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THE SPECTRAL CONTINUITY OF ESSENTIALLY HYPONORMAL OPERATORS

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Abstract. If $\mathfrak{A}$ is a unital Banach algebra, then the spectrum can be viewed as a function $\sigma : \mathfrak{A} \to \mathfrak{S}$, mapping each $T \in \mathfrak{A}$ to its spectrum $\sigma(T)$, where $\mathfrak{S}$ is the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. This paper is concerned with the continuity of the spectrum σ via Browder's theorem. It is shown that σ is continuous when σ is restricted to the set of essentially hyponormal operators for which Browder's theorem holds, that is, the Weyl spectrum and the Browder spectrum coincide.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ write $\sigma(T)$ and $\sigma_p(T)$ for the spectrum and the set of eigenvalues of T, respectively. Let $\mathfrak{S}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. If $\mathfrak{A}$ is a unital Banach algebra, then the spectrum can be viewed as a function $\sigma : \mathfrak{A} \to \mathfrak{S}$, mapping each $T \in \mathfrak{A}$ to its spectrum $\sigma(T)$. It is known that the function σ is upper semicontinuous and that in noncommutative algebras, σ does have points of discontinuity. J. Newburgh [17] gave the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [7] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is $\mathcal{B}(\mathcal{H})$. It seems to be interesting and challenging to identify classes $\mathfrak{C}$ of operators for which σ becomes continuous when restricted to $\mathfrak{C}$. The first result of this study is: σ is continuous on the set of normal operators. On the other hand, Newburgh's argument uses the fact that the inverses of normal resolvents are normaloid (cf. see Solution 105 of [10]) and this argument is extended to the set of hyponormal operators because the inverses of hyponormal resolvents are also hyponormal and hence normaloid.

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In [15], it was shown that σ is continuous when restricted to the set of p-hyponormal operators. In [9] and [14], the continuity of σ was explored when σ is restricted to certain subsets of Toeplitz operators. In [4], it was shown that σ is discontinuous on the entire manifold of Toeplitz operators. Also in [16], the first author and E. Kwon have considered the continuity of σ when restricted to the set of essentially p-hyponormal operators. The following is still challengeable:

Problem. Find subsets $\mathfrak{C}$ of $\mathcal{B}(\mathcal{H})$ for which the spectrum σ, a set-valued function, is continuous when restricted to $\mathfrak{C}$.

In this paper we show that the spectrum σ is continuous when restricted to the set of essentially hyponormal operators for which Browder's theorem holds. This set includes all commuting compact perturbations of hyponormal operators.

If $\{T_n\}$ is a sequence of elements in a unital Banach algebra $\mathfrak{A}$, then $\liminf_n \sigma(T_n)$ is the set of all limit points of convergent sequences of the form $\{\lambda_n\}$, where $\lambda_n \in \sigma(T_n)$ for each n. Because the set of invertible elements in $\mathfrak{A}$ forms an open set, we can see that $\liminf_n \sigma(T_n) \subseteq \sigma(T)$ whenever the sequence of elements T_n converges to T in $\mathfrak{A}$. Thus proving the spectral continuity is to show equality in this relation.

