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VOLUME INTEGRAL MEANS OF HARMONIC FUNCTIONS ON SMOOTH BOUNDARY DOMAINS

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Abstract. We newly define the volume integral means of harmonic functions to characterize the weighted harmonic Bergman spaces. It is based on Xiao and Zhu's results on holomorphic Bergman spaces [5].

1. Introduction

Let Ω be a bounded domain with $\mathcal{C}^\infty$-boundary in $\mathbf{R}^n$. For $x \in \Omega$, let $r(x) := dist(x, \partial\Omega)$. For $\epsilon > 0$, we define

$$\Omega_\epsilon = \{y \in \Omega : r(y) > \epsilon\}.$$

Due to the smoothness of the boundary $\partial\Omega$ there exists $\epsilon > 0$ only depending on the shape of the region Ω such that the projection map $\pi : \Omega \setminus \Omega_\epsilon \to \partial\Omega$ is well defined and smooth. Additionally, for $0 \le r \le \epsilon$, the restriction of the projection map $\pi|_{\partial\Omega_r} : \Omega \setminus \Omega_\epsilon \to \partial\Omega$ is one to one and onto, and for all $\eta \in \partial\Omega_r$ can be written as

$$\eta = \pi(\eta) + r\mathbf{n}_{\pi(\eta)}, \tag{1.1}$$

where $\mathbf{n}_\zeta$ denotes the inward unit normal to $\partial\Omega$ at $\zeta \in \partial\Omega$. Furthermore, for all $0 \le r \le \epsilon$ and for nonnegative continuous functions f on $\Omega_r \setminus \Omega_\epsilon$,

$$\int_{\Omega_r \setminus \Omega_\epsilon} f(x)dV(x) \approx \int_{\partial\Omega} \int_r^\epsilon f(\zeta + s\mathbf{n}_\zeta) ds dS(\zeta), \tag{1.2}$$

where dV denotes the Lebesque measure on Ω and dS denotes the surface area measure on $\partial\Omega$. Throughout our paper, ϵ denotes a positive real value satisfying conditions we stated above and the subset D_r is defined as

$$D_r := \Omega_r \setminus \Omega_\epsilon, \qquad 0 \le r \le \epsilon.$$

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Especially, $D := \Omega \setminus \Omega_\epsilon$.

For $\alpha > -1$, $1 < p < \infty$, the weighted harmonic Bergman space $b^p_\alpha = b^p_\alpha(\Omega)$ is the set of all complex-valued harmonic functions f on Ω such that

$$\int_\Omega |f(x)|^p dV_\alpha(x) < \infty,$$

where $dV_\alpha(x) = r^\alpha(x)dV(x)$. Since $|f|^p$ is subharmonic on Ω, the maximum principle allows us to define another norm of p-harmonic Bergman space,

$$\begin{aligned} \|f\|^p_{p,\alpha} &:= \int_D |f(x)|^p dv_\alpha(x) \\ &:= c_{\alpha,\epsilon} \int_{\partial\Omega} \int_0^\epsilon |f(\zeta + s\mathbf{n}_\zeta)|^p s^\alpha ds d\sigma(\zeta) < \infty, \end{aligned} \tag{1.3}$$

where $d\sigma$ is normalized measure $\sigma(\partial\Omega) = 1$ and $c_{\alpha,\epsilon} = (\alpha+1)/\epsilon^{\alpha+1}$ so that we normalized the measure dv_α.

