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INEQUALITIES FOR THE NON-TANGENTIAL DERIVATIVE AT THE BOUNDARY FOR HOLOMORPHIC FUNCTION

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Abstract. In this paper, we present some inequalities for the non-tangential derivative of $f(z)$. For the function $f(z) = z + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots$ defined in the unit disc, with $\Re\left(\frac{f'(z)}{\lambda f'(z)+1-\lambda}\right) > \beta$, $0 \leq \beta < 1$, $0 \leq \lambda < 1$, we estimate a module of a second non-tangential derivative of $f(z)$ function at the boundary point ξ, by taking into account their first nonzero two Maclaurin coefficients. The sharpness of these estimates is also proved.

1. Introduction

Let f be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = cz$, $|c| = 1$ ([4], p. 329).

Let $f(z) = z + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots$ be a holomorphic function in the unit disc D such that $\Re\left(\frac{f'(z)}{\lambda f'(z)+1-\lambda}\right) > \beta$, $0 \leq \beta < 1$, $0 \leq \lambda < 1$.

Consider the function

$$g(z) = \frac{\frac{f'(z)}{\lambda f'(z)+1-\lambda} - \beta}{1-\beta} = 1 + a_p z^p + \cdots$$

$g(z)$ is a holomorphic function and $\Re g(z) > 0$ for $|z| < 1$ and hence

$$\phi(z) = \frac{1-g(z)}{1+g(z)} = c_p z^p + \cdots \tag{1.1}$$

is a holomorphic function in the unit disc D, $\phi(0) = 0$ and since $\Re\left(\frac{f'(z)}{\lambda f'(z)+1-\lambda}\right) > \beta$, it also follows that $|\phi(z)| < 1$ for $|z| < 1$. Therefore, from the Schwarz

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lemma, we obtain

$$|f'(z)| \leq \frac{(1-\lambda)\left(1+(1-2\beta)\,|z|^{p}\right)}{1-\lambda-(1+(1-2\beta)\lambda)\,|z|^{p}} \tag{1.2}$$

and

$$|b_{p+1}| \leq \frac{2(1-\beta)}{(1-\lambda)\,(1+p)}. \tag{1.3}$$

Equality is achieved in (1.2) (for some nonzero $z \in D$) or in (1.3) if and only if $f(z)$ is the function of the form

$$f(z) = \int_0^z \frac{(1-\lambda)\left(1-(1-2\beta)t^p e^{i\theta}\right)}{1-\lambda+(1+(1-2\beta)\lambda)\,t^p e^{i\theta}}dt,$$

where θ is a real number.

H. Unkelbach and R. Osserman have given the inequalities which are called the boundary Schwarz lemma. They have first showed that

$$|f'(\xi)| \geq \frac{2}{1+|f'(0)|} \tag{1.4}$$

and

$$|f'(\xi)| \geq 1 \tag{1.5}$$

under the assumption $f(0) = 0$, where f is a holomorphic function mapping the unit disc into itself and ξ is a boundary point to which f extends continuously. Moreover, the equality in (1.4) holds if and only if f is of the form

$$f(z) = ze^{i\theta}\frac{z-a}{1-\overline{a}z},$$

where $\theta \in \mathbb{R}$ and $a \in D$ satisfies $\arg a = \arg \xi$. Also, the equality in (1.5) holds if and only if $f(z) = ze^{i\theta}$, $\theta \in \mathbb{R}$.

One does not need to assume that f extends continuously to ξ. For example, if f has a radial limit $f(\xi)$ at ξ, with $|f(\xi)| = 1$, and if f has a radial derivative at ξ, then that derivative also satisfies the inequalities (1.4) and (1.5). Accordingly, using the Möbius transformation, they have generalized the inequality on the case of $f(0) \neq 0$ (see [7], [9]).

