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FIBONACCI NUMBERS AND SEMISIMPLE CONTINUED FRACTION

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Abstract. The ratios of any two Fibonacci numbers are expressed by means of semisimple continued fraction.

1. Introduction

The Fibonacci sequence $\{F_n\}_{n\geq 1}$ is a series of numbers that begins with $F_1 = F_2 = 1$ and each next is the sum of the previous two terms. The Lucas sequence $\{L_n\}_{n\geq 1}$ is a modified Fibonacci sequence starting from $L_1 = 1$ and $L_2 = 3$. A number of properties of these sequences were studied by many researchers. Among them, the ratios $\frac{F_n}{F_{n-1}}$ and $\frac{L_n}{L_{n-1}}$ of two successive terms of each sequence were investigated by means of simple continued fraction ([2], [3], [4] and [6]).

This work is devoted to studying the ratio $\frac{F_n}{F_k}$ for any n and k. For the purpose, a semisimple continued fraction will be defined and compared to a simple continued fractions. We shall show that $\frac{F_n}{F_k}$ is expressed by a semisimple continued fraction more efficiently than by a simple continued fraction. And it will be seen that semisimple continued fraction may yield any large Fibonacci numbers, like F_{105} a 22 digit number.

2. Semisimple continued fractions

We begin with a lemma that provides a motivation of this work.

Lemma 2.1 ([5]). *Let k, t and r be positive integers. If $t \geq 2$ and $r \leq k$, then we have the following relations.*

1. (1) $L_{kt+r} = L_k\ L_{k(t-1)+r} + (-1)^{k-1}L_{k(t-2)+r}$, *so L_n ($n = kt + r$) is expressed by only three Lucas numbers L_k, L_r and L_{k+r}.*
2. (2) $F_{kt+r} = L_k\ F_{k(t-1)+r} + (-1)^{k-1}F_{k(t-2)+r}$, *so F_n ($n = kt + r$) is expressed by L_k, F_r and F_{k+r}.*

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Note that r may be considered as the remainder of n when divided by k, but we assume $1 \leq r \leq k$ because the sequences start from $F_1 = L_1 = 1$.

If $k = 5$, then $L_{17} = L_5 L_{12} + L_7$ and $L_{12} = L_5 L_7 + L_2$, so

$$\frac{L_{17}}{L_{12}} = \frac{L_5 L_{12} + L_7}{L_{12}} = L_5 + \frac{1}{\frac{L_{12}}{L_7}} = L_5 + \frac{1}{\frac{L_5 L_7 + L_2}{L_7}} = L_5 + \frac{1}{L_5 + \frac{1}{\frac{L_7}{L_2}}}$$

and again

$$\frac{L_{22}}{L_{17}} = \frac{L_5 L_{17} + L_{12}}{L_{17}} = L_5 + \frac{1}{\frac{L_{17}}{L_{12}}} = L_5 + \frac{1}{L_5 + \frac{1}{L_5 + \frac{1}{\frac{L_7}{L_2}}}}.$$

These fractions yield a motivation to define a sort of continued fraction composed of only Lucas numbers. Let $n \geq 2$. For real numbers $a_0, b_0 \geq 0$ and $a_i > 0, b_i > 1$ $(i = 1, \ldots, n)$, the fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, a_{n-2} + \cfrac{1}{\cfrac{a_{n-1}}{a_n}}}} \quad \text{and} \quad b_0 - \cfrac{1}{b_1 - \cfrac{1}{\ddots\, b_{n-2} - \cfrac{1}{\cfrac{b_{n-1}}{b_n}}}}$$

are called (*minus*) *semisimple continued fractions* denote by $\langle\langle a_0; a_1, \ldots, a_n \rangle\rangle$ the former and $[[b_0; b_1, \ldots, b_n]]$ the latter respectively. We also define

$$\langle\langle a_0; a_1 \rangle\rangle = [[a_0; a_1]] = \frac{a_0}{a_1}.$$

When every a_i are Lucas [resp. Fibonacci] numbers, the semisimple continued fraction is called Lucas [resp. Fibonacci] continued fraction. For instance, $\frac{L_{22}}{L_{17}}$ equals the Lucas continued fraction $\langle\langle L_5; L_5, L_5, L_7, L_2 \rangle\rangle$. This provides a good reason to define the semisimple continued fraction.

We may compare these fractions to the (minus) simple continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} \quad \text{and} \quad b_0 - \cfrac{1}{b_1 - \cfrac{1}{\ddots\, \cfrac{1}{b_{n-1} - \cfrac{1}{b_n}}}},$$

where $a_0, b_0 \geq 0$ and $a_i > 0$, $b_i > 1$ $(i = 1, \ldots, n)$, denoted by $\langle a_0; a_1, \ldots, a_n \rangle$ and $[b_0; b_1, \ldots, b_n]$, respectively (refer to [1]).

Theorem 2.2. *Let $n \geq 2$. For the (minus) semisimple continued fractions,*

(1)

$$\begin{aligned} \langle\langle a_0; \ldots, a_n \rangle\rangle &= a_0 + \langle\langle 0; a_1, \ldots, a_n \rangle\rangle \\ &= a_0 + \frac{1}{\langle\langle a_1; \ldots, a_n \rangle\rangle} \end{aligned}$$

$$= \frac{1}{\langle\langle 0; a_0, \dots, a_n \rangle\rangle}$$
$$= \langle\langle a_0; \dots, a_{n-3}, a_{n-2}a_{n-1} + a_n, a_{n-1} \rangle\rangle$$
$$= \langle\langle a_0; \dots, a_{n-2}, \frac{a_{n-2}a_{n-1}}{a_{n-2}a_{n-1} + a_n} \rangle\rangle.$$

(2)

$$\begin{aligned}[[a_0; \dots, a_n]] &= a_0 + [[0; a_1, \dots, a_n]] \\ &= a_0 - \frac{1}{[[a_1; \dots, a_n]]} \\ &= -\frac{1}{[[0; a_0, \dots, a_n]]} \\ &= \langle\langle a_0; \dots, a_{n-2}, \lfloor \frac{a_{n-1}}{a_n} \rfloor, a_n, a_{n-1} - \lfloor \frac{a_{n-1}}{a_n} \rfloor a_n \rangle\rangle.\end{aligned}$$

