

THE AUTOMORPHISM GROUP OF COMMUTING GRAPH OF A FINITE GROUP

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ABSTRACT. Let G be a finite group and X be a union of conjugacy classes of G . Define $C(G, X)$ to be the graph with vertex set X and $x, y \in X$ ($x \neq y$) joined by an edge whenever they commute. In the case that $X = G$, this graph is named commuting graph of G , denoted by $\Delta(G)$. The aim of this paper is to study the automorphism group of the commuting graph. It is proved that $\text{Aut}(\Delta(G))$ is abelian if and only if $|G| \leq 2$; $|\text{Aut}(\Delta(G))|$ is of prime power if and only if $|G| \leq 2$, and $|\text{Aut}(\Delta(G))|$ is square-free if and only if $|G| \leq 3$. Some new graphs that are useful in studying the automorphism group of $\Delta(G)$ are presented and their main properties are investigated.

1. Introduction

In this section we recall some definitions that will be used in the paper. Let G be a group and X a union of its conjugacy classes. Then $C(G, X)$ is the graph whose vertex set is X with $x, y \in X$, $x \neq y$, joined by an edge whenever they commute. The graph $C(G, X)$ has been considered in many different ways, by several authors. The case $X = G \setminus Z(G)$ is denoted by $\Gamma(G)$, the complement of this graph has been studied in [1, 10]. This graph has long been an object of interest in group theory [7, 11]. We investigated the metric structure of the complement of $\Gamma(G)$, for finite groups, [8]. In the case that $X = G$, this graph is called a *commuting graph* of G and denoted by $\Delta(G)$.

We consider simple graphs which are undirected, without loops or multiple edges. For any graph Γ , we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. Suppose $v \in V(\Gamma)$ and $V_1(\Gamma) \subseteq V(\Gamma)$. Then $N(v)$ is the set of neighbors of v and $\langle V_1(\Gamma) \rangle$ is the subgraph of Γ induced by $V_1(\Gamma)$. The *complement* of Γ is the graph $\bar{\Gamma}$ on the same vertices such that two vertices of $\bar{\Gamma}$ are adjacent if and only if they are not adjacent in Γ . For two graphs with disjoint vertex sets V_1 and V_2 their *union* is the graph H in which $V(H) = V_1 \cup V_2$ and $E(H) = E_1 \cup E_2$. Define nH to be the union of n disjoint

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copies of H . The set of all automorphisms of a graph Γ forms a permutation group, $\text{Aut}(\Gamma)$, acting on the object set $V(\Gamma)$.

Let A and B be permutation groups acting on object sets X and Y , respectively. Define $B \wr A = \{(a, f) \mid a \in A, f : X \rightarrow B\}$, $(a, f)(x, y) = (ax, b_x y)$ where $f(x) = b_x$. $B \wr A$ is called the *wreath product*. It acts on $X \times Y$ as follows: for each $a \in A$ and any sequence b_1, b_2, \dots, b_n (where $n = |X|$) in B , there is a unique permutation in $A \wr B$ written $(a; b_1, \dots, b_n)$, and $(a; b_1, \dots, b_n)(x_i, y_i) = (ax_i, b_i y_i)$.

Suppose S_n denotes the symmetric group on $\{1, 2, \dots, n\}$. In what follows, we describe some important results relating the automorphism group of a graph which are crucial in this paper. Frucht [4] described if Γ is a connected graph, then $\text{Aut}(n\Gamma) \cong (\text{Aut}(\Gamma)) \wr S_n$, by [3], if no component of Γ_1 is isomorphic with a component of Γ_2 , then $\text{Aut}(\Gamma_1 \cup \Gamma_2) \cong \text{Aut}(\Gamma_1) \times \text{Aut}(\Gamma_2)$ and applying the last two theorems we have the result: Let $\Gamma = n_1\Gamma_1 \cup n_2\Gamma_2 \cup \dots \cup n_r\Gamma_r$, where n_i is the number of components of Γ isomorphic to Γ_i , then

$$\text{Aut}(\Gamma) \cong ((\text{Aut}(\Gamma_1)) \wr S_{n_1}) \times ((\text{Aut}(\Gamma_2)) \wr S_{n_2}) \times \dots \times ((\text{Aut}(\Gamma_r)) \wr S_{n_r}).$$

Throughout this paper our notation is standard and taken mainly from [2, 12, 14].

