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RELATIVE RELATION MODULES OF FINITE ELEMENTARY ABELIAN p-GROUPS

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Abstract. Let E be a free product of a finite number of cyclic groups, and S a normal subgroup of E such that $E/S \cong G$ is finite. For a prime p, $\hat{S} = S/S^{'}S^{p}$ may be regarded as an F_pG-module via conjugation in E. The aim of this article is to prove that $\hat{S}$ is decomposable into two indecomposable modules for finite elementary abelian p-groups G.

1. Introduction

Consider a short exact sequence $1 \to S \to E \xrightarrow{\psi} G \to 1$, where G is a finite group of order n generated by $X = \{g_i : 1 \leq i \leq d\}$. Let E be a free product of cyclic groups E_i, where $1 \leq i \leq d$. Let p be a (fixed) prime, and F_p the field of p-elements. If E is a free group, $\hat{S} = S/S^{'}S^{p}$, regarded as an F_pG-module, is known as a relation module of G. In general $\hat{S}$ is called a relative relation module, and it is said to be minimal if it cannot be generated by fewer than d elements.

Gaschutz [1], Gruenberg [2], Kovacs and Stohr [4], Mittal and Passi [5], and others have studied relation modules. Relative relation modules have been studied by Kimmerle [3], Yamin [8, 9], and Sharma and Yamin [7]. As a direct consequence of ([2, Theorem 2.9]), minimal relation modules of p-groups are non-projective and indecomposable. Yamin [8] has proved that relative relation modules of p-groups are non-projective. Kimmerle [3] has proved that minimal relative relation modules of finite p-groups are indecomposable if $\delta = 1$. The same result has been proved by Yamin [9] for $\delta = d$. If G is a finite elementary abelian p-group, in this article, we prove that $\hat{S}$ is decomposable into two indecomposable modules for $1 < \delta < d$.

Throughout this article, concepts related with groups and their representations are used mainly from Robinson [6], often without reference.

Let G_i be the cyclic subgroup of G generated by g_i of order n_i and E_i be the cyclic group generated by e_i of order m_i, where $m_i = k_in_i, 1 \leq i \leq d$. Let

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p be fixed prime and F_p the field of p-elements. Let $k_i < \infty$ and $p \nmid k_i$ if $i \leq \delta$ and $\delta \leq d$, and $k_i = \infty$ or $p \mid k_i$ if $\delta + 1 \leq i \leq d$.

For an F_pG-module V, define φV to be the smallest submodule of V such that $V/\varphi V$ is completely reducible. Equivalently, φV is the intersection of all maximal submodules of V. Set $\varphi^0 V = V$, $\varphi^k V = \varphi(\varphi^{k-1}V)$. ($\{\varphi^k V\}$ is known as the Loewy series of V.) From ([8, Prop 2.10]), we have an F_pG exact sequence:

$$1 \to \hat{S} \to \oplus_{i=1}^{\delta} \underline{g}_i^G \oplus_{i=\delta+1}^{d} b_i F_p G \xrightarrow{\hat{\psi}} \underline{g} \to 1, \tag{1.1}$$

where $\underline{g}$, which is same as φF_pG, denotes the augmentation ideal of F_pG and $\underline{g}_i^G$ that of $b_iF_pG_i$ induced to G. Moreover $\hat{\psi}$ is determined by $b_i(1-g_i) \to 1-g_i$ if $1 \leq i \leq \delta$, and $b_i \to 1-g_i$ if $1+\delta \leq i \leq d$.

2. Structure of relative relation modules

Theorem 2.1. *If G is a finite elementary abelian p-group and $1 < \delta < d$, then $\hat{S}$ is decomposable into two indecomposable modules.*

In order to prove the theorem, we shall construct a minimal generating set for $\hat{S}$. For, we need certain bases for the Loewy factors of b_iF_pG and $\underline{g}_i^G$.