For $\alpha = -1$, $b^p_{-1} = h^p(\Omega)$ denote the p-harmonic Hardy space and by the maximum modulus theorem it is natural that we define the norm of f in $h^p(\Omega)$ as

$$\|f\|^p_p := \sup_{0<r<\epsilon} \int_{\partial\Omega} |f(\zeta + r\mathbf{n}_\zeta)|^p d\sigma(\zeta) < \infty.$$

In this paper, we are going to extend the results of volume integral means of holomorphic functions in [5] to those of harmonic Bergman functions in general bounded smooth domains in $\mathbf{R}^n$. In [5] Xiao and Zhu define the volume integral means of the holomorphic function f in the unit ball $\mathbf{B}_n$ in $\mathbf{C}^n$ such that

$$\mathbf{M}_{p,\alpha}(f,r) = \left[\frac{1}{V_\alpha(r\mathbf{B}_n)} \int_{r\mathbf{B}_n} |f(z)|^p dV_\alpha(z)\right]^{1/p}, \quad 0 \le r < 1, \tag{1.4}$$

where dV_α is the weighted volume measure on $\mathbf{B}_n$. They showed that $\mathbf{M}_{p,\alpha}(f,r)$ is strictly increasing as a function of $r \in [0,1)$ and they characterized Hardy and Bergman spaces using $\mathbf{M}_{p,\alpha}(f,r)$ as an application. Now, we introduce the volume mean integral on smooth boundary domains. Let f be a harmonic function on Ω in $\mathbf{R}^n$. For $1 < p < \infty$, the integral means of f are defined by

$$M^p_p(f,r) = \int_{\partial\Omega} |f(\zeta + r\mathbf{n}_\zeta)|^p d\sigma(\zeta), \qquad 0 < r \le \epsilon.$$

When $p = \infty$, we define

$$M_\infty(f,r) = \sup_{\zeta\in\partial\Omega} \{|f(x)| : x = \zeta + r\mathbf{n}_\zeta, \quad 0 < r \le \epsilon\}.$$

Let α be any real number. The volume mean integral of f is defined by

$$M^p_{p,\alpha}(f,r) = \frac{1}{v_\alpha(D_r)} \int_{D_r} |f(x)|^p dv_\alpha(x), \qquad 0 \le r \le \epsilon.$$

Now, we will state our results analogue to the holomorphic version.

Theorem 1.1. *Let Ω be a bounded domain with $\mathcal{C}^\infty$-boundary in $\mathbf{R}^n$. Suppose f is a non-constant harmonic function in Ω. Then for given $1 < p \leq \infty$, $\alpha \in \mathbf{R}$, $M_{p,\alpha}(f,r)$ is strictly increasing when r tends to 0.*

With properties of $M_{p,\alpha}(f,r)$ in Theorem 1.1, we characterize the harmonic Bergman spaces and the harmonic hardy spaces in smooth boundary domains in $\mathbf{R}^n$.

Theorem 1.2. *Let Ω be a bounded domain with $\mathcal{C}^\infty$-boundary in $\mathbf{R}^n$. Suppose $p > 1$ and f is a harmonic function on Ω and there exists $\epsilon > 0$ satisfying conditions* (1.1) *and* (1.2)*.*

(a) *If $\alpha > -1$, then*

$$\sup\{M_{p,\alpha}(f,r) : 0 < r \leq \epsilon\} = \|f\|_{p,\alpha}.$$

(b) *If $\alpha \leq -1$, then*

$$\sup\{M_{p,\alpha}(f,r) : 0 < r \leq \epsilon\} = \|f\|_{p}.$$

In Section 2 we present some backgrounds and the proposition that we need and in Section 3, we prove our main results. In the last section we will introduce another kind of the volume mean integral on the upper half space in $\mathbf{R}^n$ and also show that it has the same properties with the case of smooth boundary domains.

Notation. In the rest of the paper, we use the notation $A \lesssim B$ to mean $A \leq CB$ for some positive real number C and $A \approx B$ if $A \lesssim B \lesssim A$.

2. Backgrounds

In this section we present some basic facts which we need to prove our results.

2.1. Bounded smooth domains

It is easy to show that $\partial\Omega_r$ is also smooth for any $0 < r \leq \epsilon$. For each $\eta \in \partial\Omega_r$, we will find a neighborhood U_η and a real-valued function f defined on U_η such that

(1) $f(x) = 0$ on $\partial\Omega_r \cap U_\eta$,
(2) $f(x) < 0$ on $\Omega_r \cap U_\eta$,
(3) $\nabla f(x) \neq 0$ on $\partial\Omega_r \cap U_\eta$.