2. Main result

The commuting graphs were important in the 1960s and 70s as they provided early constructions of the modern sporadic simple groups and played an important role in the classification of $\{2, 3\}$ -transposition groups. Since then, various authors have continued to study these graphs determining many combinatorial quantities associated with graphs such as their diameter and their clique number and giving a similar treatment to closely related graphs such as the commuting involution graphs. With the best of our knowledge there is nothing in literature that considered the automorphism group structure on commuting or non-commuting graphs.

Lemma 1. *Let X be a subset of G of size k such that $C_G(x) = C_G(y)$ for all $x, y \in X$. Then $\text{Aut}(\Delta(G))$ contains a symmetric group S_k that acts k -transitively on X while fixing $G \setminus X$ point-wise. We stress three specific cases we shall need:*

- (i) $X = Z(G)$;
- (ii) $X = \{x^i \mid (i, n) = 1\}$, where $x \in G$ has order n (in which case $|X| = \phi(n)$);
- (iii) $X = \{x, zx, x^{-1}, zx^{-1}\}$, where $x \in G$, $z \in Z(G)$ (in which case $|X| = 4$ provided $z \neq 1$ and $x^2 \notin \{1, z\}$).

Proof. Obvious. □

Lemma 2. *Let $|G| \geq 3$ and $|Z(G)| = 2$. Then $\text{Aut}(\Delta(G))$ is nonabelian, and $|\text{Aut}(\Delta(G))|$ is neither a prime power nor square-free.*

Proof. If there exists $x \in G$ such that $x^2 \notin Z(G)$, then we may apply Lemma 1(iii) to obtain an S_4 in $\text{Aut}(\Delta(G))$, and the result follows. Thus we assume $x^2 \in Z(G)$ for all $x \in G$. But this means $\frac{G}{Z(G)}$ is an elementary abelian 2-group, i.e., G is extraspecial of order 2^{2m+1} for some $m \geq 1$ [15]. If $m = 1$, then $G \cong D_8$ or Q_8 and we have $\text{Aut}(\Delta(G)) \cong (Z_2 \wr S_3) \times Z_2$ and $|\text{Aut}(\Delta(G))| = 864$, as desired. If $m > 2$, then $\frac{\text{Aut}(G)}{\text{Inn}(G)}$ is isomorphic to one of the two orthogonal groups $O^\pm(2m; 2)$ and by [6, Table I] these are not prime power or square-free order group. This completes the proof. \square

Lemma 3. *Let $G \neq 1$ be a finite group with $Z(G) = 1$. Then*

- (i) $\text{Aut}(\Delta(G))$ is non-abelian,
- (ii) $|\text{Aut}(\Delta(G))|$ is not a prime power,
- (iii) $|\text{Aut}(\Delta(G))|$ is not square-free.

Proof. Since G is centerless, we see at once that G cannot be abelian, nor can it be a p -group for any prime p . Moreover, G embeds in $\text{Aut}(\Delta(G))$ in this case, whence (i) and (ii) both follow. Suppose that $|\text{Aut}(\Delta(G))|$ is square-free, in which case $|G|$ is square-free as well. By Lemma 1(ii), we have $\pi(G) = \{2, 3\}$. Thus, since G is non-abelian, we have $G \cong S_3$. But $\text{Aut}(\Delta(S_3)) \cong S_3 \times Z_2$, which establishes (iii). \square

We are now ready to prove one of our main results as follows:

Theorem 1. *Let G be a finite group. Then the following are hold:*

- (i) $\text{Aut}(\Delta(G))$ is abelian if and only if $|G| \leq 2$,
- (ii) $|\text{Aut}(\Delta(G))|$ is a prime power if and only if $|G| \leq 2$,
- (iii) $|\text{Aut}(\Delta(G))|$ is square-free if and only if $|G| \leq 3$.