Let

$$B_i = \begin{cases} \underline{g}_i^G & \text{for } 1 \leq i \leq \delta, \\ b_iF_pG & \text{otherwise}; \end{cases} \quad \text{and } B = \oplus_{i=1}^{d} B_i.$$

Let

$$Y = \{y : y \in b_iF_pG; y = \prod_{i=1}^{d}(1-g_i)^{\mu_i}, (1-g_i)^0 = 1, g_i \in X, 0 \leq \mu_i \leq n_i - 1\}.$$

Clearly, Y is a set of non-zero and distinct elements of F_pG and $|Y| = |G|$. Then it can be checked that Y is an F_p basis of F_pG. Now, for $y \in Y$, define $\sum_{i=1}^{d} \mu_i$ as the length of y. It is easy to observe that the length of y varies from 0 to $l = d(p-1)$. Corresponding to each k, let Y_k be the set of all elements of Y of length k. Then it is easy to ckeck that Y_k is a minimal generating set of $\varphi^k(F_pG)$, and $\{\text{y}+\varphi^{k+1}F_pG : y \in Y_k\}$ is an F_p basis of $\varphi^k(F_pG) \,/\varphi^{k+1}(F_pG)$.

For a fixed i, $1 \leq i \leq \delta$, let $Z^i = \{z \mid z = b_i(1-g_i)y,\ y \in Y \text{ and } (1-g_i)y \neq 0\}$. Clearly, $(1-g_i)y = 0$ if and only if $y = (1-g_i)^{p-1}y'$ for some $y' \in Y$. The number of such y is exactly p^{n-1}. Therefore, $|Z^i| = p^n - p^{n-1} = p^{n-1}(p-1) = \dim(B_i)$. In fact Z^i is an F_p basis of B_i, and Z^i_k, the set of all elements of Z^i of length k, is a minimal generating set of $\varphi^{k-1}B_i$. Similarly, for a fixed i, $\delta+1 \leq i \leq d$, the set $Z^i = \{b_iy, y \in Y\}$ is an F_p basis of b_iF_pG and Z^i_k is a minimal generating set of $\varphi^k B_i$.

Let $Z = \cup_{i=1}^{d} Z^i$ and $Z_{k+1} = \cup_{i=1}^{d} Z^i_{k+1}$, $0 \leq k \leq l-1$. Then it is easy to see that Z is an F_p basis of B and Z_{k+1} is a minimal generating set of $\oplus_{i=1}^{\delta}\varphi^k B_i \oplus_{i=\delta+1}^{d} \varphi^{k+1} B_i$.

Remark. The Loewy length of $B_i = \begin{cases} l & \text{for } 1 \le i \le \delta; \\ l+1 & \text{for } 1+\delta \le i \le d. \end{cases}$

Lemma 2.2. *Let $X_1 = \{(b_i - b_j)(1-g_i)(1-g_j),\ 1 \le i < j \le \delta\}$ and $X_2 = \{b_i(1-g_j) - b_j(1-g_i),\ b_i(1-g_i)^{n_i-1},\ 1+\delta \le i < j \le d\}$. Then $X_1 \cup X_2$ is a minimal generating set of $\hat{S}$.*

Proof. Firstly, we note that $|X_1 \cup X_2| = \frac{1}{2}\delta(\delta-1) + \frac{1}{2}(d-\delta)(d-\delta+1)$. Now we show that $X_1 \cup X_2$ generate $\hat{S}$. If V is any submodule generated by $X_1 \cup X_2$, then $V \subseteq \hat{S}$. For the other inclusion, since $\oplus_{i=1}^{\delta} \varphi^k B_i \oplus_{i=\delta+1}^{d} \varphi^{k+1} B_i = \{0\}$, when $k = l$, therefore it is sufficient to show that an arbitrary element of $\hat{S}$ can be expresses as a sum of an element of V and an element of $\oplus_{i=1}^{\delta} \varphi^k B_i \oplus_{i=\delta+1}^{d} \varphi^{k+1} B_i$ for all k, $0 \le k \le l$. We shall show this by induction on k. The result is obviously true for $k = 0$. Let $x \in \hat{S}$, and for a fixed k, $0 \le k \le l-1$, suppose that $x = v + x'$, $v \in V$ and $x' \in \oplus_{i=1}^{\delta} \varphi^k B_i \oplus_{i=\delta+1}^{d} \varphi^{k+1} B_i$. To complete the induction argument, we shall show that $x' = x_0 + x_1$ for some $x_0 \in V$ and $x_1 \in \oplus_{i=1}^{\delta} \varphi^{k+1} B_i \oplus_{i=\delta+1}^{d} \varphi^{k+2} B_i$.