We know that projection $\pi : D \to \partial\Omega$ is well-defined and smooth. Since $\partial\Omega$ is smooth there exist a neighborhood $U_{\pi(\eta)}$ for $\eta \in D$ and defining function ϕ satisfying above (1), (2), and (3) on $\overline{\Omega} \cap U_{\pi(\eta)}$. Now, for fixed $r \in (0, \epsilon]$ and η on $\partial\Omega_r$, we consider the translation $\mathbf{t}(x) := x + r\mathbf{n}_{\pi(\eta)}$ for $x \in \overline{\Omega} \cap U_{\pi(\eta)}$. Then we can easily check $\mathbf{t}(\overline{\Omega} \cap U_{\pi(\eta)}) = \overline{\Omega_r} \cap U_\eta$ is one to one and onto and $\phi \circ \mathbf{t}^{-1} : \overline{\Omega_r} \cap U_\eta \to \mathbf{R}$ satisfied (1), (2) and (3). We refer to [2] for the geometric backgrounds of domains with smooth boundaries.

Proposition 2.1. *Let f be a continuous function on Ω. Then for fixed $0 < r < \epsilon$, there exist constants $C_1, C_2 > 0$ depending only on ϵ such that*

$$C_1 \int_{\partial\Omega} f(\zeta + r\mathbf{n}_\zeta)dS(\zeta) \leq \int_{\partial\Omega_r} f(\eta)dS(\eta) \leq C_2 \int_{\partial\Omega} f(\zeta + r\mathbf{n}_\zeta)dS(\zeta).$$

Proof. Since $\pi|_{\partial\Omega_r} : \partial\Omega_r \to \partial\Omega$ is a bijection and smooth map defined by $\pi|_{\partial\Omega_r}(\zeta + r\mathbf{n}_\zeta) = \zeta$, the determinant of the Jacobian $J\pi^{-1}|_{\partial\Omega_r} \approx 1$. Thus, the proof is complete by the following equation,

$$\begin{aligned} \int_{\partial\Omega_r} f(\eta)dS(\eta) &= \int_{\partial\Omega_r} f(\pi(\eta) + r\mathbf{n}_{\pi(\eta)})dS(\eta) \\ &= \int_{\partial\Omega} f(\zeta + r\mathbf{n}_\zeta)J\pi^{-1}|_{\partial\Omega_r} dS(\zeta) \\ &\approx \int_{\partial\Omega} f(\zeta + r\mathbf{n}_\zeta)dS(\zeta). \end{aligned}$$

□

2.2. Green identity

Let Ω be a bounded open subset with smooth boundary in $\mathbf{R}^n$. For $u, v \in \mathcal{C}^2(\overline{\Omega})$ the Green's formula states that

$$\int_\Omega (u\Delta v - v\Delta u)dV = \int_{\partial\Omega} (uD_{\mathbf{n}^+}v - vD_{\mathbf{n}^+}u)\, dS,$$

where $\mathbf{n}^+$ is the outward unit normal vector and $(D_{\mathbf{n}^+}u)(\zeta) = \nabla u(\zeta) \cdot \mathbf{n}^+(\zeta)$. The details for the statements above can be found in [1].

2.3. Subharmonic functions

Let f be a real-valued continuous function on Ω. For each $z \in \Omega$, if there exists a closed ball $\overline{B}(z, R) \subset \Omega$ such that

$$f(z) \leq \int_{S_n} f(z + r\zeta)d\sigma(\zeta)$$

whenever $0 < r < R$, then f is a subharmonic function on Ω. Especially, when $f \in \mathcal{C}^2(\overline{\Omega})$, f is subharmonic on Ω if and only if $\Delta f \geq 0$ on Ω. As we mentioned in the previous section, it is well known that subharmonic functions satisfy the maximum principle on the connected regions.

