We refer to Sato (1999) and Protter (2005) for basic results concerning Lévy processes and stochastic integration.

2. Stationarity and Exponential Ergodicity of $(V_t)_{t\geq 0}$

A Lévy process $(\xi_t)_{t\geq 0}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ is a stochastic process in the real numbers R with càdlàg paths, $\xi_0 = 0$, and stationary independent increments. Consider the volatility process $(V_t)_{t\geq 0}$ given by

$$V_t = e^{-\xi_t}\left(\beta \int_0^t e^{\xi_{s-}} ds + V_0\right), \quad t \geq 0. \tag{2.1}$$

Assume that V_0 is independent of $(\xi_t)_{t\geq 0}$. Let

$$A_t^s = e^{-(\xi_t - \xi_s)}, \qquad B_t^s = e^{-\xi_t}\beta \int_s^t e^{\xi_{u-}} du. \tag{2.2}$$

Then

$$V_{(n+1)h} = A_{(n+1)h}^{nh} V_{nh} + B_{(n+1)h}^{nh}, \quad h > 0,\ n \geq 0, \tag{2.3}$$

and $(A_{(n+1)h}^{nh}, B_{(n+1)h}^{nh})_{n\geq 0}$ is a sequence of independent and identically distributed random vectors because of stationary and independent increments of the Lévy process $(\xi_t)_{t\geq 0}$. Throughout this paper, let n stand for an integer and t a real number.

We make the following assumptions:

(A1) $0 < E(\xi_1) \leq E|\xi_1| < \infty$.

(A2) $E(e^{-r\xi_1}) < \infty$ for some $r \geq 1$.

Let $r > 0$. Recall that if $E(e^{-r\xi_h}) < \infty$ for some $h > 0$, then $E(e^{-r\xi_t}) < \infty$ for all $t \geq 0$. If $E(e^{-r\xi_1}) < \infty$, we define $\psi(r) = \log E(e^{-r\xi_1})$. Stationary and independent increments of ξ_t yield that

$E(e^{-r\xi_t}) = e^{t\psi(r)}$ and $E(\xi_h) = hE(\xi_1)$ (see Sato, 1999). Therefore, if (A1) and (A2) are satisfied, then for any $h > 0$,

$$0 < E(\xi_h) \leq E|\xi_h| < \infty, \quad E(e^{-r\xi_h}) < \infty \text{ for some } r \geq 1. \tag{2.4}$$

We temporally assume that $h > 0$ is fixed. Then $(V_{nh})_{n\geq 0}$ in (2.3) can be considered as a discrete time Markov process with n-step transition probability function $P^{(nh)}(x, B) = P(V_{nh} \in B | V_0 = x), x \in R, B \in \mathcal{B}(R)$, where $\mathcal{B}(R)$ is the Borel σ-algebra on R. $(V_{nh})_{n\geq 0}$ is called the h-skeleton chain of $(V_t)_{t\geq 0}$. $(V_t)_{t\geq 0}$ is said to be simultaneously ϕ-irreducible if any h-skeleton chain is ϕ-irreducible. It is known that if $(V_t)_{t\geq 0}$ is simultaneously ϕ-irreducible, then any h-skeleton chain is aperiodic.

Theorem 1. *(1) Under the assumption (A1), V_t converges in distribution to V_∞ as $t \to \infty$ for a finite random variable V_∞ satisfying*

$$V_\infty = \beta \int_0^\infty e^{-\xi_{s-}} ds. \tag{2.5}$$

(2) If V_∞ in (2.5) exists and $V_0 \stackrel{D}{=} V_\infty$, independent of $(\xi_t)_{t\geq 0}$, then $(V_t)_{t\geq 0}$ is strictly stationary. Here $\stackrel{D}{=}$ means 'has the same distribution'. (3) Under the assumption (A1), $(V_{nh})_{n\geq 0}$ defined by the Equation (2.3) converges in distribution to a probability measure π which does not depend on V_0. Further, π is the unique invariant initial distribution for $(V_{nh})_{n\geq 0}$ and the distribution of V_∞ in (2.5) is π.

Proof: For the proofs, we may consult Theorem 1 of Lee (2012b) and Theorem 3.1 of Klüppelberg *et al.* (2004). □

The next theorem is one of our main results.