Proof. Since

$$\begin{aligned}\langle\langle a_0; \dots, a_{n-2}, a_{n-1}, a_n \rangle\rangle &= a_0 + \cfrac{1}{\ddots\, a_{n-3} + \cfrac{1}{a_{n-2} + \frac{1}{\frac{a_{n-1}}{a_n}}}} \\ &= a_0 + \cfrac{1}{\ddots\, a_{n-3} + \cfrac{1}{\frac{a_{n-2}a_{n-1} + a_n}{a_{n-1}}}} \\ &= a_0 + \cfrac{1}{\ddots\, a_{n-3} + \cfrac{1}{\frac{a_{n-2}}{x}}}\end{aligned}$$

with $x = \frac{a_{n-2}a_{n-1}}{a_{n-2}a_{n-1}+a_n}$, (1) follows immediately.

Now for (2), write $a_{n-1} = qa_n + r$ with $0 \le r < a_n$ and $q \in \mathbb{Z}$. Then

$$\begin{aligned}\langle\langle a; a_{n-1}, a_n \rangle\rangle = a + \frac{1}{\frac{a_{n-1}}{a_n}} = a + \cfrac{1}{q + \cfrac{1}{\frac{a_n}{r}}} &= \langle\langle a; q, a_n, r \rangle\rangle \\ &= \langle\langle a; \lfloor \frac{a_{n-1}}{a_n} \rfloor, a_n, a_{n-1} - \lfloor \frac{a_{n-1}}{a_n} \rfloor a_n \rangle\rangle.\end{aligned}$$

□

Theorem 2.3. *We further have the following identities.*

(1) $\langle\langle a_0; \dots, a_n, 1 \rangle\rangle = \langle a_0; \dots, a_n \rangle$, $[[a_0; \dots, a_n, 1]] = [a_0; \dots, a_n]$ *for* $n \ge 1$.

(2) $\langle\langle a_0; \dots, a_n \rangle\rangle = \langle a_0; \dots, a_{n-2}, \frac{a_{n-1}}{a_n} \rangle = \langle a_0; \dots, a_{n-3}, a_{n-2} + \frac{a_n}{a_{n-1}} \rangle$ *for* $n \ge 3$.

(3) $\langle\langle a_0; a_1, \dots, a_n \rangle\rangle = \big\langle a_0; a_1, \dots, a_k, \langle\langle a_{k+1}; \dots, a_n \rangle\rangle \big\rangle$ *for* $0 \le k \le n-2$.

Proof. The proof is not hard. □

Example. $\langle\langle 1; 2, 3, 4, 5, 6\rangle\rangle = \langle\langle 1; 2, 3, 26, 5\rangle\rangle = \langle\langle 1; 2, 83, 26\rangle\rangle = \langle\langle 1; 192, 83\rangle\rangle = \frac{275}{192}$ by Theorems 2.2 and 2.3. On the other hand, $\langle\langle 1; 2, 3, 4, 5, 6\rangle\rangle$ is also equal to $\langle\langle 1; 2, 3, 4, \frac{10}{13}\rangle\rangle = \langle\langle 1; 2, 3, \frac{78}{83}\rangle\rangle = \langle\langle 1; 2, \frac{83}{96}\rangle\rangle = \langle\langle 1; \frac{192}{275}\rangle\rangle = \frac{275}{192}$.

Similar to the usual notation $\langle a^t, b\rangle = \langle\underbrace{a; \dots, a}_{t}, b\rangle$, we denote the t times repeated semisimple continued fractions $\langle\langle a; \dots, a, b\rangle\rangle$ and $[[a; \dots, a, b]]$ by $\langle\langle a^t, b\rangle\rangle$ and $[[a^t, b]]$, respectively.

3. Convergents of a semisimple continued fraction

The section is devoted to investigating successive convergents of a semisimple continued fraction, and to studying relationships with those of a simple continued fraction.

Given a continued fraction $\langle a_0; a_1, \dots, a_n\rangle$, a numerator u_k and a denominator v_k of the kth convergent C_k are given by the recursive formulas

$$u_k = u_{k-1}a_k + u_{k-2} \quad \text{and} \quad v_k = v_{k-1}a_k + v_{k-2} \quad (k \geq 1),$$

where $u_{-1} = 1$, $u_0 = a_0$ and $v_{-1} = 0$, $v_0 = 1$. Then (refer to [1])

$$C_k = \langle a_0; a_1, \dots, a_k\rangle = \frac{u_k}{v_k} \quad \text{and} \quad u_k v_{k-1} - u_{k-1} v_k = (-1)^{k+1}.$$

The next theorem shows a recursion formula involving the semisimple continued fraction. For every $a_k > 0$, let

$$S_n = \langle\langle a_0; a_1, \dots, a_n\rangle\rangle.$$

Theorem 3.1. *Let $n \geq 2$. Then for all $2 \leq k \leq n$, the following are equivalent for the kth convergent S_k of S_n.*

(1) $S_k = \langle\langle a_0; a_1, \dots, a_k\rangle\rangle = \frac{p_k}{q_k}$.

(2) $p_k = u_{k-2}a_{k-1} + u_{k-3}a_k$ *and* $q_k = v_{k-2}a_{k-1} + v_{k-3}a_k$.