If $T \in \mathcal{B}(\mathcal{H})$ write $N(T)$ for the null space of T; $R(T)$ for the range of T; $\alpha(T)$ for the nullity of T, i.e., $\alpha(T) := \dim N(T)$; $\beta(T)$ for the deficiency of T, i.e., $\beta(T) := \dim \overline{R(T)}^{\perp}$. If $\mathfrak{S}$ is a compact subset of $\mathfrak{C}$, write $\operatorname{iso}\mathfrak{S}$ for the isolated points of $\mathfrak{S}$; $\operatorname{acc}\mathfrak{S}$ for the accumulation points of $\mathfrak{S}$; $\partial\,\mathfrak{S}$ for the topological boundary of $\mathfrak{S}$. We recall ([11]) that an operator $T \in \mathcal{B}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range with finite dimensional null space, and *lower semi-Fredholm* if it has closed range with its range of finite co-dimension. If T is either upper or lower semi-Fredholm we call it *semi-Fredholm*, and *Fredholm* if it is both. The *index* of a semi-Fredholm operator T is given by the equality $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If $T \in \mathcal{B}(\mathcal{H})$, the left-[the right-] essential spectrum $\sigma_e^+(T)$ [$\sigma_e^-(T)$] of T is the set of all complex numbers λ such that $T - \lambda$ is not upper semi-Fredholm [lower semi-Fredholm], and the essential spectrum $\sigma_e(T)$ is the union of $\sigma_e^+(T)$ and $\sigma_e^-(T)$. Those Fredholm operators that have index zero are called Weyl operators. The Weyl spectrum, denoted $\omega(T)$ of $T \in \mathcal{B}(\mathcal{H})$ is the set of all complex numbers λ for which $T - \lambda$ is not Weyl.

2. The main result

In this section we prove the main theorem. To do so we need preliminary notions and results.

Hyponormal elements in a C^*-algebra. In a unital C^*-algebra A, an element $x \in A$ is called *normal* if $x^*x = xx^*$ and is called *hyponormal* if

$x^*x \geq xx^*$. If $A = \mathcal{B}(\mathcal{H})$, then it is familiar that every hyponormal element $x \in A$ satisfies:

1. $x - \lambda$ is hyponormal for all $\lambda \in \mathbb{C}$;
2. x^{-1} is hyponormal if x is invertible;
3. x is normaloid, i.e., $||x|| = r_A(x)$, where $r_A(x) = \sup_{\lambda \in \sigma_A(x)} |\lambda|$ (called the spectral radius of x with respect to A), where $\sigma_A(x)$ of x is defined by the set of all complex numbers λ for which $x - \lambda$ has no inverse in A.

These properties are also enjoyed for hyponormal elements in a unital C^*-algebra.

Lemma 2.1. *Let A be a unital C^*-algebra. If $x \in A$ is hyponormal, then $(x-\lambda)^{-1}$ is normaloid for $\lambda \notin \sigma(x)$.*

Proof. Suppose x is hyponormal. Evidently, $x-\lambda$ is hyponormal. If in addition x is invertible, then we can see that x^{-1} is also hyponormal by using the fact that if $a, b \in A$, $0 \leq a \leq b$, and a is invertible, then b is invertible and $b^{-1} \leq a^{-1}$. Thus it suffices to show that if x is hyponormal, then it is normaloid. By the Gelfand-Naimark representation theorem there exists a Hilbert space $\mathcal{K}$ and an isometric $*$-representation $\varphi : A \to \mathcal{B}(\mathcal{K})$. Since every $*$-homomorphism preserves positivity it follows that $\varphi(x)$ is also a hyponormal operator in $\mathcal{B}(\mathcal{K})$. Thus we have

$$||x|| = ||\varphi(x)|| = r_{\mathcal{L}(\mathcal{K})}(\varphi(x)) = r_{\varphi(A)}(\varphi(x)) = r_A(x),$$

where the third equality follows from the spectral permanence for C^*-algebras and the last equality comes from the observation that $\varphi : A \to \varphi(A)$ is an isometric isomorphism. This proves the lemma. □

Let $\mathfrak{K}(\mathcal{H})$ denote the ideal of compact operators on $\mathcal{H}$ and let π denote the canonical map of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$, which is a unital C^*-algebra. From the classical Fredholm theory we have $\sigma_e(T) = \sigma(\pi(T))$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *essentially hyponormal* [*essentially normal*] if $\pi(T)$ is hyponormal [normal] in $\mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$. We write

$$\mathfrak{E}\mathfrak{H}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : \ T \text{ is essentially hyponormal}\}.$$

We then have:

