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THE PRICING OF QUANTO OPTIONS IN THE DOUBLE SQUARE ROOT STOCHASTIC VOLATILITY MODEL

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Abstract. We drive a closed-form expression for the price of a European quanto call option in the double square root stochastic volatility model.

1. Introduction

A quanto is a type of financial derivative whose pay-out currency differs from the natural denomination of its underlying financial variable, which allows that investors are to obtain exposure to foreign assets without the corresponding foreign exchange risk. A quanto option has both the strike price and the underlying asset price denominated in foreign currency. At exercise, the value of the option is calculated as the option's intrinsic value in the foreign currency, which is then converted to the domestic currency at the fixed exchange rate.

Pricing options based on the classical Black-Scholes model, on which most of the research on quanto options has focused, has a problem of assuming a constant volatility which leads to smiles and skews in the implied volatility of the underlying asset. For that reason, in valuing quanto option, it is natural to consider a stochastic volatility model. Stochastic volatility models, such as Hull-White model [4], Stein-Stein model [8] and Heston model [3], are frequently used in pricing various kinds of European options. Despite its importance, very few researches have been done on pricing quanto option using a stochastic volatility model primarily due to the sophisticated stochastic process for underlying assets and volatilities as well as the difficulty of finding analytic form of the option price.

To mention some of the work on pricing quanto options with stochastic volatilities, F. Antonelli et al. [1] used a method of expanding and approximating with respect to correlation parameters to find analytic formula of exchange options with stochastic volatilities. Using the technique developed in [1], J. Park et al. [6] got an analytic approximation value for a quanto option price in the Hull-White stochastic volatility model. A. Giese [2] provided a closed-form

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expression for the price of quanto option in the Stein-Stein stochastic volatility model, which was influenced by the previous work of R. Schöbel and J. Zhu [7].

On the other hand, extending to a general equilibrium framework of the Cox-Ingersoll-Ross process, F. A. Longstaff [5] introduced a double square root model of stochastic interest rates to describe nonlinear term structures observed in yield curves. Later, J. Zhu [9], [10] presented an innovative modification of stochastic volatility models specified as a double square root stochastic process with the restriction of parameters.

In this paper, motivated by A. Giese [2] and J. Zhu [10], we drive a closed-form expression for the price of a quanto call option in the double square root stochastic volatility model. Here, we use the double square root stochastic process with the restriction of parameters introduced in [10] to get the main theorem.

We introduce preliminary materials on a quanto and the double square root stochastic volatility model, and specify dynamics of the processes of underlying asset and its volatility under the quanto measure in Section 2. Then, in Section 3, we drive a closed-form expression of a quanto option price under the model specified in Section 2. Theorem 3.3 is the main result of the paper.

2. Model specification

For a non-dividend paying asset, the process of the asset price S_t may be assumed to be denominated in foreign currency X and to have the following dynamics:

$$dS_t = r^X S_t dt + \sqrt{v_t} S_t dB_t^{\mathbf{Q}^X}, \tag{1}$$

$$dv_t = \kappa\,(\theta - \sqrt{v_t})\,dt + \xi\sqrt{v_t} dW_t^{\mathbf{Q}^X} \tag{2}$$

under the risk-neutral probability measure $\mathbf{Q}^X$, where $B_t^{\mathbf{Q}^X}$ and $W_t^{\mathbf{Q}^X}$ are two standard Brownian motions, r^X is the foreign riskless rate and v_t follows the stochastic volatility process of S_t with constant parameters κ, θ and ξ. Since there are two square root terms in (2), it is referred to as the double square root process, whose basic features are described in Chapter 3.4 of [10]. To give a closed-form expression, we should add a restriction of parameters so that $4\kappa\theta = \xi^2$, which is the strong condition to be able to analytically calculate some special conditional expectation included v_t. This particular meaning is explained in [5] minutely. Furthermore, we assume an investor whose domestic currency is Y and who wishes to obtain exposure to the asset price S_t without carrying the corresponding foreign exchange risk. For the rest of Section 2, we mostly follow the notations of [2].