(3)
$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} a_{k-2} & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_{k-1} & a_{k-1} \\ a_n & a_{k-2} - a_{k-2}a_{k-1} \end{bmatrix}$$
$$= \begin{bmatrix} u_{k-2} & u_{k-3} \\ v_{k-2} & v_{k-3} \end{bmatrix}\begin{bmatrix} a_{k-1} & a_{k-1} \\ a_k & a_{k-2} - a_{k-2}a_{k-1} \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}.$$

Proof. If $k = 2$, then

$$S_2 = \langle\langle a_0; a_1, a_2\rangle\rangle = \frac{a_0a_1 + a_2}{a_1} = \frac{p_2}{q_2}$$

implies $p_2 = a_0a_1 + a_2 = u_0a_1 + u_{-1}a_2$ and $q_2 = a_1 = v_0a_1 + v_{-1}a_2$. Furthermore

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 & a_1 \\ a_2 & a_0 - a_0a_1 \end{bmatrix} = \begin{bmatrix} u_0 & u_{-1} \\ v_0 & v_{-1} \end{bmatrix}\begin{bmatrix} a_1 & a_1 \\ a_2 & a_0 - a_0a_1 \end{bmatrix}$$
$$= \begin{bmatrix} a_0a_1 + a_2 & a_0 \\ a_1 & a_1 \end{bmatrix} = \begin{bmatrix} p_2 & p_1 \\ q_2 & q_1 \end{bmatrix}.$$

On the other hand, Theorem 2.3(2) together with the consideration about the continued fraction give rise to

$$\begin{aligned} S_2 &= \langle\langle a_0; a_1, a_2\rangle\rangle = \langle a_0; \frac{a_1}{a_2}\rangle = \frac{u_0\frac{a_1}{a_2} + u_{-1}}{v_0\frac{a_1}{a_2} + v_{-1}} \\ &= \frac{(a_0a_1 + a_2)/a_2}{a_1/a_2} = \frac{a_0a_1 + a_2}{a_1} = \frac{p_2}{q_2}. \end{aligned}$$

We now suppose that (1), (2) and (3) hold for S_k for $k < n$. Then

$$\begin{aligned} S_k &= \langle\langle a_0; a_1, \dots, a_{k-1}, a_k\rangle\rangle = \langle a_0; a_1, \dots, a_{k-2}, \frac{a_{k-1}}{a_k}\rangle \\ &= \frac{u_{k-1}}{v_{k-1}} = \frac{u_{k-2}\frac{a_{k-1}}{a_k} + u_{k-3}}{v_{k-2}\frac{a_{k-1}}{a_k} + v_{k-3}} \\ &= \frac{u_{k-2}a_{k-1} + u_{k-3}a_k}{v_{k-2}a_{k-1} + v_{k-3}a_k} = \frac{p_k}{q_k}. \end{aligned}$$

Moreover it follows inductively that

$$\begin{aligned} &\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{k-3} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k-2} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{k-1} & a_{k-1} \\ a_k & a_{k-2} - a_{k-2}a_{k-1} \end{bmatrix} \\ = &\begin{bmatrix} u_{k-3}a_{k-2} + u_{k-4} & u_{k-3} \\ v_{k-3}a_{k-2} + v_{k-4} & v_{k-3} \end{bmatrix} \begin{bmatrix} a_{k-1} & a_{k-1} \\ a_k & a_{k-2} - a_{k-2}a_{k-1} \end{bmatrix} \\ = &\begin{bmatrix} u_{k-2} & u_{k-3} \\ v_{k-2} & v_{k-3} \end{bmatrix} \begin{bmatrix} a_{k-1} & a_{k-1} \\ a_k & a_{k-2} - a_{k-2}a_{k-1} \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} \end{aligned}$$

if and only if $p_k = u_{k-2}a_{k-1} + u_{k-3}a_k$ and $q_k = v_{k-2}a_{k-1} + v_{k-3}k_n$. □

Example. Consider $\frac{p_5}{q_5} = \langle\langle 1; 2, 3, 4, 5, 6\rangle\rangle$. Since $\langle 1; 2, 3\rangle = \frac{10}{7} = \frac{u_2}{v_2}$ and $\langle 1; 2, 3, 4\rangle = \frac{43}{30} = \frac{u_3}{v_3}$, $\left[\begin{smallmatrix} p_5 \\ q_5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 40 & 10 \\ 30 & 7 \end{smallmatrix}\right]\left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 275 \\ 192 \end{smallmatrix}\right]$ so $\langle\langle 1; 2, 3, 4, 5, 6\rangle\rangle = \frac{275}{192}$.

From now on, we denote the semisimple continued fraction $\langle\langle a_0; a_1, \dots, a_n\rangle\rangle$ by S_n and the continued fraction $\langle a_0; a_1, \dots, a_n\rangle$ by C_n where $0 < a_i \in \mathbb{Z}$. Then for every $k < n$, $S_k = \frac{p_k}{q_k}$ and $C_k = \frac{u_k}{v_k}$ are the kth convergents of S_n and C_n respectively, where u_k, v_k, p_k and q_k are in Theorem 3.1.

Lemma 3.2. *$C_k - C_{k-1} = \frac{(-1)^{k+1}}{v_{k-1}v_k}$ and $C_k - C_{k-2} = \frac{(-1)^{k+1}}{v_{k-2}v_k}a_k$ for $k \geq 1$.*

Proof. It is clear to see that

$$C_k - C_{k-1} = \frac{u_k}{v_k} - \frac{u_{k-1}}{v_{k-1}} = \frac{u_kv_{k-1} - u_{k-1}v_k}{v_{k-1}v_k} = \frac{(-1)^{k+1}}{v_{k-1}v_k}.$$

Furthermore, since $v_k = v_{k-1}a_k + v_{k-2}$ we have

$$C_k - C_{k-2} = \frac{(-1)^{k+1}}{v_{k-1}v_k} + \frac{(-1)^k}{v_{k-1}v_{k-2}} = \frac{(-1)^k v_{k-1}a_k}{v_{k-2}v_{k-1}v_k} = \frac{(-1)^k a_k}{v_{k-2}v_k}.$$

□

It shows $C_0 < C_2 < C_4 < \cdots < C_5 < C_3 < C_1$ and $\{C_k\}$ converges to C_n for large enough n ([1]). Similarly, we shall investigate the differences $S_k - S_{k-1}$ and $S_k - S_{k-2}$.

Theorem 3.3. *Let $\delta_k = a_{k-1}a_k - a_{k-1} + a_{k+1}$. Then for $k \geq 1$,*

(1) $S_k - S_{k-1} = \frac{(-1)^k}{q_{k-1}q_k} a_{k-1}\delta_{k-1}$.