If, in addition, the function f has an angular limit $f(\xi)$ at $\xi \in \partial D$, $|f(\xi)| = 1$, then by the Julia-Wolff lemma the angular derivative $f'(\xi)$ exists and $1 \leq |f'(\xi)| \leq \infty$ (see [8]).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = b_p z^p + b_{p+1}z^{p+1} + \cdots$, with a zero set $\{a_k\}$ (see [3]).

Some other types of strengthening inequalities are obtained in (see [1], [5], [6]).

In the following theorems, if we know the second and the third coeffient in the expansion of the function $f(z) = z + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots$, then we obtain more general results on the second non-tangential derivatives of certain classes of a holomorphic function in the unit disc at the boundary by taking into account b_{p+1}, b_{p+2} and critical points of $f(z)-z$ function. The sharpness of these inequalities is also proved.

Theorem 1.1. *Let $f(z)=z+b_{p+1}z^{p+1}+b_{p+2}z^{p+2}+\cdots$, $b_{p+1}\neq 0$, $p\geq 1$ be a holomorphic function in the unit disc D and $\Re\left(\frac{f'(z)}{\lambda f'(z)+1-\lambda}\right)>\beta$, $0\leq\beta<1$, $0\leq\lambda<1$ for $|z|<1$. Suppose that, for some $\xi\in\partial D$, f' has a non-tangential limit $f'(\xi)$ at ξ and $f'(\xi)=\frac{\beta(1-\lambda)}{(1-\beta\lambda)}$. Then f has the second non-tangential derivative at ξ and*

(1.6)

$$|f''(\xi)|\geq\frac{(1-\beta)(1-\lambda)}{2(1-\beta\lambda)^2}\left(p+\frac{2\left[2(1-\beta)-(1-\lambda)(1+p)\left|b_{p+1}\right|\right]^2}{4(1-\beta)^2-\left((1-\lambda)(1+p)\left|b_{p+1}\right|\right)^2-2(1-\beta)(1-\lambda)(2+p)\left|b_{p+2}\right|}\right).$$

The equality in (1.6) *occurs for the function*

$$f(z)=\int_0^z\frac{(1-\lambda)\left(1-(1-2\beta)t^p\right)}{1-\lambda+(1+(1-2\beta)\lambda)\,t^p}dt.$$

Proof. Let $\phi(z)$ be defined as in (1.1). $h(z)=z^p$ is a holomorphic function in D, $|h(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in D$, we have $|\phi(z)|\leq|h(z)|$. Therefore, $p(z)=\frac{\phi(z)}{h(z)}$ is a holomorphic function in D and $|p(z)|<1$ for $|z|<1$. In particular, we have

$$|p(0)|=\frac{(1-\lambda)(p+1)}{2(1-\beta)}\left|b_{p+1}\right|\leq 1 \tag{1.7}$$

and

$$|p'(0)|=\frac{(1-\lambda)(p+2)}{2(1-\beta)}\left|b_{p+2}\right|.$$

Moreover, since the expression $\frac{\xi\phi'(\xi)}{\phi(\xi)}$ is a real number greater than or equal to 1 ([2]) and $f'(\xi)=\frac{\beta(1-\lambda)}{(1-\beta\lambda)}$ yields $|\phi(\xi)|=1$, we get

$$\frac{\xi\phi'(\xi)}{\phi(\xi)}=\left|\frac{\xi\phi'(\xi)}{\phi(\xi)}\right|=\left|\phi'(\xi)\right|.$$

Also, since $|\phi(z)|\leq|h(z)|$, we take

$$\frac{1-|\phi(z)|}{1-|z|}\geqslant\frac{1-|h(z)|}{1-|z|}.$$

Passing to the non-tangential limit in the last inequality yield

$$\left|\phi'(\xi)\right|\geq\left|h'(\xi)\right|.$$

Therefore, we obtain

$$\frac{\xi\phi'(\xi)}{\phi(\xi)} = |\phi'(\xi)| \geq |h'(\xi)| = \frac{\xi h'(\xi)}{h(\xi)}.$$

