Commun. Korean Math. Soc. **29** (2014), No. 3, pp. 429–437
http://dx.doi.org/10.4134/CKMS.2014.29.3.429

FURTHER HYPERGEOMETRIC IDENTITIES DEDUCIBLE BY FRACTIONAL CALCULUS

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Abstract. Motivated by the recent investigations of several authors, in this paper we present a generalization of a result obtained recently by Choi *et al.* ([3]) involving hypergeometric identities. The result is obtained by suitably applying fractional calculus method to a generalization of the hypergeometric transformation formula due to Kummer.

1. Introduction

The largely investigated generalized hypergeometric function ${}_pF_q$ with p numerator parameters $a_1, \dots, a_p$ such that $a_j \in \mathbb{C}$ $(j = 1, \dots, p)$ and q denominator parameters $b_1, \dots, b_q$ such that $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \dots, q;\ \mathbb{Z}_0^- := \mathbb{Z} \cup \{0\} = \{0, -1, -2, \dots\})$ is defined by (see, for example [17, Chapter 4]; see also [19, 20, 21])

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix}\ z\right] &= {}_pF_q[\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z] \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.1}$$

$(p \leq q$ and $|z| < \infty;\ p = q+1$ and $|z| < 1;\ p = q+1,\ |z| = 1$ and $\mathrm{Re}(\omega) > 0)$, where

$$\omega := \sum_{j=1}^{q} b_i - \sum_{j=1}^{p} a_i$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbb{N}; \alpha \in \mathbb{C}) \\ 1 & (n = 0; \alpha \in \mathbb{C} \setminus \{0\}). \end{cases}$$

Received February 21, 2014; Revised May 2, 2014.
2010 *Mathematics Subject Classification.* 26A33, 33C70, 33C90, 33C05.
Key words and phrases. fractional derivatives, generalized hypergeometric function, Kummer's formula, beta integral method, generalization of Kummer's formula.

Many years ago, Kummer [9, p. 81, Entry 72] and Ramanujan [1, p. 64, Entry 21] presented independently the following relationship involving the Gauss hypergeometric function:

$$
\begin{aligned}
(1.2)\qquad & {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}(1+z)\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{1}{2}(a+1)\right)\Gamma\left(\frac{1}{2}(b+1)\right)}\, {}_2F_1\left[\begin{matrix} \frac{a}{2},\ \frac{b}{2}; \\ \frac{1}{2}; \end{matrix}\ z^2\right] \\
&\quad + \frac{2z\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}(a+b+1)\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\, {}_2F_1\left[\begin{matrix} \frac{a}{2}+\frac{1}{2},\ \frac{b}{2}+\frac{1}{2}; \\ \frac{3}{2}; \end{matrix}\ z^2\right]
\end{aligned}
$$

with $z \in \mathbb{U}$ where $\mathbb{U}$ denotes the *open* unit disk, that is,

$$
\mathbb{U} = \{z:\ z \in \mathbb{C} \text{ and } |z| < 1\}.
$$

Recently, Choi *et al.* [2, Equation (2.1)] obtained the following generalization of Kummer's formula:

(1.3)

$$
\begin{aligned}
& {}_2F_1\left[\begin{matrix} a,\ b; \\ \frac{1}{2}(a+b+\ell+1); \end{matrix}\ \frac{1}{2}(1+z)\right] \\
=& \frac{\Gamma\left(\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}|\ell|\right)} \\
& \cdot \left[\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_j\ \left(\frac{1}{2}a+\frac{1}{2}\right)_j\ \left(\frac{1}{2}b\right)_j\ \left(\frac{1}{2}b+\frac{1}{2}\right)_j}{\left(\frac{1}{2}\right)_j\ j!}\, z^{2j}\right. \\
& \cdot \left\{\frac{C_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)\ \left(\frac{1}{2}a+\frac{1}{2}\right)_j\ \left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
& \left. + \frac{D_\ell}{\Gamma\left(\frac{1}{2}a\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\ \left(\frac{1}{2}a\right)_j\ \left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\} \\
& + \frac{abz}{2}\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_j\ \left(\frac{1}{2}a+1\right)_j\ \left(\frac{1}{2}b+\frac{1}{2}\right)_j\ \left(\frac{1}{2}b+1\right)_j}{\left(\frac{3}{2}\right)_j\ j!}\, z^{2j} \\
& \cdot \left\{\frac{E_\ell}{\Gamma\left(\frac{1}{2}a+1\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+1-\left[\frac{\ell+1}{2}\right]\right)\ \left(\frac{1}{2}a+1\right)_j\ \left(\frac{1}{2}b+1+\frac{1}{2}\ell-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
& \left.\left. + \frac{F_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\ \left(\frac{1}{2}a+\frac{1}{2}\right)_j\ \left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\}\right] \\
& \qquad\qquad (z \in \mathbb{U};\ \ell = 0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5),
\end{aligned}
$$