(2) $S_k - S_{k-2} = \frac{(-1)^k}{q_{k-2}q_k}(a_{k-1}\delta_{k-1} - a_{k-2}\delta_{k-2} - \delta_{k-2}\delta_{k-1})$.

Proof. By Theorem 3.1, we have

$$\begin{bmatrix} u_{k-2} & u_{k-3} \\ v_{k-2} & v_{k-3} \end{bmatrix} \begin{bmatrix} a_{k-1} & a_{k-1} \\ a_k & a_{k-2} - a_{k-2}a_{k-1} \end{bmatrix} = \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix}.$$

Then the determinants of both sides yield

$$p_k q_{k-1} - p_{k-1}q_k = (-1)^k a_{k-1}(a_{k-2}a_{k-1} - a_{k-2} + a_k).$$

Now by setting $\delta_{k-1} = a_{k-2}a_{k-1} - a_{k-2} + a_k$, we have

$$S_k - S_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1}q_k}{q_{k-1}q_k} = \frac{(-1)^k a_{k-1}\delta_{k-1}}{q_{k-1}q_k}.$$

Again

$$\begin{aligned} S_k - S_{k-2} &= \frac{(-1)^k a_{k-1}\delta_{k-1}}{q_{k-1}q_k} + \frac{(-1)^{k-1} a_{k-2}\delta_{k-2}}{q_{k-2}q_{k-1}} \\ &= \frac{(-1)^k}{q_{k-2}q_{k-1}q_k}(a_{k-1}\delta_{k-1}q_{k-2} - a_{k-2}\delta_{k-2}q_k). \end{aligned}$$

From the identities $v_k = v_{k-1}a_k + v_{k-2}$ and $q_k = v_{k-2}a_{k-1} + v_{k-3}a_k$ in Theorem 3.1, we have

$$\begin{aligned} q_k &= (v_{k-3}a_{k-2} + v_{k-4})a_{k-1} + v_{k-3}a_k \\ &= v_{k-3}(a_{k-2}a_{k-1} + a_k) + v_{k-4}a_{k-1} \\ &= v_{k-3}(a_{k-2}a_{k-1} + a_k) + (q_{k-1} - v_{k-3}a_{k-2}) \\ &= v_{k-3}(a_{k-2}a_{k-1} - a_{k-2} + a_k) + q_{k-1} \;=\; v_{k-3}\delta_{k-1} + q_{k-1}. \end{aligned}$$

Thus

$$\begin{aligned} &a_{k-1}\delta_{k-1}q_{k-2} - a_{k-2}\delta_{k-2}q_k \\ &= q_{k-1}(a_{k-1}\delta_{k-1} - a_{k-2}\delta_{k-2}) - \delta_{k-2}\delta_{k-1}(v_{k-3}a_{k-2} + v_{k-4}a_{k-1}) \\ &= q_{k-1}(a_{k-1}\delta_{k-1} - a_{k-2}\delta_{k-2}) - \delta_{k-2}\delta_{k-1}q_{k-1} \\ &= q_{k-1}(a_{k-1}\delta_{k-1} - a_{k-2}\delta_{k-2} - \delta_{k-2}\delta_{k-1}), \end{aligned}$$

so

$$\begin{aligned} S_k - S_{k-2} &= \frac{(-1)^k}{q_{k-2}q_{k-1}q_k} q_{k-1}(a_{k-1}\delta_{k-1} - a_{k-2}\delta_{k-2} - \delta_{k-2}\delta_{k-1}) \\ &= \frac{(-1)^k}{q_{k-2}q_k}(a_{k-1}\delta_{k-1} - a_{k-2}\delta_{k-2} - \delta_{k-2}\delta_{k-1}). \end{aligned}$$

□

Theorem 3.4. *$S_3 < S_5 < S_7 < \cdots < S_6 < S_4 < S_2$, so $\{S_k\}$ converges to S_n for large enough n.*

Proof. Obviously $q_k > 0$ and $\delta_k = a_{k-1}(a_k - 1) + a_{k+1} > 0$ for every k. Thus $S_k - S_{k-1} = \frac{(-1)^k}{q_{k-1}q_k} a_{k-1}\delta_k$ shows $S_{k-1} < S_k$ if k is even, otherwise $S_k < S_{k-1}$. Furthermore we have

$$\begin{aligned}
& a_{k-1}\delta_{k-1} - a_{k-2}\delta_{k-2} - \delta_{k-2}\delta_{k-1} \\
= & - a_{k-2}a_{k-1} - a_{k-3}a_{k-2}^2 a_{k-1} + a_{k-3}a_{k-2}a_{k-1} - a_{k-3}a_{k-2}a_k + a_{k-3}a_k \\
= & - a_{k-2}a_{k-1}(1 + a_{k-3}(a_{k-2} - 1)) - a_{k-3}a_k(a_{k-2} - 1) \\
= & - (a_{k-3}(a_{k-2} - 1)(a_{k-2}a_{k-1} + a_k) + a_{k-2}a_{k-1}) < 0.
\end{aligned}$$

Thus from

$$S_k - S_{k-2} = \frac{(-1)^{k+1}}{q_k q_{k-2}}(a_{k-3}(a_{k-2} - 1)(a_{k-2}a_{k-1} + a_k) + a_{k-2}a_{k-1}),$$

we have $S_k < S_{k-2}$ for even k, otherwise $S_{k-2} < S_k$. Therefore

$$S_3 < S_5 < S_7 < \cdots < S_6 < S_4 < S_2,$$

so S_k converges when k gets larger. □

For example, from $\pi = \langle a_0; a_1, a_2, \ldots\rangle = \langle 3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots\rangle$, the first few convergents of continued fraction are

$$C_0 = \langle 3\rangle = 3 = \frac{u_0}{v_0}, \quad C_1 = \langle 3; 7\rangle = \frac{22}{7} = \frac{u_1}{v_1}.$$

