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EULER TYPE INTEGRAL INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION

Shakeel Ahmed and Mumtaz Ahmad Khan

Abstract. In this paper, we obtain some theorems on certain Euler type integrals involving generalized Mittag-Leffler function. Further, we deduce some special cases involving Wiman function, Prabhakar function and exponential and binomial functions.

1. Introduction

In 1903, the Swedish mathematician Gosta Mittag-Leffler [6] introduced the function

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ \alpha\in\mathbb{C},\ \mathrm{Re}(\alpha)>0, \tag{1.1}$$

where z is a complex variable and $\alpha \geq 0$. The Mittag-Leffler function is a direct generalization of exponential function to which it reduces for $\alpha = 1$. For $0 < \alpha < 1$ it interpolates between the pure exponential and hypergeometric function $\frac{1}{1-z}$. Its importance is realized during the last two decades due to its involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential or fractional order integral equation. The generalization of $E_\alpha(z)$ was studied by Wiman [11] in 1905 and he defined the function as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \alpha,\beta\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0, \tag{1.2}$$

which is known as Wiman function.

In 1971, Prabhakar [7] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form of

(1.3)
$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!},\ \alpha,\beta,\gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,$$

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where $(\gamma)_n$ is the Pochhammer symbol [9].

$$(\gamma)_0 = 1,\ (\gamma)_n = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1).$$

Recently, Shukla [10] introduced the function $E^{\gamma,q}_{\alpha,\beta}(z)$ which is defined for $\alpha, \beta, \gamma \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$ and $q \in (0,1) \bigcup \mathbb{N}$, as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!}. \tag{1.4}$$

The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely $\forall z$ if $q < \mathrm{Re}(\alpha)+1$ and for $|z| < 1$ if $q = \mathrm{Re}(\alpha)+1$. It is an entire function of order $(\mathrm{Re}(\alpha))^{-1}$.

Now we recall the basic Euler integral which defines the beta function

$$B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta) > 0, \tag{1.5}$$

and Gauss's hypergeometric function

$$\begin{aligned} {}_2F_1[\alpha,\beta;\gamma;z] &= \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha} dt \\ &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.6}$$

where $z \notin [1,\infty)$ and $\mathrm{Re}(\beta)$, $\mathrm{Re}(\gamma-\beta)$ are positive for the integrand and $|z| < 1$ for the series. Euler generalized the factorial function from the domain of natural numbers to the gamma function

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} \exp(-t) dt,\ \mathrm{Re}(\alpha) > 0, \tag{1.7}$$

defined over the right half of the complex plane. This led Legendre to decompose the gamma function into the incomplete gamma functions, $\gamma(\alpha,x)$ and $\Gamma(\alpha,x)$ which are obtained from Eq.(1.7) by replacing upper and lower limits by x, respectively. These functions develop singularities at the negative integers. Choudhary and Zubair [4] extended the domain of these functions to the entire complex plane by inserting a regularization factor $\exp(-\frac{A}{t})$ in the integrand of Eq.(1.7). For $A > 0$ this factor clearly removes the singularity coming from the limit $t = 0$. For $A = 0$ this factor becomes unity and thus we get the original gamma function. We note the following relation [3, p. 20 (1.2)]

$$\Gamma_A(\alpha) = \int_0^{\infty} t^{\alpha-1} \exp\left(-t - \frac{A}{t}\right) dt = 2(A)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{A}),\ \mathrm{Re(A)} > 0, \tag{1.8}$$

where $K_n(x)$ is the modified Bessel function of the second kind of order n (or Macdonald's function [2]). The relationships between the generalized gamma and Macdonald functions could not have been apparent in the original gamma function.