where $[x]$ denotes the greatest integer less than or equal to x and the coefficients C_ℓ, D_ℓ and E_ℓ, F_ℓ are, respectively, given in Tables 1 and 2 below.

TABLE 1

ℓ	C_ℓ	D_ℓ
5	$-(b+a+4j+6)^2+\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+6)$ $+11(b+a+4j+6)$ $-\frac{13}{2}(b-a+6)-20$	$(b+a+4j+6)^2-\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+6)$ $-17(b+a+4j+6)$ $-\frac{1}{2}(b-a+6)+62$
4	$\frac{1}{2}(b+a+4j+1)(b+a+4j-3)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a+4j-1)$
3	$-\frac{1}{2}(3a+b+8j-2)$	$\frac{1}{2}(3b+a+8j-2)$
2	$\frac{1}{2}(b+a++4j-1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a+4j-1)$	2
-3	$\frac{1}{2}(3a+b+8j-2)$	$\frac{1}{2}(3b+a+8j-2)$
-4	$\frac{1}{2}(b+a+4j+1)(b+a+4j-3)$ $-\frac{1}{2}(b-a+3)(b-a-3)$	$2(b+a+4j-1)$
-5	$(b+a+4j-4)^2-\frac{1}{4}(b-a-4)^2$ $-\frac{1}{2}(b+a+4j-4)(b-a-4)$ $+4(b+a+4j-4)-\frac{7}{2}(b-a-4)$	$(b+a+4j-4)^2-\frac{1}{4}(b-a-4)^2$ $+\frac{1}{2}(b+a+4j-4)(b-a-4)$ $+8(b+a+4j-4)-\frac{1}{2}(b-a-4)$ $+12$

Many authors [3, 8, 27] obtained several transformations formulas involving hypergeometric functions as well as their multi-variables analogues by using the so-called Beta integral method. The beta function $B(\alpha,\beta)$ is defined by the following integral representation:

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \quad (\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0). \tag{1.4}$$

TABLE 2

ℓ	E_ℓ	F_ℓ
5	$-(b+a+4j+8)^2+\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+8)$ $+11(b+a+4j+8)$ $-\frac{13}{2}(b-a+6)-20$	$(b+a+4j+8)^2-\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4j+8)$ $-17(b+a+4j+8)$ $-\frac{1}{2}(b-a+6)+62$
4	$\frac{1}{2}(b+a+4j+3)(b+a+4j-1)$ $-\frac{1}{2}(b-a+3)(b-a-3)$	$-2(b+a+4j+1)$
3	$-\frac{1}{2}(3a+b+8j+2)$	$\frac{1}{2}(3b+a+8j+2)$
2	$\frac{1}{2}(b+a+4j+1)$	-2
1	-1	1
0	1	0
-1	1	1
-2	$\frac{1}{2}(b+a+4j+1)$	2
-3	$\frac{1}{2}(3a+b+8j+2)$	$\frac{1}{2}(3b+a+8j+2)$
-4	$\frac{1}{2}(b+a++4j+3)(b+a+4j-1)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$2(b+a+4j+1)$
-5	$(b+a+4j-2)^2-\frac{1}{4}(b-a-4)^2$ $-\frac{1}{2}(b+a+4j-2)(b-a-4)$ $+4(b+a+4j-2)-\frac{7}{2}(b-a-4)$	$(b+a+4j-2)^2-\frac{1}{4}(b-a-4)^2$ $+\frac{1}{2}(b+a+4j-2)(b-a-4)$ $+8(b+a+4j-2)-\frac{1}{2}(b-a-4)$ $+12$

By making use of this method in conjunction with formula (1.3), Choi *et al.* [3] obtained a generalization of a result due to Krattenthaler and Rao [8].

