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INDEFINITE TRANS-SASAKIAN MANIFOLD ADMITTING AN ASCREEN HALF LIGHTLIKE SUBMANIFOLD

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Abstract. We study the geometry of indefinite trans-Sasakian manifold $\bar{M}$, of type (α, β), admitting a half lightlike submanifold M such that the structure vector field of $\bar{M}$ does not belong to the screen and co-screen distributions of M. The purpose of this paper is to prove several classification theorems of such an indefinite trans-Sasakian manifold.

1. Introduction

Oubina [14] introduced the notion of an indefinite trans-Sasakian manifold, of type (α, β). Indefinite Sasakian manifold is an important kind of indefinite trans-Sasakian manifold with $\alpha = 1$ and $\beta = 0$. Indefinite cosymplectic manifold is another kind of indefinite trans-Sasakian manifold such that $\alpha = \beta = 0$. Indefinite Kenmotsu manifold is also an example with $\alpha = 0$ and $\beta = 1$.

Alegre *et al.* [1] introduced generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Sasakian, Kenmotsu and cosymplectic space forms are important kinds of indefinite generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, \quad f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, \quad f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4},$$

respectively, where c is a constant J-sectional curvature of each space forms.

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [2] and later studied by many authors (see recent results in two books [3, 5]). The class of lightlike submanifolds of codimension 2 is compose of two classes by virtue of the rank of its radical distribution, which are called half lightlike submanifold or coisotropic submanifold [3]. Half lightlike submanifold is a particular case of r-lightlike submanifold [2] such that $r = 1$ and its geometry is more general form than that of coisotrophic submanifolds or lightlike

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hypersurfaces. Much of the works on half lightlike submanifolds will be immediately generalized in a formal way to arbitrary r-lightlike submanifolds. For this reason, we study only half lightlike submanifolds in this article.

Recently many authors have studied lightlike submanifolds M of indefinite almost contact metric manifolds $\bar{M}$ ([5]∼[13]). The authors in these papers principally assumed that the structure vector fields of $\bar{M}$ is tangent to M, which are called tangential lightlike submanifolds. There are several different types of non-tangential lightlike submanifold of indefinite almost contact metric manifolds $\bar{M}$, according to the form of the structure vector field of $\bar{M}$. We study a type of them here, named by ascreen lightlike submanifold. In this paper, we study the geometry of indefinite trans-Sasakian manifold $\bar{M}$ admitting an ascreen half lightlike submanifold M. The main result is to prove several classification theorems of such an indefinite trans-Sasakian manifold.

2. Half lightlike submanifold

It is well known [3] that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of half lightlike submanifold (M, g) of a semi-Rimannian manifold $(\bar{M}, \bar{g})$, of codimension 2, is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank 1. Thus there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen distribution* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M, by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M and by $(-.-)_i$ the i-th equation of $(-.-)$. We use same notations for any others. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly, $TM^{\perp}$ is a subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a non-degenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ of $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a non-degenerate vector bundle such that

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp}.$$

Clearly, $Rad(TM)$ is a subbundle of $S(TM^{\perp})^{\perp}$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit spacelike vector field, without loss of generality. It is well known [3] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\bar{g}(\xi, N) = 1, \ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \ \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector* *field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Then $T\bar{M}$ is decomposed as

$$T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ = \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \tag{2.2}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$, with respect to the decomposition (2.1). Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{2.3}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.4}$$
$$\bar{\nabla}_X L = -A_L X + \phi(X)N; \tag{2.5}$$
$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.6}$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad \forall\, X,\, Y \in \Gamma(TM), \tag{2.7}$$

where ∇ and ∇^* are induced connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are called the *shape operators*, and τ, ρ and ϕ are 1-forms on TM. We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental form tensor* of M.

Since the connection $\bar{\nabla}$ on $\bar{M}$ is torsion-free, the induced connection ∇ on M is also torsion-free, and B and D are symmetric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.8}$$
$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{2.9}$$
$$D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \qquad \bar{g}(A_L X, N) = \rho(X), \tag{2.10}$$

for any X, $Y \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM).$$

From (2.8), (2.9) and (2.10), we see that B and D satisfy

$$B(X,\,\xi) = 0, \quad D(X,\,\xi) = -\phi(X), \quad \forall\, X \in \Gamma(TM), \tag{2.11}$$

A^*_ξ and A_N are $S(TM)$-valued, and A^*_ξ is self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.12}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y) \tag{2.13}$$

for any X, Y, $Z \in \Gamma(TM)$. But the connection ∇^* on $S(TM)$ is metric.