Writing x' as a linear combination of an element of Z_{k+1} and then rewritting each element of F_pG as a sum of an element of F_p and an element of $\underline{g}$, we have

$$\begin{aligned} x' = &\sum a_{\lambda_1} b_1 (1-g_1) y_{\lambda_1} + \cdots + \sum a_{\lambda_\delta} b_\delta (1-g_\delta) y_{\lambda_\delta} \\ &+ \sum a_{\lambda_{\delta+1}} b_{\delta+1} y'_{\lambda_{\delta+1}} + \cdots + \sum a_{\lambda_d} b_d y'_{\lambda_d} + x_1, \end{aligned}$$

where $a_{\lambda_i} \in F_p$, $y_{\lambda_i} \in Y_k$, $y'_j \in Y_{k+1}$ and $x_1 \in \oplus_{i=1}^{\delta} \varphi^{k+1} B_i \oplus_{i=\delta+1}^{d} \varphi^{k+2} B_i$. Clearly, $(1-g_i) y_{\lambda_i} \in Y_{k+1}$, however they need not be distinct if $x_\lambda = (1-g_i) y_{\lambda_i} = (1-g_j) y_{\lambda_j} = \cdots = (1-g_r) y_{\lambda_r} = (1-g_s) y_{\lambda_s} = y'_{\lambda_k} = y'_{\lambda_l} = y'_{\lambda_m} = y'_{\lambda_n}$, where $0 \le i, j, \ldots, r, s \le \delta$ and $\delta+1 \le k, l, \ldots, m, n \le d$. Let

$$\begin{aligned} \mu_\lambda = &a_{\lambda_i} b_i + a_{\lambda_j} b_j + \cdots + a_{\lambda_r} b_r + a_{\lambda_s} b_s + a_{\lambda_k} b_k + a_{\lambda_l} b_l \\ &+ \cdots + a_{\lambda_m} b_m + a_{\lambda_n} b_n. \end{aligned}$$

Then $x_0 = \sum_\lambda \mu_\lambda x_\lambda$, where x_λ are distinct elements of Y_{k+1}. Therefore we have

$$\begin{aligned} x_0 \hat{\psi} &= [\sum_\lambda \{(a_{\lambda_i} b_i + a_{\lambda_j} b_j + \cdots + a_{\lambda_r} b_r + a_{\lambda_s} b_s + a_{\lambda_k} b_k + a_{\lambda_l} b_l \\ &\qquad + \cdots + a_{\lambda_m} b_m + a_{\lambda_n} b_n)\} x_\lambda] \hat{\psi} \\ &= \sum_\lambda \{(a_{\lambda_i} b_i + a_{\lambda_j} b_j + \cdots + a_{\lambda_r} b_r + a_{\lambda_s} b_s) + a_{\lambda_k}(1-g_k) + a_{\lambda_l}(1-g_l) \\ &\qquad + \cdots + a_{\lambda_m}(1-g_m) + a_{\lambda_n}(1-g_n)\} x_\lambda. \end{aligned}$$