3. Volume integral means

The following theorem is the harmonic extension of the classical integral means of a holomorphic function in the unit ball in $\mathbf{C}^n$.

Theorem 3.1. *Let Ω be a bounded domain with $\mathcal{C}^\infty$-boundary in $\mathbf{R}^n$. Suppose $1 < p \leq \infty$ and f is a non-constant harmonic function in Ω. Then $M_p(f, r)$ is strictly decreasing for r over $(0, \epsilon)$.*

Proof. Since $|f|$ is subharmonic on Ω, the case $p=\infty$ is an consequence of the maximum modulus principle. So we assume $1<p<\infty$. First, we will estimate the case $2\le p<\infty$. Since $|f(x)|^p\in\mathcal{C}^2$ when $p\ge 2$, we obtain the following equality by Green identity

$$\int_{\Omega_r}\Delta|f(x)|^p dV(x)=\int_{\partial\Omega_r}D_{\mathbf{n}}|f(\eta)|^p dS(\eta). \tag{3.1}$$

By Proposition 2.1, we have

$$\int_{\partial\Omega_r}D_{\mathbf{n}}|f(\eta)|^p dS(\eta)\le C\int_{\partial\Omega}D_{\mathbf{n}}|f(\zeta+r\mathbf{n}_\zeta)|^p dS(\zeta) \tag{3.2}$$

for some positive constant C.

Since $D_{\mathbf{n}}|f(\zeta+r\mathbf{n}_\zeta)|^p=-\frac{d}{ds}|f(\zeta+s\mathbf{n}_\zeta)|^p|_{s=r}$, we have

$$\int_{\partial\Omega}D_{\mathbf{n}}|f(\zeta+r\mathbf{n}_\zeta)|^p dS(\zeta)=-\frac{d}{dr}\int_{\partial\Omega}|f(\zeta+r\mathbf{n}_\zeta)|^p dS(\zeta). \tag{3.3}$$

Let $f(x)=u(x)+iv(x)$ where $u(x),v(x)$ are real valued functions on Ω. Then by a simple calculation, we have

$$\begin{aligned}\Delta|f|^p&=p(p-2)|f|^{p-4}|u\nabla u+v\nabla v|^2+p|f|^{p-2}|\nabla f|^2\\&\ge p|f|^{p-2}|\nabla f|^2.\end{aligned}$$

Thus, we obtain from (3.1)

$$\frac{d}{dr}\int_{\partial\Omega}|f(\zeta+r\mathbf{n}_\zeta)|^p dS(\zeta)\le-\frac{1}{C}\int_{\Omega_r}p|f|^{p-2}|\nabla f|^2 dV(x). \tag{3.4}$$

Thus $M_p(f,r)$ is a non-increasing function over r when $2\le p<\infty$. Now, we consider the case $1<p<2$. We should say that we follow the calculation in [4, p.12]. First, we take the function $(|f|^2+\delta)^{p/2}$, $0<\delta<1$ and apply Green identity to get

$$\int_{\partial\Omega_r}D_{\mathbf{n}}(|f(\eta)|^2+\delta)^{p/2}dS(\eta)=\int_{\Omega_r}\Delta(|f(x)|^2+\delta)^{p/2}dV(x). \tag{3.5}$$

Since $(|f|^2+\delta)^{p/2}$ is a $\mathcal{C}^1$-function on Ω we obtain the following inequality from (3.2) and (3.3)

$$\begin{aligned}\lim_{\delta\to 0}\int_{\partial\Omega_r}D_{\mathbf{n}}(|f(\eta)|^2+\delta)^{p/2}dS(\eta)&=\int_{\partial\Omega_r}D_{\mathbf{n}}|f(\eta)|^p dS(\eta)\\&\lesssim-\frac{d}{dr}\int_{\partial\Omega}|f(\zeta+r\mathbf{n}_\zeta)|^p dS(\zeta).\end{aligned} \tag{3.6}$$