Definition 1. A half lightlike submanifold M of $\bar{M}$ is said to be

(1) *totally umbilical* [2] if there is a smooth vector field $\mathcal{H}$ on $tr(TM)$ on any coordinate neighborhood $\mathcal{U}$ such that

$$h(X,Y) = \mathcal{H}g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0$, *i.e.*, $h = 0$ on $\mathcal{U}$, we say that M is *totally geodesic*.

(2) *screen totally umbilical* [2] if there exists a smooth function γ on $\mathcal{U}$ such that $A_{_N}X = \gamma PX$, or equivalently,

$$C(X,PY) = \gamma g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

(3) *screen conformal* [3] if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that $A_{_N} = \varphi A^*_\xi$, or equivalently,

$$C(X,PY) = \varphi B(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

It is easy to see that M is totally umbilical if and only if there exist smooth functions σ and δ on each coordinate neighborhood $\mathcal{U}$ such that

$$B(X,Y) = \sigma g(X,Y), \quad D(X,Y) = \delta g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

Denote by $\bar{R}$, R and R^* the curvature tensors of the Lavi-Civita connection $\bar\nabla$ on $\bar{M}$ and the induced connections ∇ and ∇^* on M and $S(TM)$, respectively.

Definition 2. A lightlike submanifold $M = (M, g, \nabla)$ equipped with a degenerate metric g and a linear connection ∇ is said to be of *constant curvature c* if there exists a constant c such that the curvature tensor R of ∇ satisfies

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \tag{2.14}$$

Definition 3. We say that M is *locally symmetric* [6, 7, 9] if its curvature tensor R be parallel, *i.e.*, have vanishing covariant differential, $\nabla R = 0$.

3. Indefinite trans-Sasakian manifolds

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be an *indefinite almost contact metric manifold* ([5]∼[13]) if there exists a structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$, where J is a $(1,\,1)$-type tensor field, ζ is a vector field which is called the *structure vector field* of $\bar{M}$ and θ is a 1-form such that

$$J^2X = -X + \theta(X)\zeta,\;\; \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\theta(X)\theta(Y),\;\; \theta(\zeta) = 1, \tag{3.1}$$

for any vector fields X and Y on $\bar{M}$, where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike, respectively. In this case, the structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$.

In an indefinite almost contact metric manifold, we show that $J\zeta = 0$ and $\theta \circ J = 0$. Such a manifold is said to be an *indefinite contact metric manifold* if $d\theta = \Phi$, where $\Phi(X,Y) = \bar{g}(X,\, JY)$ is called the *fundamental 2-form* of $\bar{M}$. The indefinite almost contact metric structure of $\bar{M}$ is said to be *normal* if $[J,\, J](X,Y) = -2d\theta(X,Y)\zeta$ for any vector fields X and Y on $\bar{M}$, where $[J,\, J]$ denotes the Nijenhuis (or torsion) tensor field of J given by

$$[J,J](X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY].$$

An indefinite normal contact metric manifold is called a *indefinite Sasakian manifold*. It is well known [5] that an indefinite almost contact metric manifold $(\bar{M}, \bar{g}, J, \zeta, \theta)$ is indefinite Sasakian if and only if

$$(\bar{\nabla}_X J)Y = \bar{g}(X, Y)\zeta - \epsilon\theta(Y)X.$$

Definition 4. An indefinite almost contact metric manifold $(\bar{M}, \bar{g})$, with the Levi-Civita connection $\bar{\nabla}$ with respect to $\bar{g}$, is called *indefinite trans-Sasakian manifold* if there exist two smooth functions α and β on $\bar{M}$ such that

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \epsilon\theta(Y)JX\} \tag{3.2}$$

for any vector fields X and Y on $\bar{M}$. In this case, we say that $\{J, \zeta, \theta, \bar{g}\}$ is an *indefinite trans-Sasakian structure of type* (α, β) [1, 12, 14].

By replacing Y by ζ in (3.2) and using (3.1), we get

$$\bar{\nabla}_X\zeta = -\epsilon\alpha JX + \epsilon\beta(X - \theta(X)\zeta). \tag{3.3}$$

Remark 3.1. If $\beta = 0$, then $\bar{M}$ is said to be an *indefinite α-Sasakian manifold*. Indefinite Sasakian manifolds [5, 6, 7, 11] appear as examples of indefinite α-Sasakian manifolds, with $\alpha = 1$. Another important kind of indefinite trans-Sasakian manifold is that of indefinite cosymplectic manifolds [8, 13] obtained for $\alpha = \beta = 0$. If $\alpha = 0$, then $\bar{M}$ is said to be an *indefinite β-Kenmotsu manifold*. Indefinite Kenmotsu manifolds [9, 10] are particular examples of indefinite β-Kenmotsu manifold, with $\beta = 1$.

