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WEIGHTED COMPOSITION OPERATORS ON THE MINIMAL MÖBIUS INVARIANT SPACE

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Abstract. We will characterize the boundedness and compactness of weighted composition operators on the minimal Möbius invariant space.

1. Introduction

Here and henceforth, $\mathbb{D}$ will denote the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The set of all conformal automorphisms of $\mathbb{D}$ forms a group, called a Möbius group and denoted by Aut($\mathbb{D}$). For any $\lambda \in \mathbb{D}$, let

$$\alpha_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}$$

be the Möbius transformation of $\mathbb{D}$. Let X be a linear space of analytic functions on $\mathbb{D}$. Then X is said to be Möbius invariant if $f \circ \alpha \in X$ for all $f \in X$ and all $\alpha \in$Aut($\mathbb{D}$). A typical example of Möbius invariant spaces is the Besov space. For $1 < p < \infty$, let B_p be the space of analytic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p-2} dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure on $\mathbb{D}$. Then B_p is the Banach space with the norm

$$\|f\|_{B_p} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p (1 - |z|)^{p-2} dA(z) \right)^{1/p}.$$

If $p = 2$, B_2 is the classical Dirichlet space that is minimal as Möbius invariant Hilbert space of analytic functions on $\mathbb{D}$. The analytic Besov space B_1 is the

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space of all analytic functions f for which

$$f(z) = \sum_{n=1}^{\infty} a_n \alpha_{\lambda_n}(z),$$

for some sequence $\{a_n\} \in \ell^1$ and $\{\lambda_n\}$ in $\mathbb{D}$. Then the norm $\|f\|_{B_1}$ is defined by

$$\|f\|_{B_1} = \inf\{\sum_{n=1}^{\infty} |a_n| : f(z) = \sum_{n=1}^{\infty} a_n \alpha_{\lambda_n}(z)\}.$$

It is known that B_1 is minimal, as it is contained in any Möbius invariant space and that the norm $\|f\|_{B_1}$ is equivalent to

$$|f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)| dA(z).$$

For the study of Besov spaces one can refer to [1, 2, 12, 13] and references therein.

Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then the weighted composition operator uC_φ is defined by

$$(uC_\varphi)f = u \cdot f \circ \varphi$$

for analytic functions f on $\mathbb{D}$. In these five decades, there has been much work on weighted composition operators on various spaces of analytic functions on $\mathbb{D}$. See [6, 8] for an overview of these results.

Composition operators between the Besov spaces have been investigated since Tjani [9] studied. Those operators on the minimal Möbius invariant subspace B_1 also have been studied. For example, see [3, 10]. In particular, Wulan and Xiong [10] proved that the compactness criterion of composition operators on B_p $(1 < p < \infty)$, which is Tjani's result [9], still holds for B_1. Furthermore, composition operators from the Besov spaces to any analytic function space have been characterized in [11]. Recently it is given the characterization of the weighted composition operators mapping the Besov spaces to the Bloch space in [4, 5]. However properties of each weighted composition operator acting from B_1 to B_1 are left behind. We here carry on studying this problem. That is, we will characterize the boundedness and compactness of weighted composition operators mapping the minimal Möbius invariant space B_1 to B_1.

2. Boundedness and compactness on B_1

In order to characterize boundedness and compactness on B_1, we introduce the new generalized integral type operators.

Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then we define

$$C_\varphi^u f(z) = \int_0^z (f \circ \varphi)'(\zeta) u(\zeta) d\zeta$$

and

$$D_\varphi^u f(z) = \int_0^z (f \circ \varphi)(\zeta) u'(\zeta) d\zeta$$

for analytic functions f on $\mathbb{D}$.

If $u \equiv 1$, then

$$C_\varphi^u f(z) = (f \circ \varphi)(z) - f(\varphi(0)) = C_\varphi f(z) - f(\varphi(0)) \quad \text{and} \quad D_\varphi^n f \equiv 0.$$

If $\varphi(z) \equiv z$, then

$$C_\varphi^u f(z) = \int_0^z f'(\zeta) u(\zeta) d\zeta$$

and

$$D_\varphi^u f(z) = \int_0^z f(\zeta) u'(\zeta) d\zeta.$$

At first we have the result on the boundedness of uC_φ on B_1.

Proposition 2.1. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then the following are equivalent.*

(i) *uC_φ is bounded on B_1.*

(ii) $\sup_{\lambda \in \mathbb{D}} \|uC_\varphi \alpha_\lambda\|_{B_1} < \infty$.

(iii) $\sup_{\lambda \in \mathbb{D}} \|(C_\varphi^u + D_\varphi^u)\alpha_\lambda\|_{B_1} < \infty$.