Since we have

$$\begin{aligned}\Delta[(|f|^2+\delta)^{p/2}]&=p(p-2)(|f|^2+\delta)^{p/2-2}|u\nabla u+v\nabla v|^2+p|\nabla f|^2(|f|^2+\delta)^{p/2-1}\\&\ge p(|f|^2+\delta)^{p/2-2}[(p-1)|f|^2+\delta]|\nabla f|^2,\end{aligned}$$

we obtain by Fatou's Lemma

$$\lim_{\delta\to 0}\int_{\Omega_r} \Delta(|f(x)|^2+\delta)^{p/2}dV(x) \geq \int_{\Omega_r} \liminf_{\delta\to 0} \Delta(|f(x)|^2+\delta)^{p/2}dV(x) \geq \int_{\Omega_r} p(p-1)|f|^{p-2}|\nabla f|^2 dV(x).$$

Therefore we have the following inequality from (3.5), (3.6)

$$\frac{d}{dr}\int_{\partial\Omega}|f(\zeta+r\mathbf{n}_\zeta)|^p dS(\zeta) \leq -\frac{1}{C}\int_{\Omega_r} p(p-1)|f|^{p-2}|\nabla f|^2 dV(x). \tag{3.7}$$

Consequently, (3.4) and (3.7) imply that $M_p^p(f,r)$ is non-decreasing as r tends to 0. Moreover, if there exist some r_0 such that $\frac{d}{dr}M_p^p(f,r_0)=0$, then we can notice ∇f must be 0 on Ω_{r_0} from (3.4) and (3.7). It induces a contradiction for the assumption. Therefore the proof is complete. □

Now, we will show the monotonicity of $M_{p,\alpha}(f,r)$ over r. To prove our results, we follow the argument used in [5].

Theorem 3.2. *Suppose $1<p\leq\infty$, $\alpha\in\mathbf{R}$ and f is a non-constant harmonic function in Ω. Then $M_{p,\alpha}(f,r)$ is strictly increasing when r tends to 0.*

Proof. Applying the Fubini Theorem, we have

$$\int_{D_r}|f(x)|^p\, dv_\alpha(x) = c_{\alpha,\epsilon}\int_{\partial\Omega}\int_r^\epsilon |f(\zeta+t\mathbf{n}_\zeta)|^p t^\alpha\, dt\, d\sigma(\zeta) = c_{\alpha,\epsilon}\int_r^\epsilon M_p^p(f,t)t^\alpha\, dt.$$

Since

$$v_\alpha(D_r) = \frac{c_{\alpha,\epsilon}}{\alpha+1}(\epsilon^{\alpha+1}-r^{\alpha+1}), \tag{3.8}$$

we obtain the similar form in [5] using integration by part,

$$M_{p,\alpha}^p(f,r) - M_p^p(f,r) = \frac{1}{v_\alpha(D_r)}\int_r^\epsilon \left[\frac{d}{dt}M_p^p(f,t)\right] v_\alpha(D_t)\, dt. \tag{3.9}$$

Also, we have the following equation

$$\frac{d}{dr}M_{p,\alpha}^p(f,r) = \frac{r^\alpha}{v_\alpha(D_r)}[M_{p,\alpha}^p(f,r)-M_p^p(f,r)].$$

Since the left-hand side of (3.9) is positive by Theorem 3.1, $M_{p,\alpha}^p(f,r)$ is strictly increasing when r tends to 0 unless f is constant. Thus the proof is complete. □

Corollary 3.3. *Suppose $1<p<\infty$, $\alpha\leq -1$, and f is a harmonic function on Ω. Then*

$$\lim_{r\to 0+}\int_{D_r}|f(x)|^p dv_\alpha(x) < \infty$$

if and only if f is identically 0 *on* Ω.