It is known [7, 8] that, for any half lightlike submanifold M of an indefinite almost contact metric manifold $\bar{M}$, $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$, of rank 1. In the sequel, we shall assume that ζ is a unit spacelike vector field, *i.e.*, $\epsilon = 1$, without loss of generality. Let a and b be the smooth functions given by $a = \theta(N)$ and $b = \theta(\xi)$.

Definition 5. A half lightlike submanifold M of an indefinite almost contact metric manifold $\bar{M}$ is called an *ascreen half lightlike submanifold* [8, 11] if the structure vector field ζ of $\bar{M}$ belongs to the distribution $Rad(TM) \oplus ltr(TM)$.

Example 1. Consider a submanifold M of $\bar{M} = (R^5_2,\, J,\, \zeta,\, \theta,\, \bar{g})$ given by

$$X(u_1,\, u_2,\, u_3) = (u_1,\, u_2,\, u_3,\, u_2,\, \frac{1}{\sqrt{2}}(u_1 + u_3)).$$

By direct calculations we easily check that

$$TM = \text{Span}\{\xi = \partial x_1 + \partial y_1 + \sqrt{2}\partial z,\;\; U = \partial x_1 - \partial y_1,\;\; V = \partial x_2 + \partial y_2\},$$

$$TM^{\perp} = \text{Span}\{\xi,\; L = \partial x_2 - \partial y_2\}, \quad Rad(TM) = \text{Span}\{\xi\}.$$

We obtain the lightlike transversal and transversal vector bundles

$$ltr(TM) = \text{Span}\{N = \frac{1}{4}(-\partial x_1 - \partial y_1 + \sqrt{2}\partial z)\}, \quad tr(TM) = \text{Span}\{N,\, L\}.$$

From this results, we show that $J\xi = U$, $Rad(TM) \cap J(Rad(TM)) = \{0\}$, $JN = -\frac{1}{4}U$, $JL = -V$, $JN = -\frac{1}{4}J\xi$ and $J(Rad(TM) = J(ltr(TM)$, $\zeta =$

$\frac{1}{2\sqrt{2}}\xi + \sqrt{2}N$ and $J\zeta = 0$. Thus M is an ascreen half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$.

Theorem 3.2. *Any indefinite trans-Sasakian manifold $\bar{M}$, of type (α, β), admitting an ascreen half lightlike submanifold M satisfies $\alpha = 1$. Therefore $\bar{M}$ is neither indefinite β-Kenmotsu manifold nor indefinite cosymplectic manifold.*

Proof. As M is ascreen half lightlike submanifold of $\bar{M}$, ζ is decomposed by

$$\zeta = a\xi + bN. \tag{3.4}$$

As $\bar{g}(\zeta, \zeta) = 1$, we have $2ab = 1$. This implies that $a \neq 0$ and $b \neq 0$. Taking the scalar product with X to (3.4), we show that $\theta(X) = b\eta(X)$ for all $X \in \Gamma(TM)$. Consider a unit timelike vector field V on $S(TM)$ and its 1-form v defined by

$$V = -b^{-1}J\xi, \quad v(X) = -g(X, V), \quad \forall\, X \in \Gamma(TM). \tag{3.5}$$

Applying J to (3.4) and using $(3.5)_1$ and the fact that $J\zeta = 0$, we have

$$JN = aV. \tag{3.6}$$

Applying J to $(3.5)_1$ and using $(3.1)_1$, (3.4) and the fact that $2ab = 1$, we have

$$JV = a\xi - bN. \tag{3.7}$$

Consider a unit spacelike vector field U on $S(TM)$ and its 1-form u defined by

$$U = JL, \quad u(X) = g(X, U), \quad \forall\, X \in \Gamma(TM). \tag{3.8}$$

As $JN = -\frac{a}{b}J\xi$, we show that $J(Rad(TM)) = J(tr(TM))$. From the fact $J(Rad(TM))$ is vector subbundle of $S(TM)$, there exists a non-degenerate almost complex distribution D with respect to J such that

$$TM = Rad(TM) \oplus_{orth} \{J(Rad(TM)) \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} D\}. \tag{3.9}$$