We note that the Riemann's zeta function $\zeta(x)$ defined by the series [1, p. 102 (2.101)]

$$\zeta(x)=\sum_{n=1}^{\infty}\frac{1}{n^{x}},\ x>1, \tag{1.9}$$

is useful in providing convergence or divergence of other series by means of comparison test. Zeta function is closely related to the logarithm of gamma function and to the polygamma functions. The regularizer $\exp(-\frac{A}{t})$ also proved very useful in extending the domain of Riemann's zeta function,there by providing relationships that could not have been obtained with the original zeta function. In view of the effectiveness of the above regularizer for gamma and zeta functions, Choudhary et al. [3] obtained an extension of Euler's beta function $B(\alpha,\beta)$ in the form

$$B(\alpha,\beta;A)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\exp\left(\frac{-A}{u(1-u)}\right)du,\ \mathrm{Re(A)}>0. \tag{1.10}$$

Clearly, when $A=0$ Eq.(1.10) reduces to the original beta function. The extended beta function (EBF) $B(\alpha,\beta;A)$ is extremely useful in the sense that most of the properties of the beta function carry over naturally and simply for it.This function is also related to other special functions.

In this paper, we obtain some theorems on Euler type integral involving generalized Mittag-Leffler function and discusses their special cases.

2. Theorems

Theorem 1. *If* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(\rho)>0$, $\mathrm{Re(A)}>0$ *and* $q\in N$, *then*

$$\int_0^1 u^{\rho-1}(1-u)^{\delta-1}\exp\left(\frac{-A}{u(1-u)}\right)E_{\alpha,\beta}^{\gamma,q}(zu^{\alpha})du$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)n!}B(\alpha n+\rho,\delta;A).$$

Proof. Denote L.H.S. of Theorem 1 by I_1, then

$$I_1=\int_0^1 u^{\rho-1}(1-u)^{\delta-1}\exp\left(\frac{-A}{u(1-u)}\right)E_{\alpha,\beta}^{\gamma,q}(zu^{\alpha})du.$$

Using the definition of generalized Mittag-Leffler function (1.4), we get

$$I_1=\int_0^1 u^{\rho-1}(1-u)^{\delta-1}\exp\left(\frac{-A}{u(1-u)}\right)\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(zu^{\alpha})^n}{\Gamma(\alpha n+\beta)n!}du$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)n!}\int_0^1 u^{\alpha n+\rho-1}(1-u)^{\delta-1}\exp\left(\frac{-A}{u(1-u)}\right)du$$

which further on using the definition of the extended beta integral (1.10), we get

$$I_1 = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} B(\alpha n+\rho, \delta; A). \qquad \square$$

Corollary 2.1. *For $A = 0$, Theorem 1 reduces to the following result:*

$$\frac{1}{\Gamma(\delta)} \int_0^1 u^{\rho-1}(1-u)^{\delta-1} E_{\alpha,\beta}^{\gamma,q}(zu^{\alpha}) du = E_{\alpha,\beta+\delta}^{\gamma,q}(z).$$

Remark 2.1. If we set $\rho = \beta$, then the corollary 1.1 reduces to the known relation [10, Eq.(2.4.1)].

Theorem 2. *If $\alpha, \beta, \gamma, \delta, \rho, \lambda, A \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\delta) > 0$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re(A)} > 0$; $|\arg(\frac{bc+d}{ac+d})| < \pi$, and $q \in N$, then*

$$\int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda} \exp\left(\frac{-A}{(u-a)(b-u)}\right) E_{\alpha,\beta}^{\gamma,q}(z(b-u)^f) du$$

$$= (ac+d)^{\lambda} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-A)^m}{m!} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} B(\rho-m, \delta+fn-m)$$

$$\times (b-a)^{\rho+\delta+fn-1} {}_2F_1 \left[\begin{matrix} \rho-m, \ -\lambda \ ; \\ \rho+\delta+fn-2m \ ; \end{matrix} \ \frac{-(b-a)c}{ac+d}\right].$$

Proof. Denote L.H.S. of Theorem 2 by I_2, then

$$I_2 = \int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda} \exp\left(\frac{-A}{(u-a)(b-u)}\right) E_{\alpha,\beta}^{\gamma,q}[z(b-u)^f] du.$$