Let $Z_t^{Y/X}$ denote the price of one unit of currency Y in units of currency X and we assume that $Z_t^{Y/X}$ follows the standard Black-Scholes type dynamics under $\mathbf{Q}^X$ such as

$$dZ_t^{Y/X} = \left(r^X - r^Y\right) Z_t{}^{Y/X} dt + \sigma Z_t^{Y/X} d\hat{B}_t^{\mathbf{Q}^X},$$

where $\hat{B}_t^{\mathbf{Q}^X}$ is a standard Brownian motion under $\mathbf{Q}^X$, r^Y is the domestic riskless rate and σ is the constant volatility of the foreign exchange rate Z_t. This model allows three constant correlations ρ, ν and β satisfying

$$dB_t^{\mathbf{Q}^X} dW_t^{\mathbf{Q}^X} = \rho dt, \quad dB_t^{\mathbf{Q}^X} d\hat{B}_t^{\mathbf{Q}^X} = \nu dt, \quad dW_t^{\mathbf{Q}^X} d\hat{B}_t^{\mathbf{Q}^X} = \beta dt.$$

Using the change of measure from $\mathbf{Q}^X$ to the domestic risk-neutral probability measure $\mathbf{Q}^Y$ with the Radon-Nikodým derivative

$$\left.\frac{d\mathbf{Q}^Y}{d\mathbf{Q}^X}\right|_{\mathcal{F}_t} = \frac{Z_t^{Y/X}}{Z_0^{Y/X}} e^{\left(r^Y - r^X\right)t} = e^{-\frac{1}{2}\sigma^2 t + \sigma \hat{B}_t^{\mathbf{Q}^X}},$$

the Girsanov's Theorem implies that the processes $B_t^{\mathbf{Q}^Y}$, $W_t^{\mathbf{Q}^Y}$ and $\hat{B}_t^{\mathbf{Q}^Y}$ defined by

$$\begin{aligned} dB_t^{\mathbf{Q}^Y} &= dB_t^{\mathbf{Q}^X} - \nu\sigma dt, \\ dW_t^{\mathbf{Q}^Y} &= dW_t^{\mathbf{Q}^X} - \beta\sigma dt, \\ d\hat{B}_t^{\mathbf{Q}^Y} &= d\hat{B}_t^{\mathbf{Q}^X} + \sigma dt \end{aligned}$$

are again standard Brownian motions under the domestic risk-neutral probability measure $\mathbf{Q}^Y$, so called the quanto measure. Thus, the foreign exchange rate $Z_t^{X/Y}$ denoting the price in currency X per unit of the domestic currency Y follows

$$dZ_t^{X/Y} = \left(r^Y - r^X\right) Z_t{}^{X/Y} dt + \sigma Z_t^{X/Y} d\hat{B}_t^{\mathbf{Q}^Y}.$$

Also, we obtain the following dynamics of S_t and v_t under $\mathbf{Q}^Y$:

$$dS_t = \left(r^X + \nu\sigma\sqrt{v_t}\right) S_t dt + \sqrt{v_t} S_t dB_t^{\mathbf{Q}^Y}, \tag{3}$$

$$dv_t = \hat{\kappa}\left(\hat{\theta} - \sqrt{v_t}\right) dt + \xi\sqrt{v_t} dW_t^{\mathbf{Q}^Y} \tag{4}$$

with $\hat{\kappa} = \kappa - \beta\sigma\xi$ and $\hat{\theta} = \frac{\kappa\theta}{\kappa - \beta\sigma\xi}$. We notice that (4) maintains the same form as (2) so that $4\hat{\kappa}\hat{\theta} = \xi^2$ also has to be satisfied.

3. A closed-form quanto option price

Here, using the model specified in previous section, we drive a closed-form expression of a quanto option price. The following two lemmas are about some special conditional expectations under the measure $\mathbf{Q}^Y$, both of which are crucial ingredients to the main result of the paper.

Lemma 3.1. *Under the assumption of* (4), *together with constants* m_1, m_2 *and* m_3, *we get the following equality*:

$$\mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (m_1 v_s + m_2\sqrt{v_s})ds + m_3 v_T}\middle|\mathcal{F}_t\right] = A(t)\, e^{B(t)v_t + C(t)\sqrt{v_t}},$$