The function

$$\Theta(z) = \frac{p(z) - p(0)}{1 - \overline{p(0)}p(z)}$$

is a holomorphic function in D, $|\Theta(z)| < 1$ for $|z| < 1$, $\Theta(0) = 0$ and $|\Theta(\xi)| = 1$ for $\xi \in \partial D$. It can be easily shown a non-tangential derivative of Θ at $\xi \in \partial D$ (see, [8]). Therefore, the second non-tangential derivative of f at ξ is obtained. From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1 + |\Theta'(0)|} \leq |\Theta'(\xi)| &= \frac{1 - |p(0)|^2}{\left|1 - \overline{p(0)}p(\xi)\right|^2} |p'(\xi)| \\
&\leq \frac{1 + |p(0)|}{1 - |p(0)|} \left| \frac{\phi'(\xi)h(\xi) - h'(\xi)\phi(\xi)}{(h(\xi))^2} \right| \\
&= \frac{1 + |p(0)|}{1 - |p(0)|} \left| \frac{\phi(\xi)}{\xi h(\xi)} \right| \left| \frac{\xi\phi'(\xi)}{\phi(\xi)} - \frac{\xi h'(\xi)}{h(\xi)} \right| \\
&= \frac{1 + |p(0)|}{1 - |p(0)|} \left\{ |\phi'(\xi)| - |h'(\xi)| \right\}.
\end{aligned}$$

Since

$$\Theta'(z) = \frac{1 - |p(0)|^2}{\left(1 - \overline{p(0)}p(z)\right)^2} p'(z)$$

and

$$\begin{aligned}
|\Theta'(0)| = \frac{|p'(0)|}{1 - |p(0)|^2} &= \frac{\frac{(1-\lambda)(p+2)}{2(1-\beta)} |b_{p+2}|}{1 - \left(\frac{(1-\lambda)(p+1)}{2(1-\beta)} |b_{p+1}|\right)^2} \\
&= \frac{2(1-\beta)(1-\lambda)(p+2)|b_{p+2}|}{4(1-\beta)^2 - ((1-\lambda)(p+1)|b_{p+1}|)^2},
\end{aligned}$$

we take

$$\frac{2}{1 + \frac{2(1-\beta)(1-\lambda)(p+2)|b_{p+2}|}{4(1-\beta)^2 - ((1-\lambda)(p+1)|b_{p+1}|)^2}} \leq \frac{1 + \frac{(1-\lambda)(p+1)}{2(1-\beta)} |b_{p+1}|}{1 - \frac{(1-\lambda)(p+1)}{2(1-\beta)} |b_{p+1}|} \left\{ \frac{2|g'(\xi)|}{|1 + g(\xi)|^2} - p \right\}.$$

Since

$$|1 + g(\xi)|^2 = \left| 1 + \frac{\frac{f'(\xi)}{\lambda f'(\xi) + 1 - \lambda} - \beta}{1 - \beta} \right|^2$$

$$= \left| 1 + \frac{1}{1-\beta} \left(\frac{\frac{\beta(1-\lambda)}{(1-\beta\lambda)}}{\lambda\frac{\beta(1-\lambda)}{(1-\beta\lambda)} + 1 - \lambda} - \beta \right) \right|^2 = 1$$

and

$$|g'(\xi)| = \frac{1}{1-\beta} \left(\frac{(1-\lambda)\,|f''(\xi)|}{|\lambda f'(\xi) + 1 - \lambda|^2} \right) = \frac{1}{1-\beta} \frac{(1-\beta\lambda)^2\,|f''(\xi)|}{(1-\lambda)},$$

we get

$$|\phi'(\xi)| = \frac{2\,|g'(\xi)|}{|1+g(\xi)|^2} = \frac{2\,(1-\beta\lambda)^2\,|f''(\xi)|}{(1-\beta)\,(1-\lambda)}.$$

So, we obtain the inequality (1.6).