Theorem 2. *Suppose that (A1)and (A2) hold and the limiting distribution π in Theorem 2.1 has a probability density function. Then (1) For any $h > 0$, $(V_{nh})_{n\geq 0}$ given in (2.3) satisfies the drift condition. (2) The process $(V_t)_{t\geq 0}$ in (2.1) is simultaneously π-irreducible. (3) $(V_t)_{t\geq 0}$ is exponentially ergodic and holds the exponentially β-mixing property.*

Proof:

(1) The proof of (1) has the same lines as those of Theorem 2.2 of Lee (2012b) and thereby, we omit it.

(2) By Theorem 1(3), $(V_{nh})_{n\geq 0}$ converges in distribution to π, which implies that for any $x \in R$, $P^{(nh)}(x, \cdot)$ converges weakly to $\pi(\cdot)$ as $n \to \infty$. From the assumption that π has a probability density function, we have that $P^{(nh)}(x, I) \to \pi(I)$ for every x and interval I in R, since $\pi(\partial I) = 0$. Take $C = \{A \in \mathcal{B}(R) \mid P^{(nh)}(x, A) \to \pi(A)\ as\ n \to \infty, \forall x \in R\}$. Then obviously any interval I is in C. If $A_n \in C$ and $A_n \downarrow A$ or $A_n \uparrow A$, then $A \in C$ by the continuity property of a probability measure. So it follows from the monotone class theorem that $C = \mathcal{B}(R)$ and then $(V_{nh})_{n\geq 0}$ is π-irreducible. Therefore, for any $h > 0$, h-skeleton chain $(V_{nh})_{n\geq 0}$ is π-irreducible and hence $(V_t)_{t\geq 0}$ is simultaneously π-irreducible and $(V_{nh})_{n\geq 0}$ is aperiodic.

(3) Note that $(V_{nh})_{n\geq 0}$ is a Feller chain, *i.e.*, $E(f(V_{(n+1)h})|V_{nh} = x)$ is a continuous function of x whenever f is continuous and bounded. Therefore, any nontrivial compact set is a small set. Moreover, Theorem 2(1) ensures that $(V_{nh})_{n\geq 0}$ holds the drift condition. Hence Theorem 2.2(2)

and Theorem 2.4.3 in Doukhan (1994) (see also Meyn and Tweedie, 1993) imply that $(V_{nh})_{n\geq 0}$ is geometrically ergodic, *i.e.*, there exists a constant $\rho \in (0,1)$ such that

$$\left\| P^{(nh)}(x,\cdot) - \pi(\cdot) \right\| = O\left(\rho^n\right), \tag{2.6}$$

π-a.a. x as $n \to \infty$. Under simultaneous π-irreducibility condition of $(V_t)_{t\geq 0}$, the Equation (2.9) and Theorem 5 in Tuominen and Tweedie (1979) guarantee the exponential ergodicity of $(V_t)_{t\geq 0}$ in the following sense:

$$\left\| P^{(t)}(x,\cdot) - \pi(\cdot) \right\| = O\left(e^{-\alpha t}\right), \tag{2.7}$$

as $t \to \infty$, for some $\alpha > 0$ and π-a.a. x. From exponential ergodicity, exponentially β-mixing property for the continuous time process $(V_t)_{t\geq 0}$ is also obtained.

□

3. Continuous Time Approximations to GARCH(1, 1)-Family Models

In this section, we study continuous time versions of GARCH(1, 1)-family processes as special cases of the process $(V_t)_{t\geq 0}$ given by the Equation (2.1) and obtain the exponential ergodicity and mixing conditions