Now for $\delta+1 \le k, l, \ldots, m, n \le d$, we have $(1-g_k) y'_{\lambda_k} = (1-g_l) y'_{\lambda_l}$ if and only if $y'_{\lambda_k} = (1-g_l) y$ and $y'_{\lambda_l} = (1-g_k) y$ for some $y \in Y_{k+1}$. Let $x_{\lambda'} = (1-g_k) y'_{\lambda_k} = (1-g_l) y'_{\lambda_l} = \cdots = (1-g_m) y'_{\lambda_m} = (1-g_n) y'_{\lambda_n}$, where the

$x_{\lambda'}$ are distinct elements of Y_{k+1}. Then we have

$$\begin{aligned} x_0\hat{\psi} = &\sum_{\lambda}(a_{\lambda_i} + a_{\lambda_j} + \cdots + a_{\lambda_r} + a_{\lambda_s})x_\lambda \\ &+ \sum_{\lambda'}(a_{\lambda_k} + a_{\lambda_l} + \cdots + a_{\lambda_m} + a_{\lambda_n})x_{\lambda'} \\ = &\sum_{\lambda}\beta_\lambda x_\lambda + \sum_{\lambda'}\beta_{\lambda'}x_{\lambda'}, \end{aligned}$$

where $\beta_\lambda = \alpha_{\lambda_i} + \alpha_{\lambda_j} + \cdots + \alpha_{\lambda_r} + \alpha_{\lambda_s}$ and $\beta_{\lambda'} = \alpha_{\lambda_k} + \alpha_{\lambda_l} + \cdots + \alpha_{\lambda_m} + \alpha_{\lambda_n}$. Clearly, $0 = x\hat{\psi} = v\hat{\psi} + x'\hat{\psi} = x'\hat{\psi} = x_0\hat{\psi} + x_1\hat{\psi}$, where $x_1\hat{\psi} \in \oplus_{i=1}^{\delta}\varphi^{k+1}B_i \oplus_{i=\delta+1}^{d} \varphi^{k+2}B_i$.

Now $(\oplus_{i=1}^{d}\varphi^{k+1}B_i)\hat{\psi} \subseteq \varphi^{k+1}\underline{g}$ and $(\oplus_{i=1}^{d}\varphi^{k+2}B_i)\hat{\psi} \subseteq \varphi^{k+2}\underline{g}$. Moreover,

$$\varphi^{k+2}\underline{g} \subseteq \varphi^{k+1}\underline{g} \text{ implies that } (\oplus_{i=1}^{\delta}\varphi^{k+1}B_i \oplus_{i=\delta+1}^{d} \varphi^{k+2}B_i) \subseteq \varphi^{k+1}\underline{g}.$$

But $\varphi^{k+1}\underline{g} = (x_0\hat{\psi}+x_1\hat{\psi})+\varphi^{k+1}\underline{g} = x_0\hat{\psi}+\varphi^{k+1}\underline{g} = \sum\beta_\lambda x_\lambda+\sum\beta_{\lambda'}x_{\lambda'}+\varphi^{k+1}\underline{g}$. Since each x_λ and $x_{\lambda'}$ are distinct elements of Y_{k+1} and Y_{k+2}, respectively, therefore, each $\beta_\lambda = \beta_{\lambda'} = 0$. Therefore, $\alpha_{\lambda_s} = -\alpha_{\lambda_i} - \alpha_{\lambda_j} - \cdots - \alpha_{\lambda_r}$ and $\alpha_{\lambda_n} = -\alpha_{\lambda_k} - \alpha_{\lambda_l} - \cdots - \alpha_{\lambda_m}$. Thus