Proof. Since $\text{Aut}(\Delta(Z_2)) \cong Z_2$ and $\text{Aut}(\Delta(Z_3)) \cong S_3$, all statements are verified in the reverse direction. To establish these statements in the forward direction, we may assume $Z(G) \neq 1$ by virtue of Lemma 3. First assume that $\text{Aut}(\Delta(G))$ is either abelian or a p -group. As $\text{Aut}(\Delta(G))$ cannot contain S_3 , we conclude from Lemma 1(i) that $|Z(G)| = 2$. The result now follows from Lemma 2. To establish (iii), we assume that $|\text{Aut}(\Delta(G))|$ is square-free. Then $|Z(G)| \leq 3$ follows at once from Lemma 1(i), as does $\pi(G) = \{2, 3\}$ from Lemma 1(ii). Hence, as $\frac{G}{Z(G)} \cong \text{Inn}(G)$ must also have square-free order, it is clear that $|\frac{G}{Z(G)}|$ divides 6. Suppose that $\frac{G}{Z(G)} \cong S_3$. Let $\theta \in \text{Aut}(\Delta(G))$ interchange 1 and z and fix all remaining vertices of $\Delta(G)$, where $Z(G) = \langle z \rangle$. Then $\langle \theta, \iota_x \rangle \cong Z_2 \times Z_2$, where ι_x is any involution in $\text{Inn}(G)$. Thus 4 divides $|\text{Aut}(\Delta(G))|$, and we reach a contradiction. This proves that $\frac{G}{Z(G)}$ must be cyclic, whence G is abelian, and the proof now follows since $|G| = |Z(G)| \leq 3$. \square

Lemma 4. *Let G be a finite group. Then*

- (1) $\text{Aut}(G) = \text{Aut}(\Delta(G))$ if and only if $|G| = 1$.

$$(2) \text{Aut}(\Delta(G)) \cong \text{Aut}(\Gamma(G)) \times S_{|Z(G)|}.$$

Proof. (1) It is easily seen that the map $\phi : x \rightarrow x^{-1}$ is an automorphism of the graph $\Delta(G)$, but it is an automorphism of the group G if and only if G is abelian. If $\text{Aut}(G) = \text{Aut}(\Delta(G))$ and $|G| = n$, then G is abelian and $\text{Aut}(G) = \text{Aut}(\Delta(G)) \cong S_n$. We conclude that $|G| = 1$.

(2) Suppose $\sigma \in S_{|Z(G)|}$. Define $A = \{f \in \text{Aut}(\Delta(G)) \mid \forall z \in Z(G) ; f(z) = z\}$ and $B = \{f_\sigma \mid \sigma \in S_{|Z(G)|}\}$, where $f_\sigma : G \rightarrow G$ is defined by

$$f_\sigma(x) = \begin{cases} x & x \notin Z(G), \\ \sigma(x) & x \in Z(G). \end{cases}$$

Then by the definition of $\Gamma(G)$ and $\Delta(G)$, $\text{Aut}(\Gamma(G)) \cong A \trianglelefteq \text{Aut}(\Delta(G))$, $B \trianglelefteq \text{Aut}(\Delta(G))$ and $A \cap B = 1$. Since $\text{Aut}(\Delta(G)) = \langle A \cup B \rangle$, $\text{Aut}(\Delta(G)) \cong A \times B \cong \text{Aut}(\Gamma(G)) \times S_{|Z(G)|}$, proving the result. \square

Suppose that G is a finite group. Then it is clear that $\text{Aut}(\Gamma(G))$ and $\text{Aut}(G)$ are subgroups of $\text{Aut}(\Delta(G))$. Define two permutations $\Phi_{x,y}, \phi : G \rightarrow G$ as follows: $\Phi_{x,y} = (x, y)$ and the permutation ϕ is inverse mapping $x \rightarrow x^{-1}$. We also define $\text{Aut}^*(G) = \langle \text{Aut}(G), \phi \rangle$ and restrict our attention to the equality of the subgroup and the main group.