Proof. The equivalence of (i) and (ii) is trivial. As

$$(uC_\varphi \alpha_\lambda)'' = ((C_\varphi^u + D_\varphi^u)\alpha_\lambda)'',$$

we obtain the equivalence of (ii) and (iii). □

In the proof of characterization of compact (weighted) composition operators we usually need the so-called "weak convergence theorem", which we can show by the similar way as in the proof of Proposition 3.11 in [6].

Proposition 2.2. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Suppose that uC_φ is bounded on B_1. Then uC_φ is compact on B_1 if and only if $\|uC_\varphi f_n\|_{B_1} \to 0$ as $n \to \infty$ for every sequence $\{f_n\}_n$ in B_1 with $\|f_n\|_{B_1} \leq 1$ satisfying $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$.*

Thus we could characterize the compactness.

Theorem 2.3. *Let u be a fixed analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$ with $\|\varphi\|_\infty = 1$. Suppose that uC_φ is bounded on B_1. Then the following are equivalent.*

(i) *uC_φ is compact on B_1.*

(ii) $\lim_{|\lambda| \to 1} \|uC_\varphi(\alpha_\lambda - \lambda)\|_{B_1} = 0$.

(iii) $\lim_{|\lambda| \to 1} \|(C_\varphi^u + D_\varphi^u)(\alpha_\lambda - \lambda)\|_{B_1} = 0$.

(iv) $\lim_{r \to 1} \sup_{\lambda \in \mathbb{D}} \int_{\{|\varphi(z)| > r\}} |(uC_\varphi(\alpha_\lambda - \lambda))''(z)| dA(z) = 0$.

Proof. At first, by the boundedness we can note that u'', $2u'\varphi'+u\varphi''$ and $u(\varphi')^2$ are $L^1(\mathbb{D})$-summable since $uC_\varphi 1, uC_\varphi z$ and $uC_\varphi z^2$ are in B_1.

The implication (i)⇒(ii) is shown because $\alpha_\lambda - \lambda$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $|\lambda| \to 1$. The equivalence of (ii) and (iii) is implied since $(uC_\varphi(\alpha_\lambda - \lambda))'' = ((C_\varphi^u + D_\varphi^u)(\alpha_\lambda - \lambda))''$.

Next we will prove the implication (ii)⇒(iv). By (ii), for any $\varepsilon > 0$, there exists a constant δ, $0 < \delta < 1$, such that

$$\sup_{|\lambda|>\delta} \int_{\mathbb{D}} |(uC_\varphi(\alpha_\lambda - \lambda))''(z)| dA(z) < \varepsilon.$$

Moreover for all $r, 0 < r < 1$,

$$\sup_{|\lambda|>\delta} \int_{\{|\varphi(z)|>r\}} |(uC_\varphi(\alpha_\lambda - \lambda))''(z)| dA(z) < \varepsilon.$$

On the other hand,

$$\begin{aligned}
&\sup_{|\lambda|\le\delta} \int_{\{|\varphi(z)|>r\}} |(uC_\varphi(\alpha_\lambda - \lambda))''(z)| dA(z) \\
&= \sup_{|\lambda|\le\delta} \int_{\{|\varphi(z)|>r\}} |u''(z)(\alpha_\lambda(\varphi(z)) - \lambda) \\
&\qquad + 2u'(z)\alpha_\lambda'(\varphi(z))\varphi'(z) + u(z)\alpha_\lambda''(\varphi(z))(\varphi'(z))^2 \\
&\qquad\qquad + u(z)\alpha_\lambda'(\varphi(z))\varphi''(z)| dA(z) \\
&\le C\Big(\int_{\{|\varphi(z)|>r\}} |u''(z)| dA(z) \\
&\qquad + \int_{\{|\varphi(z)|>r\}} |2u'(z)\varphi'(z) + u(z)\varphi''(z)| dA(z) \\
&\qquad\qquad + \int_{\{|\varphi(z)|>r\}} |u(z)(\varphi'(z))^2| dA(z)\Big),
\end{aligned}$$

where $C = \max\{2, \sup\{|\alpha_\lambda'(\varphi(z))| : |\lambda| \le \delta, z \in \mathbb{D}\}, \sup\{|\alpha_\lambda''(\varphi(z))| : |\lambda| \le \delta, z \in \mathbb{D}\}\}$. Considering that u'', $2u'\varphi' + u\varphi''$ and $u(\varphi')^2$ are $L^1(\mathbb{D})$-summable,

$$\lim_{r\to 1} \sup_{|\lambda|\le\delta} \int_{\{|\varphi(z)|>r\}} |(uC_\varphi(\alpha_\lambda - \lambda))''(z)| dA(z) = 0.$$

Consequently,

$$\begin{aligned}
&\lim_{r\to 1} \sup_{\lambda\in\mathbb{D}} \int_{\{|\varphi(z)|>r\}} |(uC_\varphi\alpha_\lambda)''(z)| dA(z) \\
&\le \lim_{r\to 1} \sup_{|\lambda|\le\delta} \int_{\{|\varphi(z)|>r\}} |(uC_\varphi\alpha_\lambda)''(z)| dA(z) \\
&\qquad + \lim_{r\to 1} \sup_{|\lambda|>\delta} \int_{\{|\varphi(z)|>r\}} |(uC_\varphi\alpha_\lambda)''(z)| dA(z) \\
&< \varepsilon.
\end{aligned}$$

As ε is arbitrary,

$$\lim_{r\to 1}\sup_{\lambda\in\mathbb{D}}\int_{\{|\varphi(z)|>r\}}|(uC_\varphi\alpha_\lambda)''(z)|dA(z)=0.$$

So we obtain condition (iv).