$$\begin{aligned} x_0 = &\sum_{\lambda}\{a_{\lambda_i}b_i + a_{\lambda_j}b_j + \cdots + a_{\lambda_r}b_r - (a_{\lambda_i} + a_{\lambda_j} + \cdots + a_{\lambda_r})b_s\}x_\lambda \\ &+ \{a_{\lambda_k}b_k + a_{\lambda_l}b_l + \cdots + a_{\lambda_m}b_m - (a_{\lambda_k} + a_{\lambda_l} + \cdots + a_{\lambda_m})b_n\}x_\lambda \\ = &\sum_{\lambda}\{a_{\lambda_i}(b_i - b_s) + a_{\lambda_j}(b_j - b_s) + \cdots + a_{\lambda_r}(b_r - b_s)\}x_\lambda \\ &+ \{a_{\lambda_k}(b_k y'_{\lambda_n} - b_n y'_{\lambda_k}) + a_{\lambda_l}(b_l y'_{\lambda_n} - b_n y'_{\lambda_l}) + \cdots \\ &+ a_{\lambda_m}(b_m y'_{\lambda_n} - b_n y'_{\lambda_m})\}, \end{aligned}$$

which is an element of V. Therefore, $\hat{S} \subseteq V$. Since X_1 and X_2 lie in different direct summands of the middle term $\oplus_{i=1}^{\delta}\underline{g}_i^G \oplus_{i=\delta+1}^{d} b_iF_pG$ of (1.1), therefore $X_1 \cap X_2 = 0$. Thus in order to show that $X_1 \cup X_2$ is a minimal generating set of $\hat{S}$, we only need to show that X_1 and X_2 are minimal sets of U and V, respectively. First we shall show that X_1 is a minimal generating set for U. For, since we are dealing with a finite elementary p-group case, it is sufficient to show that no proper subset of X_1 generates U. For, on the contrary suppose that X_1 is not minimal, and let $(b_\mu - b_\nu)(1 - g_\mu)(1 - g_\nu) = \sum_{(\mu,\nu)\neq(i,j),1\leq i<j\leq\delta}(b_i - b_j)(1 - g_i)(1 - g_j)a_{ij} = x$, say, $a_{ij} \in F_pG$. Then $b_\mu(1 - g_\mu)(1 - g_\nu) = x + b_\nu(1 - g_\mu)(1 - g_\nu)$, which is a contradiction because $\cup_{i=1}^{\delta}Z_2^i$ is a minimal generating set of $\oplus_{i=1}^{\delta}\varphi^iB$. This shows that X_1 is minimal. For proving that X_2 is a minimal generating set, for convenient reference, we shall call the elements $b_i(1 - g_j) - b_j(1 - g_i)$ of type-I and $b_i(1 - g_i)^{n_i-1}$ of type-II. Then it is clear that type-I elements cannot be generated from type-II elements and vice-versa. Suppose on the contrary that the set X_2 is not minimal and let $b_\mu(1-g_\nu)-b_\nu(1-$

$g_\mu) = \sum_{(\mu,\nu)\neq(i,j),\delta+1\leq i<j\leq d}\{b_i(1-g_j) - b_j(1-g_i)\}a_{ij} = y$, say, $a_{ij} \in F_pG$. Then $b_\mu(1-g_\nu) = y + b_\nu(1-g_\mu)$, which is a contradiction, because an element of Z_1 can not be expressed as a linear combination of the remaining elements of Z_1. □

Proposition 2.3. *If U and V are F_pG modules generated by X_1 and X_2, respectively, then $\hat{S} \cong U \oplus V$.*

Proof. By Lemma 2.2, it follows that $\hat{S} = U + V$. Moreover, since X_1 and X_2 lie in different direct summands of (1.1), $U \cap V = \{0\}$, and hence $\hat{S} \cong U \oplus V$. □

In view of Proposition 2.3, to complete the proof of Theorem 2.1, we only need to prove that U and V are indecomposable. For, consider a set

$$C = \{c_{ij} : c_{ij} = (b_i - b_j)\prod_{\mu=1}^{d}(1-g_\mu)^{n_\mu - 1},\ 1 \leq i < j \leq \delta\}.$$