where

$$A(t) = \frac{1}{\sqrt{\gamma_4}}\exp\left[\frac{\left(\gamma_3^2 - \hat{\kappa}^2\gamma_1^2\right)(T-t)}{2\xi^2\gamma_1^2} + \frac{\left(\gamma_2\gamma_3 - 2\hat{\kappa}\gamma_1^2\right)\gamma_3}{2\xi^2\gamma_1^4}\left(\frac{1}{\gamma_4} - 1\right)\right.$$
$$\left. + \frac{\sinh\{\gamma_1(T-t)\}\left\{\hat{\kappa}^2\gamma_1^2 - \hat{\kappa}\gamma_2\gamma_3 - \gamma_3^2 + \frac{1}{2}\left(\frac{\gamma_2\gamma_3}{\gamma_1}\right)^2\right\}}{2\xi^2\gamma_1^3\gamma_4}\right],$$

$$B(t) = -\frac{2\gamma_1}{\xi^2}\cdot\frac{2\gamma_1\sinh\{\gamma_1(T-t)\} + \gamma_2\cosh\{\gamma_1(T-t)\}}{2\gamma_1\cosh\{\gamma_1(T-t)\} + \gamma_2\sinh\{\gamma_1(T-t)\}}$$

and

$$C(t) = \frac{2\sinh\left\{\frac{\gamma_1(T-t)}{2}\right\}}{\xi^2\gamma_1\gamma_4}$$
$$\times\left[(\hat{\kappa}\gamma_2 - 2\gamma_3)\cosh\left\{\frac{\gamma_1(T-t)}{2}\right\} + \left(2\hat{\kappa}\gamma_1 - \frac{\gamma_2\gamma_3}{\gamma_1}\right)\sinh\left\{\frac{\gamma_1(T-t)}{2}\right\}\right]$$

with

$$\gamma_1 = \frac{\sqrt{2m_1\xi^2}}{2},\quad \gamma_2 = -m_3\xi^2,\quad \gamma_3 = \frac{m_2\xi^2}{2},$$
$$\gamma_4 = \cosh\{\gamma_1(T-t)\} + \frac{\gamma_2}{2\gamma_1}\sinh\{\gamma_1(T-t)\}.$$

Proof. Let us define

$$y(t, v_t) = \mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (m_1 v_s + m_2\sqrt{v_s})ds + m_3 v_T}\middle|\mathcal{F}_t\right].$$

Then according to the Feynman-Kač formula, y is the solution of the following PDE:

$$\frac{\xi^2}{2}v\frac{\partial^2 y}{\partial v^2} + \hat{\kappa}\left(\hat{\theta} - \sqrt{v}\right)\frac{\partial y}{\partial v} - \left(m_1 v + m_2\sqrt{v}\right)y + \frac{\partial y}{\partial t} = 0$$

with the terminal condition

$$y(T, v_T) = e^{m_3 v_T}.$$

Now, putting our solution as the following functional form:

$$y(t, v_t) = A(t)\, e^{B(t)v_t + C(t)\sqrt{v_t}},$$

we have the following ODEs[1]:

$$\begin{aligned} A'(t) &= -\frac{\xi^2}{8}A(t)C(t)^2 - \frac{\xi^2}{4}A(t)B(t) + \frac{\hat{\kappa}}{2}A(t)C(t), \\ B'(t) &= -\frac{\xi^2}{2}B(t)^2 + m_1, \\ C'(t) &= -\frac{\xi^2}{2}B(t)C(t) + \hat{\kappa}B(t) + m_2 \end{aligned}$$

with terminal conditions

$$A(T) = 1, \quad B(T) = m_3, \quad C(T) = 0.$$

By solving these ODEs, we complete the proof of the lemma. □

Using the result obtained in Lemma 3.1, we can compute $\mathbf{E}_{\mathbf{Q}^Y}[S_T|\mathcal{F}_t]$ which represents, from the risk-neutral valuation, the value of a quanto forward contract.

Lemma 3.2. *Under the assumptions of* (3) *and* (4), *we get the following equality*:

$$\mathbf{E}_{\mathbf{Q}^Y}[S_T|\mathcal{F}_t] = S_t e^{r^X(T-t) - \frac{\rho}{\xi}\left\{v_t + \hat{\kappa}\hat{\theta}(T-t)\right\}} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (c_1 v_s + c_2\sqrt{v_s})ds + c_3 v_T} \middle| \mathcal{F}_t \right],$$

where c_1, c_2 *and* c_3 *are constants with*

$$c_1 = \frac{\rho^2}{2}, \quad c_2 = -\nu\sigma - \frac{\rho\hat{\kappa}}{\xi}, \quad c_3 = \frac{\rho}{\xi}.$$