Theorem 2. $\text{Aut}^*(G) = \text{Aut}(\Delta(G))$ if and only if $G \cong S_3$.

Proof. We first notice that if $G \cong S_3$, then $\text{Aut}(\Delta(S_3)) = \text{Aut}(\Gamma(S_3))$. Since $\Gamma(S_3)$ is a union of three isolated vertices and one edge, $\text{Aut}(\Delta(S_3)) \cong Z_2 \times S_3$. On the other hand, $\text{Aut}^*(S_3) = \langle S_3, \phi \rangle \cong Z_2 \times S_3$, as desired.

Conversely, suppose that $\text{Aut}^*(G) = \text{Aut}(\Delta(G))$. We first prove $|Z(G)| = 1$. If there exists $1 \neq a \in Z(G)$, then the map $f_a : G \rightarrow G, x \rightarrow ax$, is an element of $\text{Aut}(\Delta(G))$. Since ϕ is an automorphism of order 2 and each $f \in \text{Aut}(G)$ commute with ϕ , $f_a = f_1 \circ \phi$, where $f_1 \in \text{Aut}(G)$. By definition of f_a and ϕ it can easily be seen that for each $x \in G$, $f_1(x) = x^{-1}a^{-1}$, a contradiction. Therefore $|Z(G)| = 1$. Consider an element $x \in G$ and define $\psi_x : G \rightarrow G$ which sends the elements $x, x^{-1} \in G$ to their inverses and assigns each element of $G \setminus \{x, x^{-1}\}$ to itself. We claim that $\psi_x \in \text{Aut}(\Delta(G)) \setminus \text{Aut}(G)$, when $o(x) > 2$. To do this, we notice that $\psi_x \in \text{Aut}(G)$ if and only if $o(x) = 2$. Suppose $\psi_x \in \text{Aut}(G)$. Since $|Z(G)| = 1$ and G is a non-abelian group, $|G| > 5$. Choose $t \in G \setminus \{1, x^{-2}\}$. Then $xt = \psi_x(xt) = \psi_x(x)\psi_x(t) = x^{-1}t$ and so $x^2 = 1$. Notice that if G is an elementary abelian group of order 2^m , then we have a contradiction by $|Z(G)| = 1$. Now consider $x \in G$ with $o(x) \geq 3$. Since $\psi_x \in \text{Aut}(\Delta(G)) = \text{Aut}^*(G)$, $\psi_x = f \circ \phi$, where $f \in \text{Aut}(G)$. Thus, $f(x) = x, f(x^{-1}) = x^{-1}$, and $f(y) = y^{-1}$ for $y \in G \setminus \{x, x^{-1}\}$. We claim that G has exactly two elements of order three and other non-identity elements are involutions. If there exists $x \in G$ with $o(x) \geq 4$, then we choose $t \in G \setminus \{x, x^{-1}\}$ and consider the image of x^2t under f . If $x^2t = x$, then $t = x^{-1}$, a contradiction. If $x^2t = x^{-1}$, then $x^{-1} = f(x^2t) = f(x^2)f(t) = x^{-2}t$ and $x = t$, which is impossible. Finally, $x^2t \neq x, x^{-1}$ and so $f(x^2t) = t^{-1}x^{-2}$ and $f(x^2)f(t) = x^{-2}t^{-1}$ which implies

that $x^{-2}t^{-1} = t^{-1}x^{-2}$. The later shows that $x^2 \in Z(G) = 1$, leads to our final contradiction. Hence, G does not have elements of order ≥ 4 .