Consider the following discrete time GARCH(1, 1) type models:

$$X_n = \sigma_n e_n, \tag{3.1}$$

$$\Lambda(\sigma_n) = \omega + \beta\Lambda(\sigma_{n-1}) + \alpha z(e_{n-1})\Lambda(\sigma_{n-1}), \tag{3.2}$$

where $\omega > 0, \beta \geq 0, \alpha \geq 0$ and the noise sequence $(e_n)_{n\geq 0}$ is a sequence of independent and identically distributed random variables. Assume that $\Lambda(\cdot)$ and $z(\cdot)$ are nonnegative measurable functions from R to R. Modified GARCH(1, 1) models such as GJR GARCH, TS-GARCH, APARCH, TGARCH, NGARCH, BL-GARCH as well as the classical GARCH model can be verified as special cases of the model of (3.1) and (3.2). For example, when $\Lambda(x) = z(x) = x^2$, the process given by (3.1) and (3.2) becomes the classical GARCH model. If $\Lambda(x) = x^{2\delta}, z(x) = (|x| - \gamma x)^{2\delta}$ $(\delta > 0,\ 0 \leq \gamma \leq 1)$, then (3.1) and (3.2) generate the APARCH(1, 1) model.

3.1. Diffusion approximations

Nelson (1990) showed that the classical GARCH(1, 1) model converges weakly to the following bivariate diffusion models :

$$dG_t = \sigma_t dW_t^{(1)},$$
$$d\sigma_t^2 = \lambda\left(a - \sigma_t^2\right)dt + \rho\sigma_t^2 dW_t^{(2)}, \quad t \geq 0,$$

where $\lambda, a, \rho > 0$, $(W^{(1)})_{t\geq 0}$ and $(W^{(2)})_{t\geq 0}$ are independent Brownian motions, independent of (G_0, σ_0^2). In Nelson's diffusion model, the volatility process σ_t^2 is given by

$$\sigma_t^2 = e^{-\xi_t}\left(\lambda a \int_0^t e^{\xi_s} ds + \sigma_0^2\right), \quad t \geq 0, \tag{3.3}$$

where $\xi_t = -\rho W_t^{(2)} + (\rho^2/2 + \lambda)t$ (see Theorem 52 in Protter, 2005).

Theorem 3. *$(\sigma_t^2)_{t\geq 0}$ in (3.3) is exponentially ergodic and β-mixing.*

Proof: Since $E(W_1^{(2)}) = 0, E|W_1^{(2)}| = \sqrt{2/\pi}$ and $E(e^{\rho r W_1^{(2)}}) = e^{\rho^2 r^2/2}$, the assumptions (A1) and (A2) hold. Moreover, $\xi_t \to \infty$ as $t \to \infty$ and ξ_t is clearly continuous. Now for any $h > 0$, let $T_h = \inf\{t : \xi_t = h\}$. Then T_h is finite a.s. and

$$\begin{aligned}\sigma_\infty^2 &= \lambda a \int_0^\infty e^{-\xi_u} du \\ &= \lambda a \int_0^{T_h} e^{-\xi_u} du + \lambda a \int_{T_h}^\infty e^{-\xi_u} du := A_h + B_h.\end{aligned}$$

A_h and B_h are independent and

$$\begin{aligned}B_h &= \lambda a \int_{T_h}^\infty e^{-\xi_u} e^{-h} e^{\xi_{T_h}} du \\ &= \lambda a e^{-h} \int_{T_h}^\infty e^{-(\xi_u - \xi_{T_h})} du \stackrel{D}{=} e^{-h}\sigma_\infty^2.\end{aligned}$$

Therefore, $\sigma_\infty^2 \stackrel{D}{=} A_h + e^{-h}\sigma_\infty^2$, which shows that σ_∞^2 is self-decomposable and σ_∞^2 has a density (see Theorem 5 in Klüppelberg *et al.*, 2006). The exponential ergodicity and β-mixing property of the volatility process $(\sigma_t^2)_{t\geq 0}$ of (3.3) are obtained by Theorem 2. □

The ergodicity and mixing properties for diffusion approximations to the remaining processes can be derived by the same manner as above (see Duan, 1997).

3.2. Ergodicity and central limit theorem for Lévy-driven continuous time approximations to modified GARCH(1, 1) models

Suppose that $(L_t)_{t\geq 0}$ is a Lévy process on R with the characteristic triple $(\gamma_L, \tau_L^2, \nu_L)$, where $\gamma_L \in R, \tau_L^2 \geq 0$, and the Lévy measure ν_L satisfies $\int_R \min(1, x^2)\nu_L(dx) < \infty$ and $\nu_L(\{0\}) = 0$. If in addition $\int_R \min(1, |x|)\nu_L(dx) < \infty$, then $\gamma_{L,0} = \gamma_L - \int_{|x|<1} x\nu_L(dx)$ is called the drift of $(L_t)_{t\geq 0}$. A Lévy process is of finite variation if and only if $\int_R \min(1, |x|)\nu_L(dx) < \infty$ and $\tau_L^2 = 0$. We assume that ν_L is nonzero.