Using the definition of exponential and generalized Mittag-Leffler function, we get

$$I_3 = \int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda} \sum_{m=0}^{\infty} \frac{(-A)^m}{(u-a)^m(b-u)^m m!} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n (b-u)^{fn}}{\Gamma(\alpha n+\beta) n!} du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-A)^m}{m!} \frac{(\gamma)_{qn} z^n (b-u)^{fn}}{\Gamma(\alpha n+\beta) n!} \int_a^b (u-a)^{\rho-m-1}(b-u)^{\delta+fn-m-1}(cu+d)^{\lambda} du$$

which further on using the integral [8, p. 263]

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma} dt$$

$$= B(\alpha,\beta)(b-a)^{\alpha+\beta-1}(au+v)^{\gamma} {}_2F_1\left[\alpha, -\gamma; \alpha+\beta; \frac{-(b-a)u}{(au+v)}\right]$$

$$(\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0); \left|\arg\left(\frac{bu+v}{au+v}\right)\right| < \pi$$

yields the required result. $\square$

Corollary 2.2. *For $A = 0$, Theorem 2 reduces to the following result*:

$$\int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda} E_{\alpha,\beta}^{\gamma,q}(z(b-u)^f)du$$

$$= (ac+d)^{\lambda} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} B(\rho, \delta+fn)(b-a)^{\rho+\delta+fn-1}$$

$$\times {}_2F_1 \left[\begin{array}{lc} \rho, \ -\lambda \quad ; & \\ & \dfrac{-(b-a)c}{ac+d} \\ \rho+\delta+fn \quad ; & \end{array} \right].$$

Corollary 2.3. *For $a = 0$, $b = 1$, Theorem 2 reduces to the following result*:

$$\int_0^1 u^{\rho-1}(1-u)^{\delta-1}(cu+d)^{\lambda} \exp\left(\frac{-A}{u(1-u)}\right) E_{\alpha,\beta}^{\gamma,q}(z(1-u)^f)du$$

$$= d^{\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^m}{m!} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} B(\rho-m, \delta+fn-m)$$

$$\times {}_2F_1 \left[\begin{array}{lc} \rho-m, \ -\lambda \quad ; & \\ & \dfrac{-c}{d} \\ \rho+\delta+fn-2m \quad ; & \end{array} \right].$$

Theorem 3. *If $\alpha, \beta, \gamma, \mu, \lambda, A \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re(A)} > 0$; $\rho, \sigma \geq 0$ and $q \in N$, then*

$$\int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}(1-tu^{\rho}(1-u)^{\sigma})^{-a} \exp\left(\frac{-A}{u(1-u)}\right) E_{\alpha,\beta}^{\gamma,q}(zu^{\alpha})du$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} (a)_r B(\lambda+\alpha n+\rho r, \mu-\lambda+\sigma r; A) \frac{t^r}{r!}.$$

Proof. Denote the L.H.S. of Theorem 3 by I_3, then

$$I_4 = \int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}(1-tu^{\rho}(1-u)^{\sigma})^{-a}$$

$$\times \exp\left(\frac{-A}{u(1-u)}\right) E_{\alpha,\beta}^{\gamma,q}(zu^{\alpha})du.$$

Using the definition of generalized Mittag Leffler function (1.4), we get

$$I_3 = \int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}(1-tu^{\rho}(1-u)^{\sigma})^{-a}$$

$$\times \exp\left(\frac{-A}{u(1-u)}\right) \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n u^{\alpha n}}{\Gamma(\alpha n+\beta) n!} du$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta) n!} \int_0^1 (u)^{\lambda+\alpha n-1}(1-u)^{\mu-\lambda-1}(1-tu^{\rho}(1-u)^{\sigma})^{-a}$$

$$\times \exp\left(\frac{-A}{u(1-u)}\right) du.$$

Which further on using the integral [5, Eq.(3.5)]