Corollary 2.2. *If $T \in \mathfrak{E}\mathfrak{H}(\mathcal{H})$, then $\big(\pi(T-\lambda)\big)^{-1}$ is normaloid in $\mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ for $\lambda \notin \sigma_e(T)$.*

Proof. Immediate from Lemma 2.1. □

Browder's theorem. H. Weyl [20] examined the spectra of all compact perturbations $A + K$ of a single hermitian operator A and discovered that $\lambda \in \sigma(A + K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(A)$. Today this result is known as Weyl's theorem, and it has been extended from hermitian operators A to hyponormal operators and to Toeplitz operators by L. Coburn [6], to seminormal operators by S. Berberian [1], and to abundant classes of operators by many authors. On the other hand, Weyl's theorem is expressed as follows:

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T),$$

where

$$\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \dim N(T - \lambda) < \infty\}.$$

On the other hand, an operator $T \in \mathcal{B}(\mathcal{H})$ is called *Browder* if T is Fredholm "of finite ascent and descent": equivalently, if T is Fredholm and $T - \lambda$ is invertible for sufficiently small $|\lambda| \neq 0$ in $\mathbb{C}$ (cf. [11]). The Browder spectrum $\sigma_b(T)$ of T is defined by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : \; T - \lambda \text{ is not Browder}\}.$$

Evidently,

$$\omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T).$$

If we write

$$P_{00}(T) := \sigma(T) \setminus \sigma_b(T)$$

for the *Riesz points* of $\sigma(T)$, then we say [12] that *Browder's theorem holds for* T if

$$\sigma(T) \setminus \omega(T) = P_{00}(T).$$

Evidently Weyl's theorem implies Browder's theorem. The following statements are equivalent (cf. [12, Theorem 2]):

1. Browder's theorem holds for T;
2. $\sigma(T) = \omega(T) \cup \pi_{00}(T)$;
3. $\omega(T) = \sigma_b(T)$.

By comparison with Weyl's theorem, Browder's theorem holds for quasinilpotent operators, compact operators and algebraic operators ([12, Theorem 9]). We write

$$\mathfrak{BT}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : \text{ Browder's theorem holds for } T\}.$$

The set $\mathfrak{BT}(\mathcal{H})$ has something to do with the invariant subspace problem, which is the open question whether every operator in $\mathcal{B}(\mathcal{H})$ has nontrivial invariant subspace. If $\omega(T) \neq \sigma_b(T)$, then there exists a complex number λ such that $T - \lambda$ is Weyl but not invertible, so that λ is an eigenvalue for T; therefore T has a nontrivial invariant subspace. Thus the operators that remain to show in the invariant subspace problem are included in $\mathfrak{BT}(\mathcal{H})$. Recently, many authors have considered $\mathfrak{BT}(\mathcal{H})$ and gave interesting spectral properties via $\mathfrak{BT}(\mathcal{H})$. In this paper we provide a connection between the spectral property and $\mathfrak{BT}(\mathcal{H})$.

We now have:

Theorem 2.3. *The restriction of σ to $\mathfrak{EH}(\mathcal{H})$ is continuous at each point of $\mathfrak{BT}(\mathcal{H})$.*

Proof. If $T \in \mathcal{B}(\mathcal{H})$, write $m_e(T)$ for the essential minimum modulus of T (cf. [2], [3], [13]): i.e., $m_e(T) = \inf \sigma_e(|T|)$, where $|T|$ denotes $(T^*T)^{\frac{1}{2}}$. It is known from [2, Theorem 2] that

$$m_e(T) > 0 \iff T \text{ is upper semi-Fredholm.} \tag{1}$$

On the other hand, the essential minimum modulus can be viewed as a function $m_e : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$, mapping each operator T to its essential minimum modulus $m_e(T)$. Then we can see that m_e is continuous: indeed, if T_n converges to T, then $|T_n|$ converges to $|T|$ ([15, Lemma 1]) and $\lim \sigma_e(|T_n|) = \lim \sigma(\pi(|T_n|)) = \sigma(\pi(|T|)) = \sigma_e(|T|)$ because σ is continuous on the set of all normal elements in a unital C^*-algebra ([17, Corollary 2]), which implies that $\lim m_e(T_n) = m_e(T)$. We now claim that