Denote by Q the projection morphism of TM on D, with respect to the decomposition (3.9). Using (3.9), any vector field X on M is expressed as

$$X = QX + v(X)V + u(X)U + \eta(X)\xi. \tag{3.10}$$

Applying J to (3.10) and using $(3.5)_1$, (3.7) and $(3.8)_1$, we obtain

$$JX = fX - \theta(X)V + av(X)\xi - bv(X)N - u(X)L, \tag{3.11}$$

where f is a tensor field of type $(1, 1)$ defined on M by $f = J \circ Q$. Applying J to (3.11) and using (3.1), (3.4) $\sim$ (3.7) and the fact that $2ab = 1$, we have

$$f^2X = -X + v(X)V + u(X)U + \eta(X)\xi = -QX. \tag{3.12}$$

Applying $\bar{\nabla}_X$ to (3.4) and using (2.3) $\sim$ (2.7), (3.3) and (3.11), we have

$$aA^*_\xi X + bA_N X = \alpha fX - \beta QX - \{\alpha\theta(X) + \beta v(X)\}V - \beta u(X)U, \tag{3.13}$$

$$X[b] + b\tau(X) = b\{\alpha v(X) - \beta\theta(X)\}, \tag{3.14}$$

$$X[a] - a\tau(X) = -a\{\alpha v(X) - \beta\theta(X)\}, \tag{3.15}$$

$$\alpha u(X) = b\rho(X) - a\phi(X), \quad \forall\, X \in \Gamma(TM). \tag{3.16}$$

Taking the scalar product with V to (3.13) and then, taking $X = \xi$, we have

$$aB(X,V) + bC(X,V) = \alpha\theta(X) + \beta v(X), \quad C(\xi,V) = \alpha, \tag{3.17}$$

respectively. Using $\eta(Y) = \bar{g}(Y,N)$ and (2.4), we obtain

$$2d\eta(X,Y) = g(X, A_N Y) - g(A_N X, Y) + \tau(X)\eta(Y) - \tau(Y)\eta(X) \tag{3.18}$$

for all X, $Y \in \Gamma(TM)$. Also, using $\theta(Y) = b\eta(Y)$, we have

$$2d\theta(X,Y) = 2b\,d\eta(X,Y) + X[b]\eta(Y) - Y[b]\eta(X)$$

for all X, $Y \in \Gamma(TM)$. Substituting (3.18) and the equation $d\theta(X,Y) = \bar{g}(X, JY)$ into the last equation and using (3.14), we get

$$\begin{aligned} 2\bar{g}(X,JY) = {} & b\{g(X,A_N Y) - g(A_N X, Y)\} \\ & + b\{\alpha v(X) - \beta\theta(X)\}\eta(Y) - b\{\alpha v(Y) - \beta\theta(Y)\}\eta(X) \end{aligned} \tag{3.19}$$

for all X, $Y \in \Gamma(TM)$. Taking $X = V$ and $Y = \xi$ to this equation and using the fact that $g(A_N\xi, V) = C(\xi, V) = \alpha$, we get $2b = 2b\alpha$. Therefore $\alpha = 1$. □

Theorem 3.3. *There exist neither screen conformal nor screen totally umbilical, ascreen half lightlike submanifold of an indefinite trans-Sasakian manifold.*

Proof. Assume that M is either screen conformal or screen totally umbilical. Then we have the following impossible results:

$$\alpha = C(\xi,V) = \varphi B(\xi,V) = 0 \quad \text{or} \quad \alpha = C(\xi,V) = \gamma g(\xi,V) = 0.$$

Thus there exist neither screen conformal nor screen totally umbilical, ascreen half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. □

Applying $\bar{\nabla}_X$ to $(3.8)_1$ and using (2.5), (3.2), (3.4) and (3.11), we get

$$\begin{aligned} \bar{\nabla}_X U = {} & -f(A_L X) + \{\theta(A_L X) + a\phi(X)\}V - a\{\beta u(X) + v(A_L X)\}\xi \\ & + b\{-\beta u(X) + v(A_L X)\}N + u(A_L X)L, \quad \forall\, X \in \Gamma(TM). \end{aligned}$$