Proof. We first show that the integral is finite implies $f \equiv 0$ on Ω. Since $v_\alpha(D_r) \to +\infty$ as $r \to 0^+$ when $\alpha \le -1$, Theorem 3.2 and the assumption give us $M^p_{p,\alpha}(f,r) = 0$ for all $0 < r \le \epsilon$. Thus f must be a constant by Theorem 3.2. Moreover, the assumption implies that f is identically 0. The converse is trivial. □

Now, we induce the following applications naturally with our volume mean integrals.

Theorem 3.4. *Suppose* $1 < p \le \infty$ *and* f *is a harmonic function on* Ω.

(a) *If* $\alpha > -1$, *then*

$$\sup\{M_{p,\alpha}(f,r) : 0 < r \le \epsilon\} = \|f\|_{p,\alpha}.$$

(b) *If* $\alpha \le -1$, *then*

$$\sup\{M_{p,\alpha}(f,r) : 0 < r \le \epsilon\} = \|f\|_{p}.$$

Proof. We know that $v_\alpha(D_r) \to v_\alpha(D) = 1$ as $r \to 0^+$ when $\alpha > -1$. Thus we can obtain the result (a) by Theorem 3.2. For the proof of (b), we assume that $\alpha \le -1$ and f is not identically zero. Then Corollary 3.3 implies

$$\lim_{r\to 0+} \int_{D_r} |f(x)|^p dv_\alpha(x) = \infty.$$

Thus Theorem 3.2 and L'Hospital's rule give us that

$$\begin{aligned}
\sup_{0<r<\epsilon} M^p_{p,\alpha}(f,r) &= \lim_{r\to 0+} \frac{1}{v_\alpha(D_r)} \int_{D_r} |f(x)|^p dv_\alpha(x) \\
&= \lim_{r\to 0+} \int_{\partial\Omega} |f(\zeta + r\mathbf{n}_\zeta)|^p d\sigma(\zeta) \\
&= \sup_{0<r<\epsilon} M^p_p(f,r) = \|f\|^p_p
\end{aligned}$$

as desired. □

Corollary 3.5. *Suppose* $\alpha > -1$, $1 < p < \infty$ *and* f *is a non-constant harmonic function in* Ω. *Then* $\|f\|_{p,\alpha}$ *is strictly decreasing for* α.

Proof. Assume that $-1 < \alpha_1 < \alpha_2 < \infty$ and $\|f\|_{p,\alpha_1} < \infty$. Then we have

$$\begin{aligned}
\|f\|^p_{p,\alpha_2} &= -\int_0^\epsilon \frac{d}{dr}\left(\int_{D_r} |f|^p dv_{\alpha_2}\right) dr \\
&= -\frac{c_{\alpha_2,\epsilon}}{c_{\alpha_1,\epsilon}} \int_0^\epsilon r^{\alpha_2-\alpha_1} \frac{d}{dr}\left(\int_{D_r} |f|^p dv_{\alpha_1}\right) dr \\
&= \frac{c_{\alpha_2,\epsilon}}{c_{\alpha_1,\epsilon}} \int_0^\epsilon (\alpha_2-\alpha_1) r^{\alpha_2-\alpha_1-1} \int_{D_r} |f|^p dv_{\alpha_1} dr.
\end{aligned}$$

From Theorem 3.4(a) we know that for any $0 < r < \epsilon$,

$$\int_{D_r} |f|^p dv_{\alpha_1} \leq v_{\alpha_1}(D_r) \int_D |f|^p dv_{\alpha_1}.$$