$$\int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}(1-tu^\rho(1-u)^\sigma)^{-a} \times \exp\left(\frac{-A}{u(1-u)}\right) du$$
$$= \sum_{n=0}^{\infty}(a)_n B(\lambda+\rho n, \mu-\lambda+\sigma n; A)\frac{t^n}{n!}$$

yields the required the result. □

Corollary 2.4. *For $a = 0$, Theorem 3 reduces to the following result*:

$$\int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1} \exp\left(\frac{-A}{u(1-u)}\right) E^{\gamma,q}_{\alpha,\beta}(zu^\alpha)du$$
$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)n!}B(\lambda+\alpha n, \mu-\lambda; A).$$

Corollary 2.5. *For $a = A = 0$, Theorem 3 reduces to the following result*:

$$\int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}E^{\gamma,q}_{\alpha,\beta}(zu^\alpha)du = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)n!}B(\alpha n+\lambda, \mu-\lambda).$$

3. Special cases

Putting $\gamma = q = 1$ in Theorem 1, we get

(3.1)
$$\int_0^1 u^{\rho-1}(1-u)^{\delta-1} \exp\left(\frac{-A}{u(1-u)}\right)E_{\alpha,\beta}(zu^\alpha)du = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}B(\alpha n+\rho, \delta; A),$$

where $E_{\alpha,\beta}(z)$ is a Wiman function (1.2).

Putting $q = 1$ in Theorem 1, we get

(3.2)
$$\int_0^1 u^{\rho-1}(1-u)^{\delta-1} \exp\left(\frac{-A}{u(1-u)}\right)E^{\gamma}_{\alpha,\beta}(zu^\alpha)du = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)}B(\alpha n+\rho, \delta; A),$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is a Prabhakar function (1.3).

Putting $\alpha = \beta = \gamma = q = 1$ in Theorem 1, we get

$$\int_0^1 u^{\rho-1}(1-u)^{\delta-1} \exp\left(\frac{-A}{u(1-u)}+zu\right) du = \sum_{n=0}^{\infty}\frac{z^n}{n!}B(n+\rho, \delta; A), \tag{3.3}$$

which further for $\delta = 1$ give

$$\int_0^1 u^{\rho-1} \exp\left(\frac{-A}{u(1-u)}+zu\right) du = B(\rho, 1-z; A). \tag{3.4}$$

Putting $\gamma = q = 1$ in Theorem 2, we get

(3.5)

$$\int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda}\exp\left(\frac{-A}{(u-a)(b-u)}\right)E_{\alpha,\beta}[z(b-u)^f]du$$

$$= (ac+d)^{\lambda}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-A)^m}{m!}\frac{z^n}{\Gamma(\alpha n+\beta)}B(\rho-m,\delta+fn-m)$$

$$\times (b-a)^{\rho+\delta+fn-2m-1}{}_2F_1\left[\begin{matrix}\rho-m,\ -\lambda & ; & \\ & & \dfrac{-(b-a)c}{ac+d} \\ \rho+\delta+fn-2m & ; & \end{matrix}\right]$$

$$(\mathrm{Re}(\rho)>0,\ \mathrm{Re}(\lambda)>0;\ |\arg(\frac{bc+d}{ac+d})|<\pi),$$

where $E_{\alpha,\beta}(z)$ is a Wiman function.

Now putting $q = 1$ in Theorem 2, we get

(3.6)

$$\int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda}\exp\left(\frac{-A}{(u-a)(b-u)}\right)E^{\gamma}_{\alpha,\beta}(z(b-u)^f)du$$

$$= (ac+d)^{\lambda}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-A)^m}{m!}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}B(\rho,\delta+fn)(b-a)^{\rho+\delta+fn-1}$$

$$\times {}_2F_1\left[\begin{matrix}\rho,\ -\lambda & ; & \\ & & \dfrac{-(b-a)c}{ac+d} \\ \rho+\delta+fn & ; & \end{matrix}\right]$$

$$(\mathrm{Re}(\rho)>0,\ \mathrm{Re}(\lambda)>0;\ |\arg(\frac{bc+d}{ac+d})|<\pi),$$

where $E^{\lambda}_{\alpha,\beta}(z)$ is a Prabhakar function.