In this paper, we present a generalization of the result obtained by Choi *et al.* [3] by using fractional calculus technique. In Section 2, we give the representation of the fractional derivatives based on the Pochhammer's contour of integration. Section 3 is devoted to the proof of the main result and to the presentation of some special cases.

2. Pochhammer contour integral representation for fractional derivative and a new generalized Leibniz rule

The use of contour of integration in the complex plane provides a very powerful tool in both classical and fractional calculus. The most familiar representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [4, 6, 11, 18] that is

$$D_z^\alpha z^p f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi)\xi^p (\xi - z)^{-\alpha-1} d\xi, \tag{2.1}$$

which is valid for $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(p) > 1$ and where the integration is done along a straight line from 0 to z in the ξ-plane. By integrating by part m times, we obtain

$$D_z^\alpha z^p f(z) = \frac{d^m}{dz^m} D_z^{\alpha-m} z^p f(z). \tag{2.2}$$

This allows to modify the restriction $\mathrm{Re}(\alpha) < 0$ to $\mathrm{Re}(\alpha) < m$ [18]. Another used representation for the fractional derivative is the one based on the Cauchy integral formula widely used by Osler [13, 14, 15, 16]. These two representations have been used in many interesting research papers. It appears that the less restrictive representation of fractional derivative according to parameters is the Pochhammer's contour definition introduced in [10, 22] (see also [5, 23, 24, 25, 26]).
It is well known that [12, p. 83, Equation (2.4)]

$$D_z^\alpha z^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \qquad (\mathrm{Re}(p) > -1), \tag{2.3}$$

but adopting the Pochhammer based representation for the fractional derivative this last restriction reduces to p not a negative integer.

An important thing to mention here is the following relationship

$$B(\alpha, \beta) = \Gamma(\beta) D_z^{-\beta} z^{\alpha-1} \Big|_{z=1} \tag{2.4}$$

that exhibits the fact that the beta integral method is, in fact, a special case of the fractional calculus method.

3. Main result

In this section, we apply fractional calculus method to the generalization of the Kummer's transformation obtained by Choi *et al.* (1.3) in order to obtain the new relation more general than the one obtained by means of the beta integral method.

Theorem 3.1. *The following generalization of the Choi-Rathie-Srivastava formula* [3, p. 1677, Equation (3.2)] *holds true*:

(3.1)

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(a)_k (b)_k \; {}_2F_1\left[\begin{matrix} \alpha,\ \gamma-k; \\ \beta; \end{matrix}\; z\right]}{\left(\frac{1}{2}(a+b+\ell+1)\right)_k \; 2^k \; k!} \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\ell+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}|\ell|\right)} \\
&\cdot \left[\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_j \left(\frac{1}{2}a+\frac{1}{2}\right)_j \left(\frac{1}{2}b\right)_j \left(\frac{1}{2}b+\frac{1}{2}\right)_j \left(\frac{1}{2}\alpha\right)_j \left(\frac{1}{2}\alpha+\frac{1}{2}\right)_j z^{2j}\; {}_2F_1\left[\begin{matrix} \alpha+2j,\ \gamma; \\ \beta+2j; \end{matrix}\; z\right]}{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\beta\right)_j \left(\frac{1}{2}\beta+\frac{1}{2}\right)_j\, j!}\right. \\
&\cdot \left\{\frac{C_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)\left(\frac{1}{2}a+\frac{1}{2}\right)_j \left(\frac{1}{2}b+\frac{1}{2}\ell+\frac{1}{2}-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
&\left. +\frac{D_\ell}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\left(\frac{1}{2}a\right)_j \left(\frac{1}{2}b+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\} \\
&-\frac{abz\alpha}{2\beta}\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_j \left(\frac{1}{2}a+1\right)_j \left(\frac{1}{2}b+\frac{1}{2}\right)_j \left(\frac{1}{2}b+1\right)_j}{\left(\frac{3}{2}\right)_j \left(\frac{1}{2}\beta+\frac{1}{2}\right)_j \left(\frac{1}{2}\beta+1\right)_j\, j!} \\
&\cdot \left(\frac{1}{2}\alpha+\frac{1}{2}\right)_j \left(\frac{1}{2}\alpha+1\right)_j z^{2j} \cdot {}_2F_1\left[\begin{matrix} 1+\alpha+2j,\ \gamma; \\ 1+\beta+2j; \end{matrix}\; z\right] \\
&\cdot \left\{\frac{E_\ell}{\Gamma\left(\frac{1}{2}a+1\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\ell+1-\left[\frac{\ell+1}{2}\right]\right)\left(\frac{1}{2}a+1\right)_j \left(\frac{1}{2}b+1+\frac{1}{2}\ell-\left[\frac{\ell+1}{2}\right]\right)_j}\right. \\
&\left.\left. +\frac{F_\ell}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)\left(\frac{1}{2}a+\frac{1}{2}\right)_j \left(\frac{1}{2}b+\frac{1}{2}+\frac{1}{2}\ell-\left[\frac{\ell}{2}\right]\right)_j}\right\}\right]
\end{aligned}$$