Thus we obtain the following inequality

$$\begin{aligned} \|f\|_{p,\alpha_2}^p &\leq \frac{c_{\alpha_2,\epsilon}}{c_{\alpha_1,\epsilon}} \int_0^\epsilon (\alpha_2 - \alpha_1) r^{\alpha_2-\alpha_1-1} v_{\alpha_1}(D_r) \int_D |f|^p dv_{\alpha_1} dr \\ &= \|f\|_{p,\alpha_1}^p \frac{c_{\alpha_2,\epsilon}(\alpha_2 - \alpha_1)}{\alpha_1 + 1} \int_0^\epsilon r^{\alpha_2-\alpha_1-1}(\epsilon^{\alpha_1+1} - r^{\alpha_1+1}) dr \\ &= \|f\|_{p,\alpha_1}^p \end{aligned}$$

as we desired. □

Remark 3.6. It is easy to check that we can take $\epsilon = 1$ when Ω is the unit ball in $\mathbf{B}_n$. Then the volume integral mean of the harmonic function f is represented by

$$M_{p,\alpha}^p(f, r) = \frac{1}{v_\alpha(\mathbf{B}_{1-r})} \int_{\mathbf{B}_{1-r}} |f(x)|^p dv_\alpha(x),$$

where $\mathbf{B}_{1-r} := D_r = \{z \in \mathbf{B}_n : |z| < 1 - r\}$. Thus, if we change the variable $1 - r$ to s, then we can notice that $M_{p,\alpha}^p(f, s)$ is strictly increasing when s increases to 1 by Theorem 3.2. Thus, using the definition of volume mean integral with (1.4) under our assumptions, we can easily notice that the results and their proofs are the same with the case of holomorphic functions in [5]. Accordingly, we will use the definition (1.4) for the following section.

4. Further remark

In this section, we extend Remark 3.6 to the half-space setting. Because $\mathbf{H}$ is a unbounded domain, $V_\alpha(\mathbf{H})$ is not finite even for $\alpha > -1$. Instead we have (4.4) below as a substitute. We first introduce a modified Kelvin transform K which connects $\mathbf{B}$ and $\mathbf{H}$. Then the result for $\mathbf{H}$ follows from the result for $\mathbf{B}$.

For a fixed positive integer $n \geq 2$, let $\mathbf{H} = \mathbf{R}^{n-1} \times \mathbf{R}_+$ be the upper half-space where $\mathbf{R}_+$ denotes the set of all positive real numbers. We will write a point $x \in \mathbf{H}$ as $x = (x', x_n)$ where $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}_+$.

The map $\Phi : \mathbf{R}^n \setminus \{-\mathbf{e}\} \mapsto \mathbf{R}^n \setminus \{-\mathbf{e}\}$ is given by

$$\Phi(x) = \frac{(2x', 1 - |x|^2)}{|x + \mathbf{e}|^2}, \tag{4.1}$$

where $\mathbf{e} = (0', 1) \in \mathbf{H}$ is the standard reference point. Then the map Φ is a Möbius transform taking $\mathbf{B}$ onto $\mathbf{H}$ and $\mathbf{H}$ onto $\mathbf{B}$ with $\Phi(\mathbf{e}) = 0$ and $\Phi(0) = \mathbf{e}$. Also, Φ is an involution, i.e., $\Phi \circ \Phi$ is the identity map on $\mathbf{R}^n \setminus \{-\mathbf{e}\}$. The following identities are easily computed as

$$|\Phi(x) + \mathbf{e}| = \frac{2}{|x + \mathbf{e}|}, \qquad 1 - |\Phi(x)|^2 = \frac{4x_n}{|x + \mathbf{e}|^2} \tag{4.2}$$

and the Jacobian determinant $J\Phi$ is

$$J\Phi(x) = \left(\frac{2}{|x+\mathbf{e}|^2}\right)^n.$$

For $x, y \in \mathbf{B}$, the following identity comes from Lemma 2.2 in [3]

$$|\Phi(x) - \overline{\Phi(y)}| = \frac{2[x,y]}{|x+\mathbf{e}||y+\mathbf{e}|}, \tag{4.3}$$

where $[x,y] = \sqrt{1 - 2x\cdot y + |x|^2|y|^2}$.