Comparing the $S(TM)$-, $Rad(TM)$-, $ltr(TM)$- and $S(TM^\perp)$-components of both sides in the last equation, and using (2.3), (2.6), (2.10) and (3.4), we have

$$\nabla^*_X U = -f(A_L X) + \{b\rho(X) + a\phi(X)\}V, \tag{3.20}$$

$$C(X,U) = -a\{\beta u(X) - D(X,V)\}, \tag{3.21}$$

$$B(X,U) = -b\{\beta u(X) + D(X,V)\}. \tag{3.22}$$

Theorem 3.4. *Any indefinite trans-Sasakian manifold $\bar{M}$, of type (α,β), admitting a totally geodesic ascreen half lightlike submanifold M satisfies $\alpha = 1$ and $\beta = 0$. Thus, $\bar{M}$ is an indefinite Sasakian manifold.*

Proof. As M is totally geodesic, from(3.22) we show that $b\beta u(X) = 0$ for any $X \in \Gamma(TM)$. Taking $X = U$ and using the fact that $b \neq 0$, we get $\beta = 0$. From this result and Theorem 3.2, we obtain $\alpha = 1$ and $\beta = 0$. □

Applying $\bar{\nabla}_X$ to $bV = -J\xi$ and using (2.3), (2.6), (2.7), (2.8), (2.11), (3.1), (3.2), (3.4), (3.5), (3.8), (3.10), (3.11), (3.14), (3.16), (3.17) and (3.22), we get

$$\nabla^*_X V = \alpha QX + \beta fX + 2af(A^*_\xi X) + \{\alpha u(X) + 2a\phi(X)\}U. \tag{3.23}$$

Theorem 3.5. *Let M be a totally geodesic ascreen half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If M is locally symmetric, then M has a constant positive curvature* 1.

Proof. As M is totally geodesic, *i.e.*, $B = D = 0$, we show that $A^*_\xi = 0$ by (2.8), and $\phi = 0$ by $(2.11)_2$. As $\alpha = 1$, the equation (3.23) is reduced to

$$\nabla^*_X V = QX + u(X)U$$

for all $X \in \Gamma(TM)$. From (3.17) and the facts that $\beta = 0$ and $\theta(X) = b\eta(X)$, we get $C(X, V) = \eta(X)$. Thus, from (2.6) with $PY = V$ and (3.10), we have

$$\nabla_X V = X - v(X)V \tag{3.24}$$

for all $X \in \Gamma(TM)$. Applying ∇_Y to (3.24) and using (3.24), we have

$$\nabla_X \nabla_Y V = \nabla_X Y - v(Y)X - \{X(v(Y)) - v(X)v(Y)\}V$$

for any $X, Y \in \Gamma(TM)$. From the last two equation, we obtain

$$R(X, Y)V = v(X)Y - v(Y)X - 2dv(X, Y)V, \quad \forall\, X, Y \in \Gamma(TM).$$

As M is totally geodesic, $\bar{R}(X, Y)V = R(X, Y)V$ for all $X, Y \in \Gamma(TM)$. From this and the fact that $\bar{g}(\bar{R}(X, Y)V, V) = 0$, we obtain $dv = 0$. Thus

$$R(X, Y)V = v(X)Y - v(Y)X, \quad \forall\, X, Y \in \Gamma(TM). \tag{3.25}$$

Applying $\bar{\nabla}_X$ to $v(Y) = -g(Y, V)$ and using (3.7) and (3.24), we have

$$(\nabla_X v)(Y) = -g(X, Y) - v(X)v(Y), \quad \forall\, X, Y \in \Gamma(TM). \tag{3.26}$$

Applying ∇_Z to (3.25) and using (3.24)$\sim$(3.26) and the fact that M is locally symmetric, *i.e.*, $\nabla_Z R = 0$ for any $Z \in \Gamma(TM)$, we have

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad \forall\, X, Y, Z \in \Gamma(TM).$$

Due to (2.14), we show that M is a space of constant positive curvature 1. □

4. Indefinite generalized Sasakian space form

Definition 6. An indefinite almost contact metric manifold $\bar{M}$ is called an *indefinite generalized Sasakian space form* [1] and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \bar{R}(X, Y)Z &= f_1\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} \\ &+ f_2\{\bar{g}(X, JZ)JY - \bar{g}(Y, JZ)JX + 2\bar{g}(X, JY)JZ\} \\ &+ f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ &+ \bar{g}(X, Z)\theta(Y)\zeta - \bar{g}(Y, Z)\theta(X)\zeta\} \end{aligned} \tag{4.1}$$

for any vector fields X, Y and Z on $\bar{M}$.