$$(|z| \leq 1;\ \ell = 0,\ \pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 5),$$

for β not a negative integer and where $[x]$ denotes the greatest integer less than or equal to x and the coefficients C_ℓ, D_ℓ and E_ℓ, F_ℓ are, respectively, given in Tables 1 *and* 2 *below.*

Proof. Replace z by $-z$ in (1.3), multiply both sides of the resulting identity by $(1-z)^{-\gamma}$ where γ is an arbitrary complex numbers and apply the following operator $\frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D_z^{\alpha-\beta} z^{\alpha-1}$ on both sides of the resulting identity by expressing the ${}_2F_1$ as series and changing the order of integration and summation (which is justified by the uniform convergence of the series involved) yields the result. $\square$

Now let us examine some special cases of formula (3.1). Setting $z = 1$, $l = 1$, $a = \frac{1}{2}$ and $b = \frac{3}{2}$ in formula (3.1) and appealing to Gauss summation formula [17], we obtain the result given by Choi *et al.* in [3, p. 1677, Equation (3.2)].

If we put $l = 0$, $\gamma = 1 + \alpha - \beta$ and $z = -1$ in (3.1) and appealing to the summation formula due to Kummer [9, p. 134, Entry 1], we obtain, after some simple simplifications, the following interesting special case:

$$
\begin{aligned}
(3.2)\qquad &\sum_{k=0}^{\infty} \frac{(a)_k (b)_k \; {}_2F_1\left[\begin{matrix} \alpha,\ 1+\alpha-\beta-k; \\ \beta; \end{matrix} \; -1\right]}{\left(\frac{1}{2}(a+b+1)\right)_k \; 2^k \; k!} \\
&= \frac{2^{-\alpha}\{\Gamma\left(\frac{1}{2}\right)\}^2 \, \Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma(\beta) \; {}_3F_2\left[\begin{matrix} \frac{1}{2}a,\ \frac{1}{2}b,\ \frac{1}{2}\alpha; \\ \frac{1}{2},\ \beta-\frac{1}{2}\alpha; \end{matrix} \; 1\right]}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\beta-\frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\alpha+\frac{1}{2}\right)} \\
&\quad + \frac{2^{-\alpha-2}\, ab\alpha\{\Gamma\left(\frac{1}{2}\right)\}^2 \, \Gamma\left(\frac{1}{2}(a+b+1)\right)\Gamma(\beta) \; {}_3F_2\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2},\ \frac{1}{2}b+\frac{1}{2},\ \frac{1}{2}\alpha+\frac{1}{2}; \\ \frac{3}{2},\ \beta-\frac{1}{2}\alpha+\frac{1}{2}; \end{matrix} \; 1\right]}{\Gamma\left(\frac{1}{2}a+1\right)\Gamma\left(\beta-\frac{1}{2}\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+1\right)\Gamma\left(\frac{1}{2}\alpha+1\right)}.
\end{aligned}
$$

Putting $z = 2$, $\gamma = 0$, $\beta = 2\alpha$, $l = 0$ in (3.1), assuming that b is an even negative integer and using a result due to [7, Equation (3.8)], the following formula holds true:

$$
\begin{aligned}
(3.3)\qquad &\sum_{k=0}^{\infty} \frac{(a)_k (b)_k \; {}_2F_1\left[\begin{matrix} \alpha,\ -k; \\ 2\alpha; \end{matrix} \; 2\right]}{\left(\frac{1}{2}(a+b+1)\right)_k \; 2^k \; k!} \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(1-\alpha)\Gamma\left(\frac{1}{2}\right) \; \Gamma\left(-\frac{1}{2}k+\frac{1}{2}\right)\Gamma\left(1-\alpha-\frac{1}{2}k\right) \; (a)_k (b)_k}{\left(\frac{1}{2}(a+b+1)\right)_k (2\alpha)_k \; k!} \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \; {}_4F_3\left[\begin{matrix} \frac{1}{2}a,\ \frac{1}{2}b,\ \frac{1}{2}\alpha,\ \frac{1}{2}\alpha+\frac{1}{2}; \\ \frac{1}{2},\ \alpha,\ \alpha+\frac{1}{2}; \end{matrix} \; 4\right].
\end{aligned}
$$