Now, we present a continuous time version of modified GARCH(1, 1) model of (3.2) driven by a Lévy process $(L_t)_{t\geq 0}$, following Klüppelberg *et al.* (2004).

Iterate the Equation (3.2) to get the explicit expression for the volatility of the process of (3.1) and (3.2):

$$\Lambda(\sigma_n) = \omega \sum_{i=0}^{n-1} \Pi_{j=i+1}^{n-1}(\beta + \alpha z(e_j)) + \Pi_{i=0}^{n-1}(\beta + \alpha z(e_i))\,\Lambda(\sigma_0), \quad n \geq 0. \tag{3.4}$$

Define a càdlàg process $(\xi_t)_{t\geq 0}$ by

$$\xi_t = -t\log\beta - \sum_{0<s\leq t} \log\left(1 + \frac{\alpha}{\beta} z(\triangle L_s)\right), \tag{3.5}$$

where $0 < \beta < 1$, $\alpha \geq 0$ and $\triangle L_t = L_t - L_{t-}$, $t \geq 0$. Then with σ_0, a finite random variable, independent of $(L_t)_{t\geq 0}$, define the right continuous volatility process analogously to (3.4) by

$$\Lambda(\sigma_t) = \left(\omega \int_0^t e^{\xi_{s-}} ds + \Lambda(\sigma_0)\right) e^{-\xi_t}, \quad t \geq 0. \tag{3.6}$$

Then $(\Lambda(\sigma_t))_{t\geq 0}$ of (3.6) with the Lévy process ξ_t in (3.5) is a continuous time approximation to the volatility process for the modified GARCH(1, 1) process of (3.2). The corresponding integrated continuous time modified-GARCH(1, 1) process $(G_t)_{t\geq 0}$ is given by the càdlàg process satisfying

$$G_t = \int_0^t \sigma_{s-} dL_s, \quad t \geq 0,\ G_0 = 0. \tag{3.7}$$

$(\xi_t)_{t\geq 0}$ of (3.5) is a spectrally negative Lévy process of bounded variation with drift $\gamma_{\xi,0} = -\log\beta$ and Gaussian component $\tau_\xi^2 = 0$. The Lévy measure ν_ξ is the image measure of ν_L under the transformation $T : R \to (-\infty, 0]$ by $T(x) = -\log(1 + (\alpha/\beta)z(x))$.

If $E(e^{-r\xi_1}) < \infty$, then $\psi(r)$ is given as follows:

$$\psi(r) = r\log\beta + \int_R \left(\left(1 + \frac{\alpha}{\beta} z(y)\right)^r - 1\right) \nu_L(dy).$$

Theorem 4. *Assume that*

$$E(z(L_1))^r < \infty\ (r \geq 1), \quad \psi(1) < 0. \tag{3.8}$$

Then the process $(\Lambda(\sigma_t))_{t\geq 0}$ of (3.6) is exponentially ergodic and β-mixing with exponentially decaying rate.

Proof: Note that $E(z(L_1))^r < \infty$ if and only if $E(e^{-r\xi_t}) < \infty$ for some $t > 0$ or, equivalently, for all $t > 0$. From the second inequality in (3.8), we have that $\int_R (\alpha/\beta)z(y)\nu_L(dy) < -\log\beta$ which together with $\log(1 + (\alpha/\beta)z(y)) < (\alpha/\beta)z(y)$ implies that $E(\xi_1) = -\log\beta - \int_R \log(1 + (\alpha/\beta)z(y))\nu_L(dy) > 0$ and $E(\xi_1) > 0$ implies that $\xi_t \to \infty$ almost surely as $t \to \infty$. Therefore, the assumptions (A1) and (A2) in Section 2 hold and hence (2.4) holds for any $h > 0$. Recall that if ξ_t is a spectrally negative Lévy process of bounded variation with positive drift and $\xi_t \to +\infty$ as $t \to \infty$, then the limiting distribution π of $\Lambda(\sigma_\infty) = \omega \int_0^\infty e^{-\xi_t} dt$ is self-decomposable and π has a density function (see Theorem 5 of Klüppelberg *et al.*, 2006; Theorem 28.3 of Sato, 1999). Therefore, Theorem 2 yields the exponential ergodicity and β-mixing property of the process $(\Lambda(\sigma_t))_{t\geq 0}$ of (3.6). □