On the other hand, if $o(x) = 3$, then $f(x^2t) = f(x^2)f(t)$ and we have $t^{-1}xt = x^{-1}$. Since G is not an elementary abelian 3-group, G has an element z with $o(z) = 2$. Suppose G has an element $y \neq x, x^{-1}$ with $o(y) = 3$. Then we have three choices for $f(y^2z)$. If $y^2z = x$, then $f(y^2z) = y^2z$ and $f(y^2)f(z) = yz$. Therefore, $y = 1$ which is impossible. A similar argument shows that y^2z cannot be equal to x^{-1} . If $f(y^2z) = zy$ and $f(y^2)f(z) = yz$, then $y \in C_G(z)$ which implies that $o(yz) = 6$, a contradiction. Therefore, $\{x, x^{-1}\}$ is the unique G -conjugacy class of elements of order 3.

Our discussion above shows that $G = \langle x, t_1, t_2, \dots, t_n \rangle$ such that $o(t_i) = 2$, $o(x) = 3$ and $t_i^{-1}xt_i = x^{-1}$ where $1 \leq i \leq n$. We have two cases:

1) If for $1 \leq i \neq j \leq n$, $t_it_j = t_jt_i$ and $n \geq 2$, then $t_1^{-1}xt_1 = x^{-1}$ and $t_2^{-1}xt_2 = x^{-1}$. So $(t_1t_2)^{-1}x(t_1t_2) = x$ or $(t_1t_2)x = x(t_1t_2)$. Hence $t_1t_2 \in Z(G)$, which is a contradiction. Therefore $n = 1$ and $G \cong S_3$.

2) Suppose $t_1t_2 \neq t_2t_1$ then $o(t_1t_2) = 3$, $t_1t_2 = x$ and $t_2t_1 = x^{-1}$. If for every t_i , $t_it_1 = t_1t_i$ and $t_it_2 = t_2t_i$, then $x^{-1} = t_i^{-1}xt_i = t_i^{-1}(t_1t_2)t_i = t_1t_2 = x$, which is a contradiction. So, $t_it_2 = x^{-1}$ and $t_2t_i = x$. If $t_it_1 = t_1t_i$, then $x^{-1} = t_1^{-1}xt_1 = t_1^{-1}t_2t_it_1 = t_1^{-1}t_2t_1t_i = t_1^{-1}x^{-1}t_i = xt_1t_i$, therefore $t_1t_i = x$ which is impossible. Finally, we consider the group $G = \langle t_1, t_2, t_3 \rangle$ such that $t_1t_2 = t_2t_3 = t_3t_1 = x$. According to the above, it is obvious that $G \cong S_3$. If $n \geq 4$, then $\{t_1, t_2, t_3\} \subseteq C_G(t_4)$ and so G is isomorphic to S_3 , as desired. \square

We consider the right cosets $Z(G)x_1, Z(G)x_2, \dots, Z(G)x_{m-1}$ of the group G and define a new graph $\Delta^u(G)$ with

$$V(\Delta^u(G)) = \{x_0 = 1, x_1, \dots, x_{m-1}\},$$

$$E(\Delta^u(G)) = \{x_ix_j \mid x_ix_j = x_jx_i, 0 \leq i < j \leq m-1\}.$$

Notice when $|Z(G)| = 1$ then $\Delta(G) \cong \Delta^u(G)$ [9]. It is clear that every two elements in one of these cosets commute. Hence we have complete graph in any of these cosets. On the other hand, if there exist $x_i \in Z(G)x_i, x_j \in Z(G)x_j$ satisfying $x_ix_j = x_jx_i$, then for every $y_i \in Z(G)x_i, y_j \in Z(G)x_j$ we have $y_iy_j = y_jy_i$. Finally, the set of all $\phi \in \text{Aut}(\Delta(G))$ such that for all $a, b \in G$, $ab^{-1} \in Z(G)$ implies that $\phi(a)\phi(b)^{-1} \in Z(G)$ is denoted by T .