To show that the inequality (1.6) is sharp, take the holomorphic function

$$f(z) = \int_0^z \frac{(1-\lambda)\,(1-(1-2\beta)t^p)}{1-\lambda+(1+(1-2\beta)\lambda)\,t^p} dt.$$

Then

$$f'(z) = \frac{d}{dz} f(z) = \frac{(1-\lambda)\,(1-(1-2\beta)z^p)}{1-\lambda+(1+(1-2\beta)\lambda)\,z^p}$$

and

$$|f''(1)| = \frac{(1-\beta)\,(1-\lambda)}{2\,(1-\beta\lambda)^2} p.$$

Since $|b_{p+1}| = \frac{2(1-\beta)}{(1-\lambda)(p+1)}$, (1.6) is satisfied with equality. □

Theorem 1.2. *Let $f(z) = z + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots$, $b_{p+1} > 0$, $p \geq 1$ be a holomorphic function in the unit disc D and $f(z) - z$ has no critical point in D except $z = 0$, and $\Re\left(\frac{f'(z)}{\lambda f'(z)+1-\lambda}\right) > \beta$, $0 \leq \beta < 1$, $0 \leq \lambda < 1$ for $|z| < 1$. Suppose that, for some $\xi \in \partial D$, f' has a non-tangential limit $f'(\xi)$ at ξ, $f'(\xi) = \frac{\beta(1-\lambda)}{(1-\beta\lambda)}$. Then f has the second non-tangential derivative at ξ and*

$$|f''(\xi)| \geq \frac{(1-\beta)\,(1-\lambda)}{2\,(1-\beta\lambda)^2} \left(p - \frac{2\left[\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)\right]^2 (p+1)\,|b_{p+1}|}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)(p+1)\,|b_{p+1}| - (p+2)\,|b_{p+2}|}\right) \tag{1.8}$$

and

$$|b_{p+2}| \leq \frac{2}{(p+2)} \left| (p+1) b_{p+1} \ln\left(\frac{(1-\lambda)\,(p+1)}{2\,(1-\beta)}\,|b_{p+1}| \right) \right|. \tag{1.9}$$

Moreover, the equality in (1.8) occurs for the function

$$f(z) = \int_0^z \frac{(1-\lambda)\,(1-(1-2\beta)t^p)}{1-\lambda+(1+(1-2\beta)\lambda)\,t^p} dt$$

and the equality in (1.9) *occurs for the function*

$$f(z)=\int_0^z \frac{(1-\lambda)\left(1-(1-2\beta)t^p e^{\Upsilon}\right)}{1-\lambda+(1+(1-2\beta)\lambda)\,t^p e^{\Upsilon}}dt,$$

where $0<b_{p+1}<1$, $\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}b_{p+1}\right)<0$ *and* $\Upsilon=\frac{1+t}{1+t}\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}b_{p+1}\right)$.

Proof. Let $b_{p+1}>0$. Let $p(z)$, $\phi(z)$ and $h(z)$ be as in the proof of Theorem 1.1. Having in mind inequality (1.7), we denote by $\ln p(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln p(0)=\ln\left(\frac{(1-\lambda)\,(p+1)}{2\,(1-\beta)}b_{p+1}\right)<0.$$

The auxiliary function

$$\Gamma(z)=\frac{\ln p(z)-\ln p(0)}{\ln p(z)+\ln p(0)}$$

is a holomorphic in D, $|\Gamma(z)|<1$ for $|z|<1$, $|\Gamma(0)|=0$ and $|\Gamma(\xi)|=1$ for $\xi\in\partial D$. It can be easily shown a non-tangential derivative of Γ at $\xi\in\partial D$ (see, [8]). Thus, the second non-tangential derivative of f at ξ is obtained. From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|\Gamma'(0)|}&\leq|\Gamma'(\xi)|\\&=\frac{|2\ln p(0)|}{|\ln p(\xi)+\ln p(0)|^2}\frac{|p'(\xi)|}{|p(\xi)|}\\&=\frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(\xi)}\left|\frac{\phi'(\xi)h(\xi)-h'(\xi)\phi(\xi)}{(h(\xi))^2}\right|\\&=\frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(\xi)}\left\{|\phi'(\xi)|-|h'(\xi)|\right\}.\end{aligned}$$