Finally we show the implication (iv)⇒(i). By condition (iv) and the $L^1(\mathbb{D})$-summability of u'', for any $\varepsilon>0$, there is a constant r, $0<r<1$, such that

$$\int_{\{|\varphi(z)|>r\}}|(uC_\varphi(\alpha_\lambda-\lambda))''(z)|dA(z)<\varepsilon$$

and

$$\int_{\{|\varphi(z)|>r\}}|u''(z)|dA(z)<\varepsilon.$$

Let $\{f_n\}_n$ be a sequence of functions in B_1 with $\|f_n\|_{B_1}\le 1$ satisfying $f_n\to 0$ uniformly on compact subsets of $\mathbb{D}$. Then we have

$$f_n(z)=\sum_{k=1}^{\infty}a_{n,k}\alpha_{\lambda_{n,k}}(z),\quad \lambda_{n,k}\in\mathbb{D},$$

with

$$\|f_n\|_{B_1}\le\sum_{k=1}^{\infty}|a_{n,k}|\le 2.$$

Trivially $|(uC_\varphi f_n)(0)|+|(uC_\varphi f_n)'(0)|\to 0$ as $n\to\infty$.

Then

$$\begin{aligned}&\|uC_\varphi f_n\|_{B_1}\\ &=\int_{\mathbb{D}}|(uC_\varphi f_n)''(z)|\;dA(z)\\ &=\int_{\{|\varphi(z)|\le r\}}|(uC_\varphi f_n)''(z)|dA(z)+\int_{\{|\varphi(z)|>r\}}|(uC_\varphi f_n)''(z)|dA(z).\end{aligned}$$

Here

$$\begin{aligned}&\int_{\{|\varphi(z)|>r\}}|(uC_\varphi f_n)''(z)|dA(z)\\ &\le\int_{\{|\varphi(z)|>r\}}|(uC_\varphi(\sum_{k=1}^{\infty}a_{n,k}(\alpha_{\lambda_{n,k}}-\lambda_{n,k})))''(z)|dA(z)\\ &\qquad+\int_{\{|\varphi(z)|>r\}}|(uC_\varphi(\sum_{k=1}^{\infty}a_{n,k}\lambda_{n,k}))''(z)|dA(z)\\ &\le\sum_{k=1}^{\infty}|a_{n,k}|\int_{\{|\varphi(z)|>r\}}|(uC_\varphi(\alpha_{\lambda_{n,k}}-\lambda_{n,k}))''(z)|dA(z)\\ &\qquad+\sum_{k=1}^{\infty}|a_{n,k}|\int_{\{|\varphi(z)|>r\}}|u''(z)|dA(z)\end{aligned}$$

$$< 4\varepsilon.$$

So

$$\lim_{n\to\infty} \|uC_\varphi f_n\|_{B_1} \le \lim_{n\to\infty} \int_{\{|\varphi(z)|\le r\}} |(uC_\varphi f_n)''(z)|dA(z) + 4\varepsilon$$
$$= 4\varepsilon.$$

As ε is arbitrary,

$$\lim_{n\to\infty} \|uC_\varphi f_n\|_{B_1} = 0.$$

By Proposition 2.2, uC_φ is compact on B_1. □

Lastly we add some comment in the unweighted case. Since functions in B_1 extend continuously to the boundary of $\mathbb{D}$, a result of [7] characterizes the compactness of C_φ on B_1, so that C_φ is compact on B_1 if and only if $\varphi \in B_1$ and $\|\varphi\|_\infty < 1$.

By noticing that α'_λ and α''_λ converge uniformly to 0 on compact subsets of $\mathbb{D}$, we obtain the following ([10]).

Corollary 2.4. *For an analytic self-map φ of $\mathbb{D}$, C_φ is compact on B_1 if and only if $\varphi \in B_1$ and*

$$\lim_{|\lambda|\to 1} \int_{\mathbb{D}} |\left(C_\varphi(\alpha_\lambda - \lambda)\right)''(z)|dA(z) = \lim_{|\lambda|\to 1} \int_{\mathbb{D}} |(\alpha_\lambda \circ \varphi)''(z)|dA(z) = 0.$$

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