$$T \in \mathfrak{EH}(\mathcal{H}) \Longrightarrow m_e(T - \lambda) = \operatorname{dist}(\lambda, \sigma_e(T)) \text{ for } \lambda \notin \sigma_e(T). \tag{2}$$

To prove (2) suppose $T \in \mathfrak{EH}(\mathcal{H})$ and $0 \notin \sigma_e(T)$. Then by Corollary 2.2, $\pi(T)^{-1}$ is normaloid. Note that

$$\sigma(\pi(T)^{-1}) = \{\mu^{-1} : \mu \in \sigma(\pi(T))\} = \{\mu^{-1} : \mu \in \sigma_e(T)\}.$$

Since $\pi(x^{\frac{1}{2}}) = \big(\pi(x)\big)^{\frac{1}{2}}$ for every positive element x in $\mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$, we have that $m_e(T) = \inf\, \sigma(|\pi(T)|)$. We thus argue that if $\mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ is regarded as a C^*-subalgebra of $\mathcal{B}(\mathcal{K})$ for a Hilbert space $\mathcal{K}$, then

$$\begin{aligned} m_e(T) &= \inf \sigma(|\pi(T)|) \\ &= \inf \{||\pi(T)x|| : ||x|| = 1,\ x \in \mathcal{K}\} \\ &= \frac{1}{||\pi(T)^{-1}||} = \frac{1}{\max_{\lambda \in \sigma(\pi(T)^{-1})} |\lambda|} \\ &= \min_{\lambda \in \sigma_e(T)} |\lambda| \\ &= \operatorname{dist}(0, \sigma_e(T)). \end{aligned}$$

Applying this result with $T - \lambda$ in place of T proves (2).

Now suppose that $T_n \in \mathfrak{EH}(\mathcal{H})$ for $n \in \mathbb{Z}_+$, and $T \in \mathfrak{BT}(\mathcal{H})$ are such that T_n converges to T. Since $\liminf_n \sigma(T_n) \subseteq \sigma(T)$, it suffices to show that $\sigma(T) \subseteq \liminf_n \sigma(T_n)$. If $\lambda \in \operatorname{iso} \sigma(T)$, then by an argument of Newburgh [17, lemma 3], for every neighborhood $\mathcal{N}(\lambda)$ of λ there exists an $N \in \mathbb{Z}_+$ such that $n > N$ implies $\sigma(T_n) \cap \mathcal{N}(\lambda) \neq \emptyset$, which says that $\lambda \in \liminf_n \sigma(T_n)$. We now suppose $\lambda \in \operatorname{acc} \sigma(T)$. We assume to the contrary that $\lambda \notin \liminf_n \sigma(T_n)$. Then there exists a neighborhood $\mathcal{N}(\lambda)$ of λ such that does not intersect infinitely many $\sigma(T_n)$. Thus we can choose a subsequence $\{T_{n_k}\}_k$ of $\{T_n\}_n$ such that for some $\epsilon > 0$,

$$\operatorname{dist}(\lambda, \sigma(T_{n_k})) > \epsilon \quad \text{for all } k \in \mathbb{Z}_+.$$