Replacing $\arg^2 p(\xi)$ by zero, then

$$\frac{1}{1+|\Gamma'(0)|}\leq\frac{-1}{\ln p(0)}\left\{|\phi'(\xi)|-|h'(\xi)|\right\}.$$

Since

$$\Gamma'(z)=\frac{2\ln p(0)}{(\ln p(z)+\ln p(0))^2}\frac{p'(z)}{p(z)},$$

$$\begin{aligned}|\Gamma'(0)|&=\frac{1}{2\,|\ln p(0)|}\left|\frac{p'(0)}{p(0)}\right|=-\frac{1}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)}\frac{\frac{(1-\lambda)(p+2)}{2(1-\beta)}\,|b_{p+2}|}{\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|}\\&=-\frac{1}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)}\frac{(p+2)\,|b_{p+2}|}{(p+1)\,|b_{p+1}|}\end{aligned}$$

and

$$|\phi'(\xi)| = \frac{2\,|g'(\xi)|}{|1+g(\xi)|^2} = \frac{2\,(1-\beta\lambda)^2\,|f''(\xi)|}{(1-\beta)\,(1-\lambda)},$$

we take

$$\frac{1}{1-\frac{1}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}|b_{p+1}|\right)}\frac{(p+2)|b_{p+2}|}{(p+1)|b_{p+1}|}}$$
$$\leq \frac{-1}{\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)}\left\{\frac{2\,(1-\beta\lambda)^2\,|f''(\xi)|}{(1-\beta)\,(1-\lambda)}-p\right\},$$

$$\frac{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)(p+1)\,|b_{p+1}|}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)(p+1)\,|b_{p+1}|-(p+2)\,|b_{p+2}|}$$
$$\leq \frac{-1}{\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)}\left\{\frac{2\,(1-\beta\lambda)^2\,|f''(\xi)|}{(1-\beta)\,(1-\lambda)}-p\right\}$$

and

$$p-\frac{2\left[\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)\right]^2(p+1)\,|b_{p+1}|}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)(p+1)\,|b_{p+1}|-(p+2)\,|b_{p+2}|}$$
$$\leq \frac{2\,(1-\beta\lambda)^2\,|f''(\xi)|}{(1-\beta)\,(1-\lambda)}.$$

Thus, we obtain the inequality (1.8) with an obvious equality case.

Similarly, $\Gamma(z)$ function satisfies the assumptions of the Schwarz lemma, we obtain

$$1 \geq |\Gamma'(0)| = \left|\frac{2\ln p(0)}{(\ln p(0)+\ln p(0))^2}\frac{p'(0)}{p(0)}\right| = \frac{1}{2\,|\ln p(0)|}\left|\frac{p'(0)}{p(0)}\right|$$
$$= -\frac{1}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)}\frac{(p+2)\,|b_{p+2}|}{(p+1)\,|b_{p+1}|}$$

and

$$1 \geq -\frac{1}{2\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)}\frac{(p+2)\,|b_{p+2}|}{(p+1)\,|b_{p+1}|}.$$

Therefore, we have

$$|b_{p+2}| \leq \frac{2}{(p+2)}\left|(p+1)b_{p+1}\ln\left(\frac{(1-\lambda)\,(p+1)}{2\,(1-\beta)}\,|b_{p+1}|\right)\right|.$$

We shall show that the inequality (1.9) is sharp. Let

$$f(z)=\int_0^z \frac{(1-\lambda)\left(1-(1-2\beta)t^p e^{\Upsilon}\right)}{1-\lambda+(1+(1-2\beta)\lambda)\,t^p e^{\Upsilon}}dt.$$