Remark 1. One of the assumptions for irreducibility of the process required in Theorem 2.5 in Lee (2012b) is that

(C1) for any $h > 0$, the transition probability function $P^{(h)}(x, \cdot)$ has a probability density function $p_h(x, y)$.

To prove the exponential ergodicity by applying the Theorem 2.5 in Lee (2012b), we need to show that the processes satisfy the condition (C1) above. However, it is not easy to check if the irreducibility condition (C1) is satisfied by the processes considered in this paper, even though they are special cases of the generalized OU process. On the contrary, we can show that the limiting distribution of those processes has a probability density function and then the desired result follows directly from Theorem 2.

Theorem 5. *Assume that $E(z(L_1))^2 < \infty$ and $\psi(2) < 0$. Then the process $\Lambda(\sigma_t)$ of (3.6) with $\Lambda(\sigma_0) \stackrel{D}{=} \Lambda(\sigma_\infty)$, independent of $(\xi_t)_{t\geq 0}$ obeys the following central limit theorem:*

$$n^{-\frac{1}{2}} \int_0^{nt} \left(\Lambda(\sigma_s) + \frac{\beta}{\psi(1)}\right) ds \tag{3.9}$$

converges weakly to a Wiener measure with zero drift and variance parameter

$$\sigma^2 = \frac{-2\beta^2}{(\psi(1))^2}\left(\frac{2}{\psi(2)} - \frac{1}{\psi(1)}\right).$$

Proof: From assumption, we have $E(z(L_1)) < \infty$ and $\psi(1) < 0$. Theorem 1 and Theorem 4 ensure that $\{\Lambda(\sigma_t) : t \geq 0\}$ is strictly stationary and ergodic Markov process having transition probability $P^{(t)}(x, dy)$ and invariant initial distribution π. Note that $\int x\pi(dx) = E(\Lambda(\sigma_\infty)) = -\beta/\psi(1)$, $\int x^2\pi(dx) = 2\beta^2/\{\psi(1)\psi(2)\}$ (see Proposition 4.1 of Klüppelberg *et al.*, 2004).

Let $g(x) = ax$ for a constant a. Then by simple calculation, we have

$$\begin{aligned}\hat{A}g(x) &= \lim_{t\to 0}\frac{(T_t g)(x) - g(x)}{t}\\ &= \lim_{t\to 0}\frac{E(\Lambda(\sigma_t)|\Lambda(\sigma_0) = x) - g(x)}{t}\\ &= \lim_{t\to 0}\frac{1}{t}\left[\frac{-a\beta}{\psi(1)} + a\left(\frac{\beta}{\psi(1)} + x\right)e^{t\psi(1)} - ax\right]\\ &= a\psi(1)x + a\beta,\end{aligned}$$

where the transition operators $T_t : t > 0$ is defined by $(T_t g)(x) = \int g(y)P^{(t)}(x, dy)$ and $\hat{A}$ is the infinitesimal generator of $\{T_t : t > 0\}$.

If we take $a = 1/\psi(1)$, then $\hat{A}g(x) = x + \beta/\psi(1)$. That is, $f(x) = x + \beta/\psi(1)$ is in the range of $\hat{A}$. Hence the weak convergence of (3.9) is obtained from Theorem 2.1 of Bhattacharya (1982).

Moreover,

$$\begin{aligned}\sigma^2 &= -2\int f(x)g(x)\pi(dx)\\ &= -2\int\left(\frac{x^2}{\psi(1)} + \frac{\beta x}{(\psi(1))^2}\right)\pi(dx)\\ &= \frac{-2\beta^2}{(\psi(1))^2}\left(\frac{2}{\psi(2)} - \frac{1}{\psi(1)}\right).\end{aligned}$$

The proof is completed. □

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