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ON THE SOLUTIONS OF THE $(\lambda, n+m)$-EINSTEIN EQUATION

Seungsu Hwang

Abstract. In this paper, we study the structure of m-quasi Einstein manifolds when there exists another distinct solution to the $(\lambda, n+m)$-Einstein equation. In particular, we derive sufficient conditions for the non-existence of such solutions.

1. Introduction

Let (M, g) be a complete n-dimensional Riemannian manifold and f be a smooth real valued function on M. Recently, there has been an increasing interest in the study of the extension of the Ricci tensor called the m-Bakry-Emery Ricci tensor (c.f. [3], [5]). It is given by

$$r_g^m = r_g + D_g df - \frac{1}{m}\, df \otimes df$$

for $0 < m \le \infty$, where r_g is the Ricci tensor of g and $D_g df$ is the Hessian of f. For $\lambda \in \mathbb{R}$, (M, g, f) is called m-quasi Einstein if it satisfies the $(\lambda, n+m)$-Einstein equation

$$r_g + D_g df - \frac{1}{m} df \otimes df = \lambda\, g. \tag{1}$$

It is well known that when m is a positive integer, the m-quasi Einstein metrics correspond to certain warped product Einstein metrics (c.f. [3], [5]). It is clear that if we take m to infinity, we obtain the gradient Ricci soliton equation

$$r_g + D_g df = \lambda\, g.$$

Thus we may call a gradient Ricci soliton a (λ, ∞)-Einstein manifold. Ricci solitons are self-similar solutions of the Ricci flow.

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The main purpose of this paper is to answer the following natural question for the m-quasi Einstein metrics with $m < \infty$: What is the geometric characteristic of (M, g), if distinct solution of (1) exists?

Here, we call the solutions distinct when they do not differ by certain constants. It is clear from equation (1) that if c is a constant, $\bar{f} = f + c$ is also a solution of (1). Thus, we say that the two solution functions f and $\bar{f}$ of (1) are distinct only when the difference function $\varphi = f - \bar{f}$ is not a constant function.

The answers to the abovementioned question are given as follows.

Theorem 1.1. *Let (M, g) be a compact m-quasi Einstein manifold. Then, there exists no other distinct solution to the $(\lambda, n+m)$-Einstein equation*

$$r_g + D_g df - \frac{1}{m}\, df \otimes df = \lambda\, g.$$

Theorem 1.2. *Let (M, g) be a complete m-quasi Einstein manifold, possibly with non-empty boundary. If there exists another distinct solution to the $(\lambda, n+m)$-Einstein equation, then the scalar curvature s_g is constant. Moreover, there exists no other distinct solution to the $(\lambda, n+m)$-Einstein equation, either if $\lambda > 0$ and f has its local maximum in the interior of M, or $\lambda \leq 0$.*

When $m = \infty$, the constancy of the scalar curvature implies rigidity of the gradient Ricci solitons in some cases (c.f. [2], [4], and [6]. See also Remark 2.3). For the definition of the rigidity of gradient Ricci solitons, see [7].

2. Level sets of the difference function

Suppose that there exists another solution $\bar{f}$ to (1). Then, the difference function $\varphi = f - \bar{f}$ satisfies the equation

$$\begin{aligned} D_g d\varphi &= \frac{1}{m}\left(df \otimes df - d\bar{f} \otimes d\bar{f}\right) \\ &= \frac{1}{m}\left(df \otimes d\varphi + d\varphi \otimes df - d\varphi \otimes d\varphi\right). \end{aligned} \tag{2}$$

Consider a level set $\varphi^{-1}(c)$ for $c \in \mathbb{R}$. For any tangent vectors X, Y to $\varphi^{-1}(c)$, it is easy to see that $\langle D_X d\varphi, Y\rangle = 0$. This implies that $\varphi^{-1}(c)$ is totally geodesic if $|\nabla\varphi| \neq 0$ on $\varphi^{-1}(c)$; for $\nu = \nabla\varphi/|\nabla\varphi|$ the second fundamental form of $\varphi^{-1}(c)$ is given by

$$II(X, Y) = \langle D_X \nu, Y\rangle = \frac{1}{|\nabla\varphi|}\langle D_X d\varphi, Y\rangle = 0.$$

Further, we have the following result.

Lemma 2.1. *Each level set $\varphi^{-1}(c)$ with $|\nabla\varphi| \neq 0$ is totally geodesic. In particular, on the level set $\varphi^{-1}(c)$, we have*

$$|\nabla\varphi|^2 = k(c)\, e^{\frac{2}{m} f},$$

where $k(c)$ is a constant depending only on c.