Moreover

$$C_2 = \langle 3; 7, 15\rangle = \frac{u_1 a_2 + u_0}{v_1 a_2 + v_0} = \frac{22 \cdot 15 + 3}{7 \cdot 15 + 1} = \frac{333}{106} = \frac{u_2}{v_2},$$

and similarly

$$C_3 = \langle 3; 7, 15, 1\rangle = \frac{355}{113} = \frac{u_3}{v_3}, \; C_4 = \frac{103993}{33102}, \; C_5 = \frac{104348}{33215}, \ldots.$$

Let us consider $S = \langle\langle a_0; a_1, \ldots\rangle\rangle = \langle\langle 3; 7, 15, 1, 292, 1, 1, 1, 2, \ldots\rangle\rangle$. Then

$$S_1 = \langle\langle 3; 7\rangle\rangle = \frac{3}{7} = \frac{p_1}{q_1} \quad \text{and} \quad S_2 = \langle\langle 3; 7, 15\rangle\rangle = \frac{36}{7} = \frac{p_2}{q_2}.$$

And Theorem 3.1 shows that

$$S_3 = \langle\langle 3; 7, 15, 1\rangle\rangle = \frac{p_3}{q_3} = \frac{u_1 a_2 + u_0 a_3}{v_1 a_2 + v_0 a_3} = \frac{22 \cdot 15 + 3 \cdot 1}{7 \cdot 15 + 1 \cdot 1} = \frac{333}{106}$$

and

$$S_4 = \frac{333 \cdot 1 + 22 \cdot 292}{106 \cdot 1 + 7 \cdot 292} = \frac{6757}{2150}, \; S_5 = \frac{355 \cdot 292 + 333}{113 \cdot 292 + 106} = \frac{103993}{33102}, \ldots.$$

Now the differences of convergents of C_n are, for instance

$$C_4 - C_3 = \frac{-1}{(113)(33102)}, \; C_5 - C_4 = \frac{1}{(33102)(33215)} \; \text{ and } C_4 - C_2 = \frac{292}{(106)(33102)}.$$

Moreover Theorem 3.3 shows

$$S_5 - S_4 = \frac{-292(292+1-1)}{2150 \cdot 33102} = \frac{-292^2}{71169300}$$

and

$$S_6 - S_4 = \frac{-1}{2150 \cdot 33215}((292+1)(292-1)+292) = \frac{85555}{71412250}.$$

Corollary 3.5. *Let $S_n = \langle\langle a_0; a_1, \ldots, a_n \rangle\rangle = \frac{p_n}{q_n}$. Assume $a_k = 1$ for $k < n$. Then $S_{k+1} - S_k = \frac{(-1)^{k+1} a_{k+1}}{q_k q_{k+1}}$ and $S_{k+2} - S_k = \frac{(-1)^{k+1} a_{k+1}}{q_k q_{k+2}}$.*

Proof. From Theorem 3.3, we have

$$S_{k+1} - S_k = \frac{(-1)^{k+1}}{q_k q_{k+1}}(a_k(a_{k-1}(a_k - 1) + a_{k+1})) = \frac{(-1)^{k+1} a_{k+1}}{q_k q_{k+1}}.$$

Similarly,

$$\begin{aligned} S_{k+2} - S_k &= \frac{(-1)^{k+1}}{q_k q_{k+2}}(a_{k-1}(a_k - 1)(a_k a_{k+1} + a_{k+2}) + a_k a_{k+1}) \\ &= \frac{(-1)^{k+1} a_{k+1}}{q_k q_{k+2}}. \end{aligned}$$

□

From $S = \langle\langle 3; 7, 15, 1, 292, 1, 1, \ldots \rangle\rangle$ in the above example, since $a_3 = 1$ and $a_4 = 292$, we have $S_4 - S_3 = \frac{292}{q_3 q_4}$ and $S_5 - S_3 = \frac{292}{q_3 q_5}$.

4. Ratios of Fibonacci numbers in semisimple continued fraction

It is well known that the ratio $\frac{F_{n+1}}{F_n}$ is equal to $\langle 1^n \rangle = \langle 1; 1, \ldots, 1 \rangle$. This section is devoted to study $\frac{F_n}{F_k}$ for any n and k. The fractions $\langle \cdots \rangle$, $[\cdots]$, $\langle\langle \cdots \rangle\rangle$ or $[[\cdots]]$ will be put together to express the ratios effectively.

Lemma 4.1. *Let $n \geq 3$. Then $F_{n+1} = 2F_n - F_{n-2} = 2F_{n-1} + F_{n-2}$ and $F_{n+2} = 3F_n - F_{n-2} = L_2 F_n - F_{n-2}$.*

Proof. Obviously $F_{n+1} = F_n + F_{n-1} = F_n + (F_n - F_{n-2}) = 2F_n - F_{n-2}$, while $F_{n+1} = F_n + F_{n-1} = (F_{n-1} + F_{n-2}) + F_{n-1} = 2F_{n-1} + F_{n-2}$. And $F_{n+2} = F_{n+1} + F_n = (2F_n - F_{n-2}) + F_n = 3F_n - F_{n-2}$.

Moreover due to Lemma 2.1, $F_{n+k} = L_k F_n + (-1)^{k-1} F_{n-k}$, so we have $F_{n+2} = L_2 F_n - F_{n-2}$ if $k = 2$. □

Theorem 4.2. *Let $n \geq 3$ and write $n = 2t + r$ with $1 \leq r \leq 2$.*

(1) $$\frac{F_{n+2}}{F_n} = \langle 2; 1^{n-1} \rangle = \begin{cases} [[3^{t+1}, 1]] = [3^{t+1}] & \text{if } r = 2, \\ [[3^t, 2, 1]] = [3^t, 2] & \text{if } r = 1. \end{cases}$$

(2) $$\frac{F_{n+1}}{F_n} = \langle 1^n \rangle = \begin{cases} [[2; 3^t, 1]] & \text{if } r = 2, \\ [[2; 3^{t-1}, 2, 1]] & \text{if } r = 1. \end{cases}$$