Theorem 3. *Let G be a finite group. Then,*

- (1) $\text{Aut}(\Delta^u(G))$ is a subgroup of $\text{Aut}(\Delta(G))$. Moreover, $\text{Aut}(\Delta^u(G)) = \text{Aut}(\Delta(G))$ if and only if $|Z(G)| = 1$.
- (2) If G is not centerless, then T is a subgroup of $\text{Aut}(\Delta(G))$, and $\text{Aut}(\Delta(G)) = T$ if and only if for each pair a, b of elements of G with $C_G(a) = C_G(b)$, we have $ab^{-1} \in Z(G)$.

Proof. (1) Suppose for every $0 \leq i \leq m-1$, $Z(G)x_i = \{x_{i,1}, \dots, x_{i,|Z(G)|}\}$. For every $\phi \in \text{Aut}(\Delta^u(G))$ we define the automorphism $\phi' \in \text{Aut}(\Delta(G))$ in this way: $\phi'(x_{i,k}) = x_{j,k}$, where $\phi(x_i) = x_j$. The set of all such automorphisms is

isomorphic to $\text{Aut}(\Delta^u(G))$ and also this set is a subgroup of $\text{Aut}(\Delta(G))$. If $\text{Aut}(\Delta^u(G)) = \text{Aut}(\Delta(G))$ and $x_{0,1}, x_{0,2} \in Z(G)$, then $\Phi_{x_{0,1}, x_{0,2}} \in \text{Aut}(\Delta(G))$ but not in $\text{Aut}(\Delta^u(G))$. Therefore $|Z(G)| = 1$. The converse of equality is trivial.

(2) It is enough to prove that if $\phi_1, \phi_2 \in T$, then $\phi_1 \circ \phi_2 \in T$. Suppose $ab^{-1} \in Z(G)$ and $Z(G)\phi_1(a) = Z(G)\phi_1(b)$ then $Z(G)\phi_1 \circ \phi_2(a) = Z(G)\phi_1 \circ \phi_2(b)$ as desired. For equality suppose $\text{Aut}(\Delta(G)) = T$, $a, b \in G$ such that $C_G(a) = C_G(b)$ and $ab^{-1} \notin Z(G)$. Since $|Z(G)| > 1$, there is an element $a' \in Z(G)a$ such that $a \neq a'$. If $\phi_{a',b} \in \text{Aut}(\Delta(G))$, then $aa'^{-1} \in Z(G)$ and $\phi(a)\phi(a')^{-1} \notin Z(G)$, leads to a contradiction. Suppose $\phi \in \text{Aut}(\Delta(G))$ and $a, b \in G$ satisfying $ab^{-1} \in Z(G)$ and $C_G(a) = C_G(b)$ then $C_G(\phi(a)) = C_G(\phi(b))$ and by assumption we obtain that $\phi(a)\phi(b)^{-1} \in Z(G)$. Therefore $\phi \in T$. \square

Corollary 1. *Let $|Z(G)| \geq 2$, where G be a non-abelian group. If $T = \text{Aut}(\Delta(G))$, then $G/Z(G)$ is an elementary abelian 2-group.*

Proof. By Theorem 3, for every $x \in G$, $x^2 \in Z(G)$. Therefore $G/Z(G)$ is an elementary abelian 2-group. \square

Suppose $|\frac{G}{Z(G)}| = 4$, then $T = \text{Aut}(\Delta(G))$ if and only if $\Delta^u(G)$ is a star. So, it is natural to ask the following question:

Question. What can be the graph $\Delta^u(G)$ when $T = \text{Aut}(\Delta(G))$ and $\frac{G}{Z(G)}$ is an elementary abelian group of order 2^n , for $n > 2$?