Clearly $|C| = \frac{1}{2}\delta(\delta-1)$. Let U_1 be a submodule of U such that $\dim((U_1 + \varphi U)/\varphi U) = r$, where $0 \leq r \leq \frac{1}{2}\delta(\delta-1)$. Now we shall prove that U_1 contains at least r elements of C. For, $r = 0$, there is nothing to prove. Therefore, suppose that $r > 0$ and choose $\{u_m : u_m \in U_1, 1 \leq m \leq r\}$ such that $\{u_m + \varphi U, 1 \leq m \leq r\}$ is an F_p basis of $(U_1 + \varphi U)/\varphi U$. Then

$$u_m = \sum_{1\leq i<j\leq\delta} \alpha_{m_{ij}}(b_i - b_j)(1-g_i)(1-g_j) + w,$$

where $\alpha_{m_{ij}} \in F_p$ and $w \in \varphi U$. Clearly, in the expression of u_m, at least one $\alpha_{m_{ij}}$ is non-zero. Let $y = \prod_{\lambda=1}^{d}(1-g_\lambda)^{\nu_\lambda - 1}$, where $\nu_\lambda = \begin{cases} n_\lambda - 2, & \text{if } \lambda = i \text{ or } j; \\ n_\lambda - 1, & \text{otherwise.}\end{cases}$ Then $u_m y = \alpha_{m_{ij}}(b_i - b_j)\prod_{\lambda=1}^{d}(1-g_\lambda)^{n_\lambda - 1}$ $(wy = 0)$. So, we get $u_m y = \alpha_{m_{ij}} c_{ij}$, and so $(\alpha_{m_{ij}})^{-1} u_m y = c_{ij} \in U_1$. Since at least r of the $(b_i - b_j)(1-g_i)(1-g_j)$ are distinct, U_1 contains at least r elements c_{ij} of C.

Suppose $U = U_1 \oplus U_2$ and $\dim(U_i/\varphi U_i) = \frac{1}{2}\delta(\delta-1) = r_1 + r_2$. From above argument, we infer that U_i contains a set C_i of at least r_i elements of C. In fact $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \Phi$.

Clearly, $c_{i\delta} = c_{ik} + c_{k\delta}$ for some $1 \leq i \leq \delta - 1$ and $1 < k < \delta$. Therefore C is an F_p-linear combination of $\{c_{i\delta} : 1 \leq i \leq \delta - 1\}$. Then it is easy to check that either $C \subseteq U_1$ or $C \subseteq U_2$, and therefore, U is indecomposable.

Now we prove that V is indecomposable. If $d - \delta = 1$, then X_2 contains only one element and therefore V is indecomposable. Now we suppose that $d - \delta \geq 2$ and consider the set $C' = \{b_i \prod_{\mu=1}^{d}(1-g_\mu)^{n_\mu - 1} : 1 + \delta \leq i \leq d\}$. (Note that $|C'| = d - \delta$.)

Suppose that $V = V_1 + V_2$. As before, we shall identify the elements $b_i(1 - g_j) - b_j(1-g_i)$ of X_2 as those of type-I and $b_i(1-g_i)^{n_i - 1}$ of type-II. Then the following possibilites arise:

(i) V_1 contains all elements of X_2 of type-I and V_2 contains all elements of X_2 of type-II.

(ii) V_1 contains r_1 elements of type-I and s_1 elements of type-II, and V_2 contains r_2 elements of type-I and s_2 elements of type-II, where $r_1 + r_2 = \frac{1}{2}\,(d-\delta)(d-\delta-1)$ and $s_1 + s_2 = d - \delta$

(iii) V_1 or V_2 contain all elements of X_2.

We shall establish that only (iii) is true by eliminating the other two possibilities. If (i) was true, then both V_1 and V_2 would contain all elements of C', which would be a contradiction because $V_1 \cap V_2 = \{0\}$. Similarly if (ii) was true, then the number of elements in both V_1 and V_2 would at least be $\frac{1}{2}(d-\delta)(d-\delta-1) > (d-\delta)$, for $(d-\delta) \geq 2$, which would force V_1 and V_2 to contain some common elements of C'.

Thus only (iii) is true which implies that either $V = V_1$ or $V = V_2$, and so V is indecomposable, which completes our proof.

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