Proof. Due to Lemma 4.1, it is obvious that

$$\frac{F_{n+2}}{F_n}=\frac{2F_n+F_{n-1}}{F_n}=2+\cfrac{1}{\cfrac{F_n}{F_{n-1}}}=2+\frac{1}{\langle 1^{n-1}\rangle}=\langle 2;1^{n-1}\rangle,$$

and

$$\frac{F_{n+2}}{F_n}=\frac{3F_n-F_{n-2}}{F_n}=3-\cfrac{1}{\cfrac{F_n}{F_{n-2}}}=3-\cfrac{1}{\cfrac{3F_{n-2}-F_{n-4}}{F_{n-2}}}=3-\cfrac{1}{3-\cfrac{1}{\frac{F_{n-2}}{F_{n-4}}}}$$

$$=\cdots=3-\cfrac{1}{3-\cfrac{1}{\ddots 3-\frac{F_{n-2(v-1)}}{F_{n-2v}}}}=[[\underbrace{3;\dots,3}_{v},F_{n-2(v-1)},F_{n-2v}]]$$

for $v\le t$. Therefore, for $n=2t+r$ $(1\le r\le 2)$, Theorem 2.3 yields

$$\frac{F_{n+2}}{F_n}=[[3^t,F_{2+r},F_r]]$$
$$=\begin{cases}[[3^t,F_4,F_2]]=[[3^t,3,1]]=[3^{t+1}] & \text{if } r=2,\\ [[3^t,F_3,F_1]]=[[3^t,2,1]]=[3^t,2] & \text{if } r=1.\end{cases}$$

On the other hand, since $2F_n-F_{n-2}=F_{n+1}$, it follows from (1) that

$$\frac{F_{n+1}}{F_n}=\frac{2F_n-F_{n-2}}{F_n}=2-\cfrac{1}{\frac{F_n}{F_{n-2}}}$$
$$=\begin{cases}2-\frac{1}{[[3^t,1]]}=[[2;3^t,1]] & \text{if } r=2,\\ 2-\frac{1}{[[3^{t-1},2,1]]}=[[2;3^{t-1},2,1]] & \text{if } r=1.\end{cases}$$

□

It shows that if $n=2t+r$ $(r=1,2)$, then $\frac{F_{n+2}}{F_n}=\langle 2;1^{n-1}\rangle$ equals either $[[3^{t+1},1]]$ or $[[3^t,2,1]]$, and $\frac{F_{n+1}}{F_n}=\langle 1^n\rangle$ equals either $[[2;3^t,1]]$ or $[[2;3^{t-1},2,1]]$. This means that while n repeated computations are needed in the simple continued fraction, only about $t=\lfloor\frac{n}{2}\rfloor$ repeated computations are required in the semisimple continued fractions. For example, if $n=6$, then $[[2;3,3]]=\frac{F_7}{F_6}=\langle 1;1,1,1,1,1\rangle$. Hence if n is large, a semisimple continued fraction is more convenient than a simple continued fraction.

Theorem 4.3. *Let $n\ge 4$ and write $n=3t+r$ with $1\le r\le 3$. Then $\frac{F_{n+3}}{F_n}=\langle\langle L_3^t,F_{3+r},F_r\rangle\rangle$, which is equal to one of $\langle 4^t,3\rangle$, $\langle 4^t,5\rangle$ or $\langle 4^t,4\rangle$ according to $r=1,2,3$.*

Proof. Lemma 2.1 with $k=3$ implies $F_{n+3}=L_3F_n+F_{n-3}$, thus

$$\frac{F_{n+3}}{F_n}=L_3+\cfrac{1}{\frac{F_n}{F_{n-3}}}=4+\cfrac{1}{4+\cfrac{1}{\frac{F_{n-3}}{F_{n-6}}}}=4+\cfrac{1}{4+\cfrac{1}{\ddots 4+\cfrac{1}{\frac{F_{n-3(t-1)}}{F_{n-3t}}}}}$$

$$= \langle\langle L_3^t, F_{n-3(t-1)}, F_{n-3t}\rangle\rangle = \langle\langle 4^t, F_{3+r}, F_r\rangle\rangle.$$

Hence by making use of Theorem 2.3, it follows that

$$\frac{F_{n+3}}{F_n} = \begin{cases} \langle\langle 4^t, 3, 1\rangle\rangle = \langle 4^t, 3\rangle & \text{if } r = 1, \\ \langle\langle 4^t, 5, 1\rangle\rangle = \langle 4^t, 5\rangle & \text{if } r = 2, \\ \langle\langle 4^t, 8, 2\rangle\rangle = \langle 4^t, 4\rangle & \text{if } r = 3. \end{cases}$$ □

Now we generalize Theorem 4.3 as follows.

Theorem 4.4. *Let $k \geq 3$. Let $n \geq k+1$ and write $n = kt + r$ with $1 \leq r \leq k$. Then $\frac{F_{n+k}}{F_n}$ equals $\langle\langle L_k^t, F_{k+r}, F_r\rangle\rangle$ if k is odd, and $[[L_k^t, F_{k+r}, F_r]]$ if k is even. So*

$$\frac{F_{n+k}}{F_n} = \begin{cases} \langle L_k^t, F_{k+1}\rangle \text{ or } [L_k^t, F_{k+1}] & \text{if } r = 1, \\ \langle L_k^t, F_{k+2}\rangle \text{ or } [L_k^t, F_{k+2}] & \text{if } r = 2, \\ \langle L_k^{t+1}\rangle \text{ or } [L_k^{t+1}] & \text{if } r = k. \end{cases}$$

Proof. Let k be odd. Since $F_{n+k} = L_k F_n + F_{n-k}$ in Lemma 2.1, we have

$$\frac{F_{n+k}}{F_n} = L_k + \cfrac{1}{L_k + \cfrac{1}{\ddots\, L_k + \cfrac{1}{\frac{F_{k+r}}{F_r}}}} = \langle\langle L_k^t, F_{k+r}, F_r\rangle\rangle.$$