Proof. Applying the Itô formula to (3) together with the tower property, we get

(5)
$$\mathbf{E}_{\mathbf{Q}^Y}[S_T|\mathcal{F}_t] = S_t e^{r^X(T-t)} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{\nu\sigma\int_t^T \sqrt{v_s}ds - \frac{\rho^2}{2}\int_t^T v_s ds + \rho\int_t^T \sqrt{v_s}dW_s^{\mathbf{Q}^Y}} \middle| \mathcal{F}_t \right],$$

where we may write $B_t^{\mathbf{Q}^Y}$ as $B_t^{\mathbf{Q}^Y} = \rho W_t^{\mathbf{Q}^Y} + \sqrt{1-\rho^2}W_t$ with W_t being a $\mathbf{Q}^Y$-standard Brownian motion independent of $W_t^{\mathbf{Q}^Y}$. From (4), we have

$$\int_t^T \sqrt{v_s}dW_s^{\mathbf{Q}^Y} = \frac{1}{\xi}\left\{ v_T - v_t - \hat{\kappa}\hat{\theta}(T-t) + \hat{\kappa}\int_t^T \sqrt{v_s}ds \right\}. \tag{6}$$

Substituting (6) into (5), we obtain

$$\mathbf{E}_{\mathbf{Q}^Y}[S_T|\mathcal{F}_t] = S_t e^{r^X(T-t) - \frac{\rho}{\xi}\left\{v_t + \hat{\kappa}\hat{\theta}(T-t)\right\}} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (c_1 v_s + c_2\sqrt{v_s})ds + c_3 v_T} \middle| \mathcal{F}_t \right]$$

with

$$c_1 = \frac{\rho^2}{2}, \quad c_2 = -\nu\sigma - \frac{\rho\hat{\kappa}}{\xi}, \quad c_3 = \frac{\rho}{\xi}.$$ □

[1]Due to the restriction that $4\hat{\kappa}\hat{\theta} = \xi^2$, the coefficient of $\frac{1}{v_t}$ vanishes in the calculating course.

Using the results obtained in Lemma 3.1 and Lemma 3.2, we can obtain a closed-form expression of the quanto option price. Here is the main theorem.

Theorem 3.3. *Let us denote the log-asset price by* $x_t = \ln S_t$*. Under the assumptions of* (3) *and* (4)*, the price of a European quanto call option in currency* Y *with strike price* K *and maturity* T *is given by*

$$c_q(t, S_t) = \mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right] e^{-r^Y(T-t)} P_1 - K e^{-r^Y(T-t)} P_2,$$

where P_1*,* P_2 *are defined by*

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbf{Re}\left[\frac{e^{i\phi \ln K} f_j(\phi)}{i\phi}\right] d\phi$$

for $j = 1, 2$*, in which*

$$f_1(\phi) = \frac{e^{(1+i\phi)\left[r^X(T-t)+x_t-\frac{\rho}{\xi}\left\{v_t+\hat{\kappa}\hat{\theta}(T-t)\right\}\right]}}{\mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right]} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (m_1 v_s + m_2\sqrt{v_s})ds + m_3 v_T} \middle| \mathcal{F}_t \right]$$

with

$$m_1 = \frac{\rho^2}{2}(1+i\phi), \quad m_2 = -(1+i\phi)\left(\nu\sigma + \frac{\rho\hat{\kappa}}{\xi}\right), \quad m_3 = \frac{\rho}{\xi}(1+i\phi)$$

and

$$f_2(\phi) = \frac{e^{i\phi\left[r^X(T-t)+x_t-\frac{\rho}{\xi}\left\{v_t+\hat{\kappa}\hat{\theta}(T-t)\right\}\right]}}{\mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right]} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (n_1 v_s + n_2\sqrt{v_s})ds + n_3 v_T} \middle| \mathcal{F}_t \right]$$

with

$$n_1 = i\phi\frac{\rho^2}{2}, \quad n_2 = -i\phi\left(\nu\sigma + \frac{\rho\hat{\kappa}}{\xi}\right), \quad n_3 = i\phi\frac{\rho}{\xi}.$$