Since evidently, $\mathrm{dist}(\lambda, \sigma(T_{n_k})) \le \mathrm{dist}(\lambda, \sigma_e(T_{n_k}))$ it follows from (2) that $m_e(T_{n_k} - \lambda) > \epsilon$ for all $k \in \mathbb{Z}_+$. Since m_e is continuous we have that $m_e(T - \lambda) \ge \epsilon$ which by (1), implies that $T - \lambda$ is upper semi-Fredholm. Therefore by index continuity we have that

$$\mathrm{ind}(T - \lambda) = \lim_{k\to\infty} \mathrm{ind}(T_{n_k} - \lambda) = 0,$$

which implies that $T-\lambda$ is Weyl. Therefore $\lambda \notin \omega(T)$. But since by assumption Browder's theorem holds for T it follows that $\lambda \notin \sigma_b(T)$, which implies $\lambda \in \mathrm{iso}\,\sigma(T)$ because $\lambda \in \sigma(T)$. This contradicts our assumption $\lambda \in \mathrm{acc}\,\sigma(T)$. This completes the proof. □

Example 2.4. (a) It is known ([12, Theorem 10]) that if T is reduced by its finite-dimensional eigenspaces, then Browder's theorem holds for T. Thus by Theorem 2.3 we can see that the restriction of σ to the set of all essentially hyponormal operators which are reduced by their finite-dimensional eigenspaces is continuous.

(b) The restriction of σ to the set of all compact perturbations of hyponormal operators need not to be continuous. For example, if on $\ell_2 \oplus \ell_2$

$$T_n = \begin{pmatrix} U & \frac{1}{n}(I - UU^*) \\ 0 & U^* \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix},$$

where U is the unilateral shift on ℓ_2, then we have that T and T_n, for $n \in \mathbb{Z}_+$, are compact perturbations of a hyponormal operator (in fact, rank-one perturbations of the unitary operator $\left(\begin{smallmatrix} U & I-UU^* \\ 0 & U^* \end{smallmatrix}\right)$) and T_n converges to T, whereas $\sigma(T_n)$ is the unit circle for all n and $\sigma(T)$ is the unit disk. Note that T is essentially hyponormal, but Browder's theorem fails for T. If we denote $\mathfrak{H}(\mathcal{H})$ for the set of all hyponormal operators on $\mathcal{H}$ and let

$$\mathfrak{H}(\mathcal{H}) \uplus \mathfrak{K}(\mathcal{H}) := \{T + K : T \in \mathfrak{H}(\mathcal{H}),\ K \in \mathfrak{K}(\mathcal{H}) \text{ and } TK = KT\}$$

for a sort of "commuting sum" of $\mathfrak{H}(\mathcal{H})$ and $\mathfrak{K}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$, then Browder's theorem holds for all operators in $\mathfrak{H}(\mathcal{H}) \uplus \mathfrak{K}(\mathcal{H})$ (cf. [12, Theorem 11]). Therefore by Theorem 2.3, we can conclude that the restriction of σ to $\mathfrak{H}(\mathcal{H}) \uplus \mathfrak{K}(\mathcal{H})$ is continuous.

(c) We claim that if $T \in \mathcal{B}(\mathcal{H})$ is totally of finite ascent, in the sense that

$$\mathrm{ascent}(T - \lambda) < \infty \quad \text{for any } \lambda \in \mathbb{C},$$

then Browder's theorem holds for T. Indeed, writing T in place of $T-\lambda$ enables us to reduce the discussion to $\lambda = 0$. Suppose T is Weyl. We want to show that $0 \in \mathrm{iso}\,\sigma(T)$. The finite ascent condition says that for some $n \in \mathbb{N}$, the operator T^n has ascent $= 1$. Thus we can write

$$T^n = \begin{pmatrix} 0 & 0 \\ 0 & (T^n)^\wedge \end{pmatrix} : \begin{pmatrix} P^{-1}(0) \\ P(\mathcal{H}) \end{pmatrix} \longrightarrow \begin{pmatrix} P^{-1}(0) \\ P(\mathcal{H}) \end{pmatrix},$$

where P is a projection with $P^{-1}(0) = T^{-1}(0)$ and $T^n(P(\mathcal{H})) = T^n(\mathcal{H})$. Then $(T^n)^\wedge$ is bounded below. But since $(T^n)^\wedge$ is also Weyl, it follows that $(T^n)^\wedge$

is onto, and hence $(T^n)^\wedge$ is invertible. Thus $T^n - \lambda$ is invertible for a nonzero sufficient small $|\lambda|$. Thus $0 \in \operatorname{iso}\sigma(T^n)$, so that $0 \in \operatorname{iso}\sigma(T)$. This shows that Browder's theorem holds for T. Therefore, by Theorem 2.3, we can see that that the restriction of σ to the set of all essentially hyponormal operators which are totally of finite ascent is continuous.