If $r = 1$ or 2, then $\frac{F_{n+k}}{F_n}$ equals $\langle\langle L_k^t, F_{k+1}, 1\rangle\rangle = \langle L_k^t, F_{k+1}\rangle$ or $\langle\langle L_k^t, F_{k+2}, 1\rangle\rangle = \langle L_k^t, F_{k+2}\rangle$. If $r = k$, then since $F_{2k} = F_k L_k$, it follows from Theorem 2.3 that

$$\frac{F_{n+k}}{F_n} = \langle\langle L_k^t, F_{2k}, F_k\rangle\rangle = \langle\langle L_k^t, F_k L_k, F_k\rangle\rangle = \langle L_k^t, L_k\rangle = \langle L_k^{t+1}\rangle.$$

On the other hand, if k is even, then $F_{n+k} = L_k F_n - F_{n-k}$ thus

$$\frac{F_{n+k}}{F_n} = L_k - \cfrac{1}{\frac{F_n}{F_{n-k}}} = L_k - \cfrac{1}{L_k - \cfrac{1}{\frac{F_{n-k}}{F_{n-2k}}}} = \cdots = [[L_k^t, F_{k+r}, F_r]],$$

and the rest follows similarly. □

In particular, in case of $n = kt + r$ with $r = 3 \leq k$, the next theorem follows.

Theorem 4.5. *With the same notations as in Theorem 4.4, let $r = 3$ and write $k = 3u + v$ with $1 \leq v \leq 3$. Set $P_0 = 1$ and $P_u = 4P_{u-1} + P_{u-2}$. If k is odd, then*

$$\frac{F_{n+k}}{F_n} = \begin{cases} \left\langle L_k^t, \langle\langle 3P_u; 2\rangle\rangle\right\rangle \text{ with } P_1 = \langle 4; 3\rangle, & \text{if } v = 1, \\ \left\langle L_k^t, \langle\langle 5P_u; 2\rangle\rangle\right\rangle \text{ with } P_1 = \langle 4; 5\rangle, & \text{if } v = 2, \\ \left\langle L_k^t, \langle\langle 4P_u; 2\rangle\rangle\right\rangle \text{ with } P_1 = \langle 4; 4\rangle, & \text{if } v = 3. \end{cases}$$

If k is even, then $\frac{F_{n+k}}{F_n}$ is either $[L_k^t, \langle\langle 3P_u; 2\rangle\rangle], [L_k^t, \langle\langle 5P_u; 2\rangle\rangle]$ or $[L_k^t, \langle\langle 4P_u; 2\rangle\rangle]$.

Proof. Assume k is odd. Theorem 4.4 together with Theorem 2.3 yields

$$\frac{F_{n+k}}{F_n} = \langle\langle L_k^t, F_{k+r}, F_r\rangle\rangle = \big\langle L_k^t, \langle\langle F_{k+3}; F_3\rangle\rangle\big\rangle.$$

Since $3 \le k$, if we write $k = 3u + v$ with $1 \le v \le 3$, then we have

$$\frac{F_{k+3}}{F_k} = \frac{F_{3(u+1)+v}}{F_{3u+v}} = \langle\langle L_3^u, F_{v+3}, F_v\rangle\rangle = \langle\langle 4^u, F_{v+3}, F_v\rangle\rangle = \langle 4^u, \lambda\rangle,$$

where $\lambda = 3, 5$ or 4 according to $v = 1, 2$ or 3 by Theorem 4.3.

Let $P_0 = 1$ and $P_1 = \langle 4; \lambda\rangle$. Then we see that

$$\langle 4^2, \lambda\rangle = 4 + \frac{1}{\langle 4; \lambda\rangle} = \frac{4P_1 + 1}{P_1} = \frac{P_2}{P_1}, \text{ where } P_2 = 4P_1 + P_0,$$

$$\langle 4^3, \lambda\rangle = 4 + \frac{1}{\langle 4^2, \lambda\rangle} = \frac{4P_2 + P_1}{P_2} = \frac{P_3}{P_2}, \text{ where } P_3 = 4P_2 + P_1,$$

$$\langle 4^4, \lambda\rangle = 4 + \frac{1}{\langle 4^3, \lambda\rangle} = \frac{4P_3 + P_2}{P_3} = \frac{P_4}{P_3}, \text{ where } P_4 = 4P_3 + P_2.$$

Thus we have, for any $0 \le i \le u$,

$$\frac{F_{3(i+1)+v}}{F_{3i+v}} = \langle 4^i, \lambda\rangle = \frac{P_i}{P_{i-1}}, \quad \text{where } P_i = 4P_{i-1} + P_{i-2}$$

with $P_0 = 1$ and $P_1 = \langle 4; \lambda\rangle$. It then follows that

$$\begin{aligned}
\langle\langle F_{k+3}; F_3\rangle\rangle &= \frac{F_{3(u+1)+v}}{F_3} \\
&= \frac{F_{3(u+1)+v}}{F_{3u+v}} \; \frac{F_{3u+v}}{F_{3(u-1)+v}} \; \cdots \; \frac{F_{3\cdot 2+v}}{F_{3+v}} \; \frac{F_{3+v}}{F_3} \\
&= \langle 4^u, \lambda\rangle \; \langle 4^{u-1}, \lambda\rangle \; \langle 4^{u-2}, \lambda\rangle \; \cdots \; \langle 4; \lambda\rangle \; \langle\langle F_{v+3}; F_3\rangle\rangle \\
&= \frac{P_u}{P_{u-1}} \; \frac{P_{u-1}}{P_{u-2}} \; \cdots \frac{P_2}{P_1} \; \frac{P_1}{1} \; \langle\langle F_{v+3}; F_3\rangle\rangle \\
&= P_u \langle\langle F_{v+3}; F_3\rangle\rangle.
\end{aligned}$$

If $v = 1$, then $\lambda = 3$ and $\dfrac{F_{k+3}}{F_k} = \langle 4^u, 3\rangle$ so

$$\langle\langle F_{k+3}; F_3\rangle\rangle = P_u \langle\langle F_4; F_3\rangle\rangle = P_u \langle\langle 3; 2\rangle\rangle = \langle\langle 3P_u; 2\rangle\rangle,$$

with $P_1 = \langle 4; 3\rangle$ and $P_i = 4P_{i-1} + P_{i-2}$. Thus $\frac{F_{n+k}}{F_n} = \big\langle L_k^t, \langle\langle 3P_u; 2\rangle\rangle\big\rangle$.