Proof. It is sufficient to prove only the second statement. For a tangent vector X to $\varphi^{-1}(c)$, by (2) we have

$$\frac{1}{2}X(|\nabla\varphi|^2) = \langle D_X d\varphi, \nabla\varphi\rangle = \frac{1}{m}\, df(X)\, |\nabla\varphi|^2.$$

Thus, on $\varphi^{-1}(c)$

$$X\left(\ln|\nabla\varphi|^2 - \frac{2}{m}f\right) = 0.$$

In other words, $\ln|\nabla\varphi|^2 - \frac{2}{m}f$ is constant on $\varphi^{-1}(c)$. □

From this, we have the following result.

Proposition 2.2. *We have*

$$(m-1)\, i_{\nabla\varphi} r = (\lambda(n-1) - s_g)\, d\varphi,$$

where i_Y is the interior product with respect to the vector field Y. In particular, for any tangent vector X to $\varphi^{-1}(c)$ where $|\nabla\varphi| \neq 0$,

$$r(X, \nabla\varphi) = 0.$$

Proof. Taking the divergence of (2) gives

$$\begin{aligned} -m\, d\triangle\varphi - m\, r(\nabla\varphi, \cdot) &= -(\Delta f)\, d\varphi - D_{\nabla f} d\varphi - (\Delta\varphi) df \\ &\quad - D_{\nabla\varphi} df + (\Delta\varphi)\, d\varphi + D_{\nabla\varphi} d\varphi, \end{aligned}$$

since

$$\begin{aligned} \delta D_g d\varphi &= -d\triangle\varphi - r(\nabla\varphi, \cdot) \\ \delta(df \otimes d\varphi) &= -(\triangle f)\, d\varphi - D_{\nabla f} d\varphi, \end{aligned}$$

and so on.

Note that the trace of (1) is given by

$$\triangle f = -s_g + \lambda n + \frac{1}{m}|\nabla f|^2. \tag{3}$$

Now, since the trace of (2) is given by

$$m\,\Delta\varphi = 2\, df(\varphi) - |\nabla\varphi|^2,$$

for any vector ξ,

$$\begin{aligned} &2\langle D_\xi df, \nabla\varphi\rangle + 2\langle\nabla f, D_\xi d\varphi\rangle - 2\langle D_\xi d\varphi, \nabla\varphi\rangle + m\, r(\nabla\varphi, \xi) \\ &= (-s + \lambda n + \frac{1}{m}|\nabla f|^2)\, \xi(\varphi) + \frac{1}{m}\left(|\nabla f|^2\xi(\varphi) + df(\varphi)\xi(f) - df(\varphi)\xi(\varphi)\right) \\ &\quad + \frac{1}{m}(2\, df(\varphi) - |\nabla\varphi|^2)\, \xi(f) - r(\nabla\varphi, \xi) + \frac{1}{m} df(\varphi)\xi(f) + \lambda\, \xi(\varphi) \\ &\quad - \frac{1}{m}(2\, df(\varphi) - |\nabla\varphi|^2)\, \xi(\varphi) - \frac{1}{m}(df(\varphi)\xi(\varphi) + |\nabla\varphi|^2\xi(f) - |\nabla\varphi|^2\xi(\varphi)) \\ &= \left(-s + \lambda(n+1) + \frac{2}{m}|\nabla f|^2 + \frac{2}{m}|\nabla\varphi|^2 - \frac{4}{m} df(\varphi)\right)\xi(\varphi) - r(\nabla\varphi, \xi) \end{aligned}$$

$$+\frac{2}{m}\left(2\,df(\varphi)-|\nabla\varphi|^2\right)\xi(f).$$

On the other hand, the left-hand side of the above equation is

$$\begin{aligned}
&2\langle D_\xi df,\nabla\varphi\rangle+2\langle\nabla f,D_\xi d\varphi\rangle-2\langle D_\xi d\varphi,\nabla\varphi\rangle+m\,r(\nabla\varphi,\xi)\\
=&\;-2\,r(\xi,\nabla\varphi)+\frac{2}{m}\xi(f)df(\varphi)+\frac{2}{m}(|\nabla f|^2\xi(\varphi)+df(\varphi)\xi(f)-df(\varphi)\xi(\varphi))\\
&+2\lambda\,\xi(\varphi)-\frac{2}{m}(|\nabla\varphi|^2\xi(f)+\xi(\varphi)df(\varphi)-\xi(\varphi)|\nabla\varphi|^2)+m\,r(\nabla\varphi,\xi)\\
=&\;(m-2)\,r(\nabla\varphi,\xi)+2\left(\lambda+\frac{1}{m}|\nabla f|^2-\frac{2}{m}df(\varphi)+\frac{1}{m}|\nabla\varphi|^2\right)\xi(\varphi)\\
&+\frac{2}{m}\left(2df(\varphi)-|\nabla\varphi|^2\right)\,\xi(f).
\end{aligned}$$

By substituting this equation in the previous equation, we obtain the desired equation

$$(m-1)\,r(\nabla\varphi,\xi)=(\lambda(n-1)-s)\;\xi(\varphi).$$

The second statement follows by taking $\xi=X$. □

Remark 2.3. By Proposition 2.2, when $m=1$, the scalar curvature s_g equals to a constant $\lambda\,(n-1)$ unless φ is trivial.