Now putting $\alpha = \beta = \gamma = q = 1$ in Theorem 2, we get

(3.7)

$$\int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda}\exp\left(\frac{-A}{(u-a)(b-u)}+z(b-u)^f\right)du$$

$$= (ac+d)^{\lambda}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-A)^m}{m!}\frac{z^n}{n!}B(\rho-m,\delta+fn-m)(b-a)^{\rho+\delta+fn-2m-1}$$

$$\times {}_2F_1\left[\begin{matrix}\rho,\ -\lambda & ; & \\ & & \dfrac{-(b-a)c}{ac+d} \\ \rho+\delta+fn & ; & \end{matrix}\right]$$

$$(\mathrm{Re}(\rho)>0,\ \mathrm{Re}(\lambda)>0;\ |\arg(\frac{bc+d}{ac+d})|<\pi),$$

where $\exp(z)$ is the exponential function.

Lastly, putting $\alpha = 0$, $\beta = 1$, $q = 1$ in Theorem 2, we get

(3.8)
$$\int_a^b (u-a)^{\rho-1}(b-u)^{\delta-1}(cu+d)^{\lambda}\exp\left(\frac{-A}{(u-a)(b-u)}\right)[1-z(b-u)^f]^{-\gamma}du$$
$$= (ac+d)^{\lambda}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{n!}B(\rho-m,\rho+\delta+fn-2m)(b-a)^{\rho+\delta+fn-1}$$
$$\times\, {}_2F_1\left[\begin{array}{ll}\rho,\ -\lambda & ; \\ & \dfrac{-(b-a)c}{ac+d} \\ \rho+\delta+fn & ;\end{array}\right].$$

Putting $\gamma = q = 1$ in Theorem 3, we get

(3.9)
$$\int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}(1-tu^{\rho}(1-u)^{\sigma})^{-a}\exp\left(\frac{-A}{u(1-u)}\right)E_{\alpha,\beta}(zu^{\alpha})du$$
$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}(a)_r B(\lambda+\alpha n+\rho r,\mu-\lambda+\sigma r;A)\frac{t^r}{r!},$$

where $E_{\alpha,\beta}(z)$ is a Wiman function.

Putting $q = 1$ in Theorem 3, we get

(3.10)
$$\int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}(1-tu^{\rho}(1-u)^{\sigma})^{-a}\exp\left(\frac{-A}{u(1-u)}\right)E^{\gamma}_{\alpha,\beta}(zu^{\alpha})du$$
$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}(a)_r B(\lambda+\alpha n+\rho r,\mu-\lambda+\sigma r;A)\frac{t^r}{r!},$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is a Prabhakar function.

Putting $\alpha = \beta = \gamma = q = 1$ in Theorem 3, we get

$$\begin{aligned}&\int_0^1 u^{\lambda-1}(1-u)^{\mu-\lambda-1}(1-tu^{\rho}(1-u)^{\sigma})^{-a}\exp\left(\frac{-A}{u(1-u)}+zu\right)du\\ &= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{z^n}{n!}(a)_r B(\lambda+\alpha n+\rho r,\mu-\lambda+\sigma r;A)\frac{t^r}{r!}.\end{aligned} \tag{3.11}$$

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Shakeel Ahmed
Department of Applied Mathematics
Faculty of Engineering
Aligarh Muslim University
Aligarh-202002, India
E-mail address: shakeelamu81@gmail.com

Mumtaz Ahmad Khan
Department of Applied Mathematics
Faculty of Engineering
Aligarh Muslim University
Aligarh-202002, India
E-mail address: mumtaz_ahmad_khan_2008@yahoo.com