References

[1] B. C. Berndt, *Ramanujan's Notebooks. Parts II*, Springer-Verlag, Berlin, Heidelberg and New York, 1989.

[2] J. Choi, A. K. Rathie, and H. M. Srivastava, *A generalization of a formula due to Kummer*, Integral Transforms Spec. Funct. **22** (2011), no. 11, 851–859.

[3] ———, *Certain hypergeometric identities deducible by using the beta integral method*, Bull. Korean Math. Soc. **50** (2013), no. 5, 1673–1681.

[4] A. Erdélyi, *An integral equation involving Legendre polynomials*, SIAM J. Appl. Math. **12** (1964), 15–30.

[5] S. Gaboury, *Some relations involving generalized Hurwitz-Lerch zeta function obtained by means of fractional derivatives with applications to Apostol-type polynomials*, Adv. Difference Equ. **2013** (2013), 361.

[6] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[7] Y. S. Kim and A. K. Rathie, *Some results for terminating ${}_2F_1(2)$ series*, J. Inequal. Appl. **365** (2011), 1–12.
[8] C. Krattenthaler and K. S. Rao, *Automatic generation of hypergeometric identities by the beta integral method*, J. Comput. Appl. Math. **160** (2003), no. 1-2, 159–173.
[9] E. E. Kummer, *Über die hypergeometrische Reihe* $1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+\cdots$, J. Reine Angew. Math. **15** (1836), 39–83; 127–172.
[10] J.-L. Lavoie, T. J. Osler, and R. Tremblay, *Fundamental properties of fractional derivatives via Pochhammer integrals*, Lecture Notes in Mathematics, Springer-Verlag, 1975.
[11] J. Liouville, *Mémoire sur le calcul des différentielles à indices quelconques*, J. de l'École Polytechnique **13** (1832), 71–162.
[12] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, New York, Chichester, Brisbane, Toronto and Singapore, John Wiley and Sons, 1993.
[13] T. J. Osler, *Fractional derivatives of a composite function*, SIAM J. Math. Anal. **1** (1970), 288–293.
[14] ———, *Leibniz rule for fractional derivatives and an application to infinite series*, SIAM J. Appl. Math. **18** (1970), 658–674.
[15] ———, *Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives*, PhD thesis, New York University, 1970.
[16] ———, *Fractional derivatives and Leibniz rule*, Amer. Math. Monthly **78** (1970), 645–649.
[17] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960.
[18] M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta. Math. **81** (1949), 1–223.
[19] H. M. Srivastava and J. Choi, *Series Associated with Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
[20] ———, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
[21] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwoor Limited), John Wiley and Sons, Chichester and New York, 1985.
[22] R. Tremblay, *Une contribution à la théorie de la dérivée fractionnaire*, PhD thesis, Laval University, Canada, 1974.
[23] R. Tremblay and B.-J. Fugère, *The use of fractional derivatives to expand analytical functions in terms of quadratic functions with applications to special functions*, Appl. Math. Comput. **187** (2007), no. 1, 507–529.
[24] R. Tremblay, S. Gaboury, and B.-J. Fugère, *A new leibniz rule and its integral analogue for fractional derivatives*, Integral Transforms Spec. Funct. **24** (2013), no. 2, 111–128.
[25] ———, *A new transformation formula for fractional derivatives with applications*, Integral Transforms Spec. Funct. **24** (2013), no. 3, 172–186.
[26] ———, *Taylor-like expansion in terms of a rational function obtained by means of fractional derivatives*, Integral Transforms Spec. Funct. **24** (2013), no. 1, 50–64.
[27] C. Wei, X. Wang, and Y. Li, *Certain transformations for multiple hypergeometric functions*, Adv. Difference Equ. **360** (2013), 1–13.

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