For a finite group G we define a labeled graph $\Delta^v(G)$ as follows. For $a, b \in G$ let $a \sim b$ if $C_G(a) = C_G(b)$. Clearly, \sim is an equivalence relation, the equivalence class of $a \in G$ is $A(a) = \{x \mid C_G(x) = C_G(a)\}$. Let us denote the equivalence classes by A_1, \dots, A_k , these are the vertices of $\Delta^v(G)$. Two vertices A_i and A_j are connected if and only if $a_i a_j = a_j a_i$ for some $a_i \in A_i, a_j \in A_j$. At first we note that if there exist $a_i \in A_i, a_j \in A_j$ satisfying $a_i a_j = a_j a_i$, then for every $b_i \in A_i, b_j \in A_j$ we have $a_j \in C_G(a_i) = C_G(b_i)$. So, $b_i \in C_G(a_j) = C_G(b_j)$ implies that $b_i b_j = b_j b_i$. Each equivalence class is the union of some sets of the form $tZ(G)$, hence there exists a positive integers c_i such that $|A_i| = c_i |Z(G)|$. Let $\alpha(A_i) = c_i$ be the label of the vertex A_i in $\Delta^v(G)$. We say that $\phi : V(\Delta^v(G)) \rightarrow V(\Delta^v(G))$ is an automorphism of the labeled graph $\Delta^v(G)$ if ϕ is a bijection, it preserves the edges (and the non-edges) and it preserves the labels. The automorphism group formed by these automorphisms is denoted by $\text{Aut}(\Delta^v(G))$.

Define $S_{A_i} = \{f_\sigma \mid \sigma \in S_{|A_i|}, \forall x \in A_i, f_\sigma(x) = \sigma(x), \forall x \notin A_i, f_\sigma(x) = x\}$, $1 \leq i \leq k$. Clearly, S_{A_i} is a subgroup of $\text{Aut}(\Delta(G))$. The connection between $\text{Aut}(\Delta(G))$ and $\text{Aut}(\Delta^v(G))$ is described by the following theorem:

Theorem 4. *There is a subgroup A of $\text{Aut}(\Delta(G))$ such that $A \cong \text{Aut}(\Delta^v(G))$ and $\text{Aut}(\Delta(G)) = \langle S_{A_1}, \dots, S_{A_k} \rangle \rtimes A$.*

Proof. Suppose $A_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,|A_i|}\}$, $1 \leq i \leq k$. For each $f \in \text{Aut}(\Delta^v(G))$, define $\bar{f}(a_{i,r}) = a_{j,r}$ such that $f(A_i) = A_j$, $1 \leq i, j \leq k$, $1 \leq r \leq |A_i|$. Then $\bar{f} \in \text{Aut}(\Delta(G))$. Set A to be the set of all such automorphisms. Clearly, $A \cong \text{Aut}(\Delta^v(G))$ and A is a subgroup of $\text{Aut}(\Delta(G))$. We prove for every $f \in \text{Aut}(\Delta(G))$, f is a composition of some automorphisms of A and S_{A_i} , $1 \leq i \leq k$. Define $\psi : V(\Delta^v(G)) \rightarrow V(\Delta^v(G))$, such that $\psi(A_i) = f(A_i)$, for each i . Note that if $f(a_{i,r}) \in A_j$, then $f(A_i) = A_j$, because for each $b \in A_i$, $C_G(b) = C_G(a_{i,r})$ and $C_G(f(b)) = C_G(f(a_{i,r}))$. So, $f(b) \in A_j$ and ψ is an element of $\text{Aut}(\Delta^v(G))$, because if $A_r A_s$ is an edge of $\Delta^v(G)$, then there exist $a_r \in A_r$ and $a_s \in A_s$ such that $a_r a_s$ is an edge of $\Delta(G)$. Thus, $f(a_r) f(a_s) \in E(\Delta(G))$. We conclude that $\psi(A_r) \psi(A_s) = f(A_r) f(A_s) \in E(\Delta^v(G))$. On the other hand, ψ preserves the labels, because $|A_i| = |f(A_i)| = |\psi(A_i)|$. Therefore $\psi \in \text{Aut}(\Delta^v(G))$. Suppose $\bar{\psi} \in A$ corresponds to ψ . If $a_{i,r}$ is arbitrary and $f(a_{i,r}) = a_{j,s}$, then there exists $f_{\sigma_j} \in S_{A_j}$ such that $f_{\sigma_j}(a_{j,r}) = a_{j,s}$. For $1 \leq i \leq k$ we can choose $f_{\sigma_i} \in S_{A_i}$ such that $f = f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_k} \circ \bar{\psi}$. Therefore, $\text{Aut}(\Delta(G)) = \langle S_{A_1}, \dots, S_{A_k} \rangle A$. Finally, for each $g \in \text{Aut}(\Delta(G))$ and $f \in \langle S_{A_1}, \dots, S_{A_k} \rangle$, there exists i , $1 \leq i \leq k$ such that $g^{-1} f g$ is a permutation on A_i and we conclude that $\langle S_{A_1}, \dots, S_{A_k} \rangle$ is a normal subgroup of $\text{Aut}(\Delta(G))$. This completes our proof. \square