(d) The Toeplitz operator T_φ with symbol $\varphi \in L^\infty(\mathbb{T})$ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ is defined by

$$T_\varphi g := P(\varphi g) \quad (g \in H^2(\mathbb{T})),$$

where P denotes the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. It was known ([6]) that every Toeplitz operator obeys Weyl's theorem, and hence Browder's theorem it follows from again Theorem 2.3 that the restriction of σ to the set of all essentially hyponormal Toeplitz operators is continuous.

(e) An examination of the proof of Theorem 2.3 shows that if we write

$$\mathfrak{M}(\mathcal{H}) := \{T \in \mathcal{B}(\mathcal{H}) : m_e(T - \lambda) = \operatorname{dist}(\lambda, \sigma_e(T)) \text{ for } \lambda \notin \sigma_e(T)\},$$

then the restriction of σ to $\mathfrak{M}(\mathcal{H})$ is continuous at each point of $\mathfrak{B}\mathfrak{T}(\mathcal{H})$. In particular $\mathfrak{E}\mathfrak{H}(\mathcal{H}) \subseteq \mathfrak{M}(\mathcal{H})$.

(f) The following are the basic properties of Toeplitz operators:

1. Every Toeplitz operator has connected spectrum ([21]);
2. $\sigma_p(T_\varphi) \cap \overline{\sigma_p(T_{\overline{\varphi}})} = \emptyset$ for every non-constant Toeplitz operator T_φ ([6]);
3. $\sigma(T_\varphi) = \omega(T_\varphi)$ (by (2)).

The properties for Toeplitz operators which we require in the below are described in [5], [8], and [18]. Let $\mathfrak{T}_w$ denote the set of all Toeplitz operators with symbols in $w \subseteq L^\infty(\mathbb{T})$. Write $C(\mathbb{T})$ for all continuous complex-valued functions on $\mathbb{T}$ and $H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$. Then the subspace $H^\infty + C(\mathbb{T})$ is a closed subalgebra of $L^\infty(\mathbb{T})$. If $\varphi \in H^\infty + C(\mathbb{T})$, then $T_\psi T_\varphi - T_{\psi\varphi} \in \mathfrak{K}(H^2)$ for any $\psi \in L^\infty(\mathbb{T})$, so that $\sigma_e(|T_\varphi|) = \big(\sigma_e(T_{|\varphi|^2})\big)^{\frac{1}{2}} = [\operatorname{ess\,inf}|\varphi|,\ \operatorname{ess\,sup}|\varphi|]$, which implies that $m_e(T_\varphi) = \operatorname{ess\,inf}|\varphi|$. By (1), $\sigma_e^+(T_\varphi) = \{\lambda \in \mathbb{C} : m_e(T_{\varphi-\lambda}) = 0\} = \text{ess-ran}(\varphi)$, which implies that if $0 \notin \sigma_e(T_\varphi)$ then $m_e(T_\varphi) = \operatorname{ess\,inf}|\varphi| = \operatorname{dist}(0, \sigma_e^+(T_\varphi)) = \operatorname{dist}(0, \sigma_e(T_\varphi))$, and therefore we can determine

$$\mathfrak{T}_{H^\infty + C(\mathbb{T})} \subseteq \mathfrak{M}(H^2). \tag{3}$$

This together with (e) above recaptures [14, Theorem 10].

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