So, we get

$$f'(z)=\frac{(1-\lambda)\left(1-(1-2\beta)z^p e^{\Upsilon}\right)}{1-\lambda+(1+(1-2\beta)\lambda)\,z^p e^{\Upsilon}}$$

and

$$f'(z)=1+z^p\varpi(z),$$

where

$$\varpi(z)=-2\left(1-\beta\right)\frac{e^{\frac{1+z}{1-z}\ln\left(\frac{(1-\lambda)(1+p)}{2(1-\beta)}b_{p+1}\right)}}{1-\lambda+(1+(1-2\beta)\lambda)\,z^p e^{\frac{1+z}{1-z}\ln\left(\frac{(1-\lambda)(1+p)}{2(1-\beta)}b_{p+1}\right)}}.$$

Then

$$\varpi(0)=(p+1)\,b_{p+1}$$

and

$$\varpi'(0)=(p+2)\,b_{p+2}.$$

Under the simple calculations, we obtain

$$(p+2)\,b_{p+2}=-2\,(p+1)\,b_{p+1}\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}b_{p+1}\right)$$

and

$$|b_{p+2}|=\frac{2}{p+2}\left|(p+1)\,b_{p+1}\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}\,|b_{p+1}|\right)\right|. \qquad \square$$

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, we have*

$$|f''(\xi)|\geq\frac{(1-\beta)\,(1-\lambda)}{2\,(1-\beta\lambda)^2}\left(p-\frac{1}{2}\ln\left(\frac{(1-\lambda)\,(p+1)}{2\,(1-\beta)}\,|b_{p+1}|\right)\right). \tag{1.10}$$

The equality in (1.10) *holds if and only if*

$$f(z)=\int_0^z \frac{(1-\lambda)\left(1-(1-2\beta)t^p e^{\Upsilon}\right)}{1-\lambda+(1+(1-2\beta)\lambda)\,t^p e^{\Upsilon}}dt,$$

where $0<b_{p+1}<1$, $\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}b_{p+1}\right)<0$, $\Upsilon=\frac{1+te^{i\theta}}{1+te^{i\theta}}\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)}b_{p+1}\right)$ *and* θ *is a real number.*

Proof. The proof that f has the second non-tangential derivative at ξ is similar to the proof of Theorem1.2. Let $b_{p+1}>0$. Using the inequality (1.5) for the function $\Gamma(z)$, we obtain

$$1\leq|\Gamma'(\xi)|=\frac{|2\ln p(0)|}{|\ln p(\xi)+\ln p(0)|^2}\frac{|p'(\xi)|}{|p(\xi)|}=\frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(\xi)}\left\{|\phi'(\xi)|-|h'(\xi)|\right\}.$$

Replacing $\arg^2 p(\xi)$ by zero, then

$$1 \leq |\Gamma'(\xi)| = \frac{-2}{\ln p(0)} \left\{ \frac{2(1-\beta\lambda)^2 |f''(\xi)|}{(1-\beta)(1-\lambda)} - p \right\}$$

and

$$1 \leq \frac{-2}{\ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)} |b_{p+1}| \right)} \left\{ \frac{2(1-\beta\lambda)^2 |f''(\xi)|}{(1-\beta)(1-\lambda)} - p \right\}. \tag{1.11}$$

Therefore, we obtain the inequality (1.10).

If $|f''(\xi)| = \frac{(1-\beta)(1-\lambda)}{2(1-\beta\lambda)^2} \left(p - \frac{1}{2} \ln\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)} |b_{p+1}| \right) \right)$ from (1.11) and $|\Gamma'(\xi)| = 1$, we obtain

$$f(z) = \int_0^z \frac{(1-\lambda)\left(1-(1-2\beta)t^p e^{\Upsilon}\right)}{1-\lambda+(1+(1-2\beta)\lambda)\, t^p e^{\Upsilon}} dt.$$