On the other hand, if there are two solutions f and $\bar f$ to the gradient Ricci soliton equation

$$r_g+D_gdf=\lambda g,$$

the difference function $\varphi=f-\bar f$ satisfies $D_gd\varphi=0$, and thus $\nabla\varphi$ is a parallel vector field on M, which splits a line thus decomposing M as $M=\mathbb{R}\times N$ for some $(n-1)$-dimensional manifold N. In particular,

$$0=\delta D_gd\varphi=-d\Delta\varphi-r(\nabla\varphi,\cdot)=-r(\nabla\varphi,\cdot),$$

which is the gradient Ricci soliton version of Proposition 2.2. Further, if the second function $\bar f$ satisfies

$$r_g+D_gdf=\bar\lambda g$$

with $\bar\lambda\neq\lambda$, then $\nabla\varphi$ is a non-Killing homothetic vector field, and it is known by [8] that the universal cover of M is flat.

3. Existence of distinct solutions

As a consequence of Proposition 2.2, we have:

Theorem 3.1. *Let (M,g) be a complete m-quasi Einstein manifold, possibly with boundary. If there exists another distinct solution $\bar f$ to* (1)*, then the scalar curvature s_g is constant, and when $m\neq 1$,*

$$r(\nabla\varphi,\cdot)=0$$

for the difference function $\varphi=f-\bar f$.

Proof. Suppose that φ is not trivial. Then, if $m = 1$, s_g is constant by Remark 2.3. Now we assume that $m \neq 1$. Then, on $\varphi^{-1}(c)$ where $|\nabla\varphi| \neq 0$, from the well-known Riccati equation we have

$$-\nu(m) = r_g(\nu, \nu) + ||II||^2,$$

where $\nu = \nabla\varphi/|\nabla\varphi|$. Thus, by Lemma 2.1, we have

$$\nu(m) = ||II||^2 \equiv 0.$$

Therefore,

$$r_g(\nu, \nu) = 0$$

on each level set $\varphi^{-1}(c)$. However, by Proposition 2.2,

$$0 = r(\nu, \nu) = \frac{1}{m-1}\left(\lambda(n-1) - s_g\right).$$

This implies that

$$s_g = \lambda(n-1). \tag{4}$$

In other words, s_g equals to $\lambda(n-1)$ on the level set of φ where $|\nabla\varphi| \neq 0$. Note that on the whole manifold M, the metric tensor g and the functions $f, \bar{f}$ are real analytic by Proposition 2.4 of [5]. In particular, φ is real analytic, implying that the set $\nabla\varphi = 0$ is not open unless φ is trivial. Hence, by continuity we may conclude that s_g is constant on all of M. Combining these facts and Proposition 2.2 with continuity gives our theorem. □

Since a compact m-quasi Einstein metric with constant scalar curvature is trivial by Proposition 2.1 of [3], we may deduce the following result, which is Theorem 1.1. Here, however, we include a different proof.

Corollary 3.2. *Let* (M, g) *be a compact* m*-quasi Einstein manifold* (*without boundary*). *Then, there exists no other distinct solution to* (1).

Proof. Suppose that there exists another distinct solution $\bar{f}$ to (1). Then, $\varphi = f - \bar{f}$ is not trivial. Therefore, by (3) and Theorem 3.1 with (4)

$$\Delta f = \frac{1}{m}|\nabla f|^2 + \lambda. \tag{5}$$

If $\lambda > 0$, then f is a subharmonic function on M. Moreover, if $\lambda \leq 0$, then f is trivial by [5]. In either case, f should be trivial and g, Einstein. Thus, $\bar{f}$ is also trivial, implying that φ is trivial. This contradiction proves our corollary. □

Corollary 3.3. *Let* (M, g) *be a complete* m*-quasi Einstein manifold, possibly with non-empty boundary. Then, there exists no other distinct solution to* (1), *either if* $\lambda > 0$ *and* f *has its local maximum in the interior of* M, *or* $\lambda \leq 0$.

Proof. As in the proof of Corollary 3.2, suppose that φ is not trivial. Then $\lambda > 0$. Our corollary follows immediately from equation (5). □

Theorem 3.1 and Corollary 3.3 constitute Theorem 1.2. We remark that if $\lambda > 0$ and M is complete, possibly with non-empty boundary, M is known to be compact by [5].

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Department of Mathematics
Chung-Ang University
Seoul 156-756, Korea
E-mail address: seungsu@cau.ac.kr