Proof. From the risk-neutral valuation, the price $c_q(t, S_t)$ of a European quanto call option in currency Y with strike price K and maturity T is given by

$$c_q(t, S_t) = e^{-r^Y(T-t)}\mathbf{E}_{\mathbf{Q}^Y}\left[\max(S_T - K, 0) \middle| \mathcal{F}_t \right].$$

For a new risk-neutral probability measure $\tilde{\mathbf{Q}}^Y$, the Radon-Nikodým derivative of $\tilde{\mathbf{Q}}^Y$ with respect to $\mathbf{Q}^Y$ is defined by

$$\frac{d\tilde{\mathbf{Q}}^Y}{d\mathbf{Q}^Y} = \frac{S_T}{\mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right]}.$$

Thus, the price of a quanto call option can be rewritten as

$$\begin{aligned} c_q(t, S_t) &= e^{-r^Y(T-t)}\mathbf{E}_{\mathbf{Q}^Y}\left[S_T \mathbf{1}_{\{S_T > K\}} - K\mathbf{1}_{\{S_T > K\}} \middle| \mathcal{F}_t \right] \\ &= \mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right] e^{-r^Y(T-t)}\tilde{\mathbf{Q}}^Y(S_T > K) - K e^{-r^Y(T-t)}\mathbf{Q}^Y(S_T > K) \\ &= \mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right] e^{-r^Y(T-t)} P_1 - K e^{-r^Y(T-t)} P_2 \end{aligned}$$

with the risk-neutralized probabilities P_1 and P_2. Now, putting $x_t = \ln S_t$, the corresponding characteristic functions f_1 and f_2 can be represented as

$$f_1(\phi) = \mathbf{E}_{\tilde{\mathbf{Q}}^Y}\left[e^{i\phi x_T} \middle| \mathcal{F}_t \right]$$

$$= \frac{1}{\mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right]} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{(1+i\phi)x_T} \middle| \mathcal{F}_t \right],$$

$$f_2(\phi) = \mathbf{E}_{\mathbf{Q}^Y}\left[e^{i\phi x_T} \middle| \mathcal{F}_t \right].$$

On the other hand, applying the Itô formula to (3), we have

$$dx_t = \left(r^X + \nu\sigma\sqrt{v_t} - \frac{1}{2}v_t \right) dt + \rho\sqrt{v_t}dW_t^{\mathbf{Q}^Y} + \sqrt{1-\rho^2}\sqrt{v_t}dW_t.$$

From (6), we obtain

$$f_1(\phi) = \frac{e^{(1+i\phi)\left[r^X(T-t)+x_t-\frac{\rho}{\xi}\left\{v_t+\hat{\kappa}\hat{\theta}(T-t)\right\}\right]}}{\mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right]} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (m_1 v_s + m_2\sqrt{v_s})ds + m_3 v_T} \middle| \mathcal{F}_t \right]$$

with

$$m_1 = \frac{\rho^2}{2}(1+i\phi), \quad m_2 = -(1+i\phi)\left(\nu\sigma + \frac{\rho\hat{\kappa}}{\xi}\right), \quad m_3 = \frac{\rho}{\xi}(1+i\phi).$$

Similarly, we also obtain

$$f_2(\phi) = \frac{e^{i\phi\left[r^X(T-t)+x_t-\frac{\rho}{\xi}\left\{v_t+\hat{\kappa}\hat{\theta}(T-t)\right\}\right]}}{\mathbf{E}_{\mathbf{Q}^Y}\left[S_T \middle| \mathcal{F}_t \right]} \mathbf{E}_{\mathbf{Q}^Y}\left[e^{-\int_t^T (n_1 v_s + n_2\sqrt{v_s})ds + n_3 v_T} \middle| \mathcal{F}_t \right]$$

with

$$n_1 = i\phi\frac{\rho^2}{2}, \quad n_2 = -i\phi\left(\nu\sigma + \frac{\rho\hat{\kappa}}{\xi}\right), \quad n_3 = i\phi\frac{\rho}{\xi}.$$

Here, each value of risk-neutral expectation above was obtained in previous Lemmas.

By having closed-form solutions for the characteristic functions f_1 and f_2, the Fourier inversion formula allows us to compute the probabilities P_1 and P_2 as follows:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathbf{Re}\left[\frac{e^{i\phi \ln K} f_j(\phi)}{i\phi} \right] d\phi$$

for $j = 1, 2$. □

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