A modified Kelvin transform K with respect to the point $-\mathbf{e}$ is defined by

$$K[f](x) = 2^{(n-2)/2}\frac{f\circ\Phi(x)}{|x+\mathbf{e}|^{n-2}}.$$

In case $n = 2$, note that $K[f](x) = f \circ \Phi(x)$. Then K is its own inverse and preserves harmonicity. That is, f is harmonic on $\mathbf{H}$ if and only if $K[f]$ is harmonic on $\mathbf{B}$. See Chapter 7 in [1] for details.

The pseudohyperbolic distance between two points $x, y \in \mathbf{H}$ is defined by

$$\rho(x,y) = \frac{|x-y|}{|x-\overline{y}|}.$$

For $x \in \mathbf{H}$, and $0 < r < 1$, let $E_r(x)$ denote the pseudohyperbolic ball of radius r centered at x.

Now, we define the volume integral mean on the upper half-space. If f is harmonic on $\mathbf{H}$ and $1 < p < \infty$, we define

$$M_{p,\alpha,\mathbf{H}}(f,r) = \frac{1}{\mu_\alpha(E_r(\mathbf{e}))}\int_{E_r(\mathbf{e})} |f(x)|^p \left(\frac{|x+\mathbf{e}|}{2}\right)^{(n-2)p} d\mu_\alpha(x), \tag{4.4}$$

where the weighted measure $d\mu_\alpha(x)$ is

$$d\mu_\alpha(x) = \frac{2^{2\alpha+n}x_n^\alpha}{|x+\mathbf{e}|^{2(n+\alpha)}}\, dV(x).$$

Theorem 4.1. *Suppose $1 < p < \infty$, $\alpha \in \mathbf{R}$ and f is a non-constant harmonic function on $\mathbf{H}$. Then the volume integral mean $M_{p,\alpha,\mathbf{H}}(f,r)$ is strictly increasing when r tends to 1.*

Proof. Note that from (4.3), we have

$$\rho(\Phi(x),\mathbf{e}) = \frac{|\Phi(x)-\Phi(0)|}{|\Phi(x)-\overline{\Phi(0)}|} = |x|$$

for $x \in \mathbf{B}_r$. It means

$$\Phi(\mathbf{B}_r) = E_r(\mathbf{e}).$$

Since f is a non-constant harmonic on $\mathbf{H}$, $K[f]$ is non-constant harmonic on $\mathbf{B}$. Using (4.2), we have

$$M_{p,\alpha}(K[f],r) = \frac{1}{V_\alpha(B_r)}\int_{B_r} 2^{(n-2)/2}\frac{|f\circ\Phi(x)|^p}{|x+\mathbf{e}|^{(n-2)p}}(1-|x|^2)^\alpha\, dV(x)$$

$$= \frac{1}{V_\alpha(B_r)} \int_{E_r(\mathbf{e})} \frac{|f(x)|^p}{|\Phi(x)+\mathbf{e}|^{(n-2)p}} (1-|\Phi(x)|^2)^\alpha J\Phi(x)\, dV(x)$$
$$= \frac{1}{\mu_\alpha(E_r(\mathbf{e}))} \int_{E_r(\mathbf{e})} |f(x)|^p \left(\frac{|x+\mathbf{e}|}{2}\right)^{(n-2)p} d\mu_\alpha(x).$$

Consequently, we get

$$M_{p,\alpha,\mathbf{H}}(f,r) = M_{p,\alpha}(K[f],r). \tag{4.5}$$

Thus the note above Remark 3.6 implies that $M_{p,\alpha,\mathbf{H}}(f,r)$ is strictly increasing when r tends to 1. The proof is complete. □

In case $n = 2$, this result gives us that the following integral mean on the upper half-plane is strictly increasing when r tends to 1,

$$M_{p,\alpha,\mathbf{H}}(f,r) = \frac{1}{\mu_\alpha(E_r(\mathbf{e}))} \int_{E_r(\mathbf{e})} |f(x)|^p \, d\mu_\alpha(x).$$

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