□

Theorem 1.4. *Let $f(z) = z + b_{p+1}z^{p+1} + b_{p+2}z^{p+2} + \cdots$, $b_{p+1} \neq 0$, $p \geq 1$ be a holomorphic function in the unit disc D and $\Re\left(\frac{f'(z)}{\lambda f'(z)+1-\lambda} \right) > \beta$, $0 \leq \beta < 1$, $0 \leq \lambda < 1$ for $|z| < 1$. Suppose that, for some $\xi \in \partial D$, f' has a non-tangential limit $f'(\xi)$ at ξ and $f'(\xi) = \frac{\beta(1-\lambda)}{(1-\beta\lambda)}$. Let $a_1, a_2, \ldots, a_n$ be critical points of the function $f(z) - z$ in D that are different from zero. Then f has the second non-tangential derivative at ξ and*

$$|f''(\xi)| \geq \frac{(1-\beta)(1-\lambda)}{2(1-\beta\lambda)^2} \left(p + \sum_{k=1}^{n} \frac{1-|a_k|^2}{|\xi - a_k|^2} + \frac{2\left[2(1-\beta)\prod_{k=1}^{n}|a_k| - (1-\lambda)(p+1)|b_{p+1}| \right]^2}{\left(2(1-\beta)\prod_{k=1}^{n}|a_k| \right)^2 - ((1-\lambda)(p+1)|b_{p+1}|)^2 + 2(1-\beta)\prod_{k=1}^{n}|a_k|(p+2)|b_{p+2}|} \right). \tag{1.12}$$

In addition, the equality in (1.12) occurs for the function

$$f(z) = \int_0^z \frac{(1-\lambda)\left(1-(1-2\beta)t^p \prod_{k=1}^{n} \frac{t-a_k}{1-\overline{a_k}t}\right)}{1-\lambda+(1+(1-2\beta)\lambda)\, t^p \prod_{k=1}^{n} \frac{t-a_k}{1-\overline{a_k}t}} dt.$$

Proof. Let $\phi(z)$ be as in (1.1) and $a_1, a_2, \ldots, a_n$ be critical points of the function $f(z) - z$ in D that are different from zero. $B(z) = z^p \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}$ is a holomorphic function in the unit disc D and $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have $|\phi(z)| \leq |B(z)|$. Also, the function $\omega(z) = \frac{\phi(z)}{B(z)}$ is a holomorphic in D and $|\omega(z)| < 1$ for $|z| < 1$. In particular, we have

$$|\omega(0)| = \frac{(1-\lambda)(p+1)}{2(1-\beta)\prod\limits_{k=1}^{n}|a_k|}|b_{p+1}|$$

and

$$|\omega'(0)| = \frac{(1-\lambda)(p+2)}{2(1-\beta)\prod\limits_{k=1}^{n}|a_k|}|b_{p+2}|.$$

Moreover, it can be seen that

$$\frac{\xi\phi'(\xi)}{\phi(\xi)} = |\phi'(\xi)| \geq |B'(\xi)| = \frac{\xi B'(\xi)}{B(\xi)}.$$

It is obviously that

$$|B'(\xi)| = \frac{\xi B'(\xi)}{B(\xi)} = p + \sum_{k=1}^{n}\frac{1-|a_k|^2}{|\xi-a_k|^2}.$$

The composite function

$$\Phi(z) = \frac{\omega(z)-\omega(0)}{1-\overline{\omega(0)}\omega(z)}$$

is a holomorphic in the unit disc D, $|\Phi(z)| < 1$ for $|z| < 1$, $\Phi(0) = 0$ and $|\Phi(\xi)| = 1$ for $\xi \in \partial D$. It can be easily shown a non-tangential derivative of Φ at $\xi \in \partial D$ (see, [8]). Thus, the second non-tangential derivative of f at ξ is obtained. From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1+|\Phi'(0)|} \leq |\Phi'(\xi)| &= \frac{1-|\omega(0)|^2}{\left|1-\overline{\omega(0)}\omega(\xi)\right|^2}|\omega'(\xi)| \\
&\leq \frac{1+|\omega(0)|}{1-|\omega(0)|}\left|\frac{\phi'(\xi)B(\xi)-B'(\xi)\phi(\xi)}{(B(\xi))^2}\right| \\
&= \frac{1+|\omega(0)|}{1-|\omega(0)|}\left\{|\phi'(\xi)|-|B'(\xi)|\right\}.
\end{aligned}$$