If $v = 2$, then $\lambda = 5$ and $\frac{F_{k+3}}{F_k} = \langle 4^u, 5\rangle$ so $\frac{F_{n+k}}{F_n} = \big\langle L_k^t, \langle\langle 5P_u; 2\rangle\rangle\big\rangle$ with $P_1 = \langle 4; 5\rangle$. Similarly if $v = 3$, then $\lambda = 4$ so $\frac{F_{n+k}}{F_n} = \big\langle L_k^t, \langle\langle 4P_u; 2\rangle\rangle\big\rangle$ with $P_1 = \langle 4; 4\rangle$.

On the other hand, assume that k is even. Then Theorem 4.4 implies

$$\frac{F_{n+k}}{F_n} = [[L_k^t, F_{k+r}, F_r]],$$

which is equal to $[L_k^t, [[F_{k+3}; F_3]]] = [L_k^t, \langle\langle F_{k+3}; F_3\rangle\rangle]$. Hence for $k = 3u + v$ with $v = 1, 2, 3$, the above calculations yield

$$\langle\langle F_{k+3}, F_3\rangle\rangle = P_u\langle\langle F_{v+3}, F_3\rangle\rangle = \begin{cases} \langle\langle 3P_u; 2\rangle\rangle & \text{with } P_1 = \langle 4; 3\rangle, & \text{if } v = 1, \\ \langle\langle 5P_u; 2\rangle\rangle & \text{with } P_1 = \langle 4; 5\rangle, & \text{if } v = 2, \\ \langle\langle 4P_u; 2\rangle\rangle & \text{with } P_1 = \langle 4; 4\rangle, & \text{if } v = 3. \end{cases} \qquad \square$$

Example. In order to find F_{105} and F_{88}, let $n = 88$ and $k = 17$. Then

$$\frac{F_{105}}{F_{88}} = \frac{F_{17\cdot 6+3}}{F_{17\cdot 5+3}} = \langle\langle L_{17}^5, F_{20}, F_3\rangle\rangle = \left\langle L_{17}^5, \langle\langle F_{20}; F_3\rangle\rangle\right\rangle.$$

Now for $\langle\langle F_{20}; F_3\rangle\rangle$, since $17 = 3u + v$ with $u = 5$, $v = 2$,

$$\frac{F_{20}}{F_{17}} = \frac{F_{3\cdot 6+2}}{F_{3\cdot 5+2}} = \langle\langle L_3^5, F_5, F_2\rangle\rangle = \langle 4^5, 5\rangle.$$

Thus by letting $P_0 = 1$, $P_1 = \langle 4; 5\rangle = \dfrac{21}{5}$ and $P_i = 4P_{i-1} + P_{i-2}$ $(i \le 5)$, we get $P_2 = \frac{89}{5}$, $P_3 = \frac{377}{5}$, $P_4 = \frac{1597}{5}$ and $P_5 = \frac{6765}{5}$. Hence

$$\langle\langle F_{20}; F_3\rangle\rangle = \langle\langle P_5F_5; F_3\rangle\rangle = \langle\langle 6765; 2\rangle\rangle = \frac{6765}{2}.$$

Therefore

$$\frac{F_{105}}{F_{88}} = \langle L_{17}^5, \frac{6765}{2}\rangle = 3571 + \cfrac{1}{3571 + \cfrac{1}{3571 + \cfrac{1}{3571 + \cfrac{1}{3571 + \cfrac{1}{\frac{6765}{2}}}}}},$$

which is $\frac{3{,}928{,}413{,}764{,}606{,}871{,}165{,}730}{1{,}100{,}087{,}778{,}366{,}101{,}931}$. Here the numerator is F_{105} and the denominator is F_{88}, which are 22 digit and 19 digit numbers respectively.

Corollary 4.6. *Let $n \ge 6$ and write $n = 5t + r$ with $1 \le r \le 5$. Then we have*

$$\frac{F_{n+5}}{F_n} = \langle\langle L_5^t, F_{5+r}, F_r\rangle\rangle = \begin{cases} \langle\langle L_5^t, 8, 1\rangle\rangle = \langle 11^t, 8\rangle & \text{if } r = 1, \\ \langle\langle L_5^t, 13, 1\rangle\rangle = \langle 11^t, 13\rangle & \text{if } r = 2, \\ \langle\langle L_5^t, 21, 2\rangle\rangle = \langle 11^t, 10, 2\rangle & \text{if } r = 3, \\ \langle\langle L_5^t, 34, 3\rangle\rangle = \langle 11^{t+1}, 3\rangle & \text{if } r = 4, \\ \langle\langle L_5^t, 55, 5\rangle\rangle = \langle 11^{t+1}\rangle & \text{if } r = 5. \end{cases}$$

Proof. This is mainly due to Theorem 4.4. □

References

[1] D. Burton, *Elementary Number Theory*, 3rd ed. WCB, Oxford, England, 1994.

[2] M. Drmota, *Fibonacci numbers and continued fraction expansions*, in G. E. Vergum et. al (eds.) Applications of Fibonacci numbers, vol 5, 2nd ed. Kluwer Academic Publishers, Netherlands, 1993.

[3] S. Kalia, *Fibonacci numbers and continued fractions*, MIT PRIMES, 2011. (retrived from web.mit.edu/primes/materials/2011/2011-conf-booklet.pdf)

[4] S. Katok, *Continued fractions, hyperbolic geometry, quadratic froms*, in MASS Selecta, Teaching and learning advanced undergraduate mathematics (S. Katok, A. Sossinsky, S. Tabachnikov eds.) American Math. Soc. 2003.
[5] F. Koken and D. Bozkurt, *On Lucas numbers by the matrix method*, Hacet. J. Math. Stat. **39** (2010), no. 4, 471–475.
[6] T. E. Phipps, *Fibonacci and continued fractions*, Aperion **15** (2008), no. 4, 534–550.

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