Example 2. An *indefinite Sasakian space form*, *i.e.*, an indefinite Sasakian manifold with constant J-sectional curvature c, such that the structure vector field ζ is spacelike, is an indefinite generalized Sasakian space form with

$$f_1 = \frac{c+3}{4}, \quad f_2 = f_3 = \frac{c-1}{4}.$$

Example 3. An *indefinite Kenmotsu space form*, *i.e.*, an indefinite Kenmotsu manifold with constant J-sectional curvature c, such that the structure vector field ζ is spacelike, is an indefinite generalized Sasakian space form with

$$f_1 = \frac{c-3}{4}, \quad f_2 = f_3 = \frac{c+1}{4}.$$

Example 4. An *indefinite cosymplectic space form*, *i.e.*, an indefinite cosymplectic manifold with constant J-sectional curvature c, such that the structure vector field ζ is spacelike, is an indefinite generalized Sasakian space form with

$$f_1 = f_2 = f_3 = \frac{c}{4}.$$

We need the following three Gauss-Codazzi equations for M and $S(TM)$ (for a full set of these equations see [3]): For any $X, Y, Z \in \Gamma(TM)$, we get

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,\xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\ &\quad + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,N) &= \bar{g}(R(X,Y)Z,N) \\ &\quad + D(X,Z)\rho(Y) - D(Y,Z)\rho(X), \end{aligned} \tag{4.3}$$

$$\begin{aligned} g(R(X,Y)PZ,N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X). \end{aligned} \tag{4.4}$$

Theorem 4.1. *Any indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, equipped with indefinite trans-Sasakian structure of type (α, β), admitting a totally geodesic ascreen half lightlike submanifold M satisfies*

$$\alpha = 1, \quad \beta = 0, \quad f_1 = 1 \quad \textit{and} \quad f_2 = f_3 = 0.$$

Thus, $\bar{M}(f_1, f_2, f_3)$ is an indefinite Sasakian manifold of constant curvature 1.

Proof. As M is totally geodesic, we have $B = D = \phi = A^*_\xi = 0$ and $g(A_L X, Y) = 0$ for all $X, Y \in \Gamma(TM)$. By Theorem 3.2 and Theorem 3.4, we get $\alpha = 1$ and $\beta = 0$. Substituting (4.1) into (4.2), we obtain

$$\begin{aligned} &bf_2\{v(X)\bar{g}(Y,JZ) - v(Y)\bar{g}(X,JZ) - 2v(Z)\bar{g}(X,JY)\} \\ &+ bf_3\{g(X,Z)\theta(Y) - g(Y,Z)\theta(X)\} = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \end{aligned}$$

Taking $X = Z = U$ and $Y = \xi$ and using the fact that $b \neq 0$, we have $f_3 = 0$. Also, taking $X = Z = V$ and $Y = \xi$ and using (3.5) and (3.7), we get $f_2 = 0$.

From (3.12), (3.17), (3.23) and the fact that $\alpha = 1$ and $\beta = 0$, we have

$$C(X,V) = 2a\theta(X), \qquad \nabla^*_X V = X - v(X)V - \eta(X)\xi. \tag{4.5}$$

Taking $X = V$ and $Y = \xi$ to (3.19) and using (3.5), we get

$$g(V, A_N \xi) = 1. \tag{4.6}$$

Substituting (4.1) and (4.4) into (4.3) and using $f_2 = f_3 = 0$, we have

$$\begin{aligned} &f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &= (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X). \end{aligned}$$

Replacing PZ by V to the last equation and using $(4.5)_1$, we have

$$\begin{aligned} &f_1\{v(X)\eta(Y) - v(Y)\eta(X)\} \\ &= (\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) + 2a\{\theta(X)\tau(Y) - \theta(Y)\tau(X)\}. \end{aligned} \tag{4.7}$$

Applying ∇_X to $C(Y, V) = 2a\theta(Y)$ and using $(4.5)_2$, we have

$$\begin{aligned} (\nabla_X C)(Y, V) = 2\{X[a] + av(X)\}\theta(Y) - g(A_N Y, X) \\ + 2a\{X(\theta(Y)) - \theta(\nabla_X Y)\}. \end{aligned}$$

Substituting this equation into (4.7) and using (3.15), we get

$$f_1\{v(X)\eta(Y) - v(Y)\eta(X)\} = g(A_N X, Y) - g(X, A_N Y) + 4a\bar{g}(X, JY).$$

Taking $X = V$ and $Y = \xi$ and using (4.6) and $2ab = 1$, we have $f_1 = 1$. □

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