Since

$$\Phi'(z) = \frac{1-|\omega(0)|^2}{\left(1-\overline{\omega(0)}\omega(z)\right)^2}\omega'(z)$$

and

$$|\Phi'(0)| = \frac{|\omega'(0)|}{1-|\omega(0)|^2} = \frac{\frac{(1-\lambda)(p+2)}{2(1-\beta)\prod\limits_{k=1}^{n}|a_k|}|b_{p+2}|}{1-\left(\frac{(1-\lambda)(p+1)}{2(1-\beta)\prod\limits_{k=1}^{n}|a_k|}|b_{p+1}|\right)^2}$$

$$= \frac{2\left(1-\beta\right)\prod\limits_{k=1}^{n}\left|a_k\right|\left(p+2\right)\left|b_{p+2}\right|}{\left(2\left(1-\beta\right)\prod\limits_{k=1}^{n}\left|a_k\right|\right)^2-\left(\left(1-\lambda\right)\left(p+1\right)\left|b_{p+1}\right|\right)^2},$$

we may write

$$\frac{2}{1+\frac{2(1-\beta)\prod\limits_{k=1}^{n}|a_k|(p+2)|b_{p+2}|}{\left(2(1-\beta)\prod\limits_{k=1}^{n}|a_k|\right)^2-((1-\lambda)(p+1)|b_{p+1}|)^2}}$$

$$\leq \frac{1+\frac{(1-\lambda)(p+1)}{2(1-\beta)\prod\limits_{k=1}^{n}|a_k|}\left|b_{p+1}\right|}{1-\frac{(1-\lambda)(p+1)}{2(1-\beta)\prod\limits_{k=1}^{n}|a_k|}\left|b_{p+1}\right|}\left\{\frac{2\left(1-\beta\lambda\right)^2\left|f''(\xi)\right|}{\left(1-\beta\right)\left(1-\lambda\right)}-\left(p+\sum_{k=1}^{n}\frac{1-\left|a_k\right|^2}{\left|\xi-a_k\right|^2}\right)\right\}.$$

Thus, we obtain the inequality (1.12) with an obvious equality case. □

References

[1] T. Aliyev Azeroğlu and B. N. Örnek, *A refined Schwarz inequality on the boundary*, Complex Var. Elliptic Equ. **58** (2013), no. 4, 571–577.

[2] H. P. Boas, *Julius and Julia*: *Mastering the art of the Schwarz lemma*, Amer. Math. Monthly **117** (2010), no. 9, 770–785.

[3] V. N. Dubinin, *The Schwarz inequality on the boundary for functions regular in the disk*, J. Math. Sci. **122** (2004), no. 6, 3623–3629.

[4] G. M. Golusin, *Geometric Theory of Functions of Complex Variable*, [in Russian], 2nd edn., Moscow, 1966.

[5] H. T. Kaptanoğlu, *Some refined Schwarz-Pick lemmas*, Michigan Math. J. **50** (2002), no. 3, 649–664.

[6] B. N. Örnek, *Sharpened forms of the Schwarz lemma on the boundary*, Bull. Korean Math. Soc. **50** (2013), no. 6, 2053–2059.

[7] R. Osserman, *A sharp Schwarz inequality on the boundary*, Proc. Amer. Math. Soc. **128** (2000), no. 12, 3513–3517.

[8] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin, 1992.

[9] H. Unkelbach, *Über die Randverzerrung bei konformer Abbildung*, Math. Z. **43** (1938), no. 1, 739–742.

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