In the end of this paper the automorphism group of $\Delta(G)$ is computed, when G is an AC group. To describe this concept, we first review some important properties of dihedral groups.

The dihedral group D_{2n} , can be presented as follows:

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1, y^{-1} x y = x^{-1} \rangle.$$

If n is odd, then $|Z(D_{2n})| = 1$, $C_{D_{2n}}(a^i) = \langle a \rangle$ and $C_{D_{2n}}(a^i b) = \langle a^i b \rangle$, $0 \leq i \leq n - 1$. If n is even, then $Z(D_{2n}) = \langle a^{\frac{n}{2}} \rangle$, $C_{D_{2n}}(a^i) = \langle a \rangle$, $1 \leq i \neq \frac{n}{2} \leq n - 1$, and $C_{D_{2n}}(a^i b) = \langle a^i b, a^{\frac{n}{2}} \rangle$, $0 \leq i \leq n - 1$. By above calculations, the graph $\Gamma(D_{2n})$ is:

- The union of complete graph K_{n-1} and n isolated vertices, when n is odd. In this case, $\text{Aut}(\Gamma(D_{2n})) \cong S_n \times S_{n-1}$.
- The union of complete graph K_{n-2} and $\frac{n}{2}$ copies of K_2 , which all vertices are adjacent by the elements of the center of G , when n is even. In this case, $\text{Aut}(\Gamma(D_{2n})) \cong S_{n-2} \times (S_2 \wr S_{\frac{n}{2}}) \times S_2$.

For the dihedral group, the map $\theta_{i,t} : \{x, y\} \rightarrow D_{2n}$ such that $x \rightarrow x^t$ and $y \rightarrow x^i y$, $0 \leq i \leq n - 1$, $1 \leq t \leq n - 1$ and $(t, n) = 1$ can be extended to an automorphism $\gamma_{i,t}$ of D_{2n} . Therefore, $\text{Aut}(D_{2n}) = \{\gamma_{i,t} \mid 0 \leq i \leq n - 1, 1 \leq t \leq n - 1, (t, n) = 1\}$. Thus $|\text{Aut}(D_{2n})| = n\phi(n)$, where $\phi(n)$ is the Euler's totient function. On the other hand, we proved that $|\text{Aut}(\Gamma(D_{2n}))| = n!(n - 1)!$, when n is odd and $|\text{Aut}(\Gamma(D_{2n}))| = (n - 2)!(\frac{n}{2})!2^{\frac{n}{2} + 1}$, otherwise.

We say a group G has abelian centralizers, if for each non-central element $x \in G$, the centralizer of x in G is abelian. A group G is an AC-group, whenever the centralizers of non-central elements are abelian. The dihedral group D_{2n}

is an example of an AC-group. In [13], Roche proved that the following are equivalent:

- (1) G has abelian centralizers;
- (2) If $xy = yx$, then $C_G(x) = C_G(y)$ whenever $x, y \notin Z(G)$;
- (3) If $xy = yx$ and $xz = zx$, then $yz = zy$ whenever $x \notin Z(G)$;
- (4) If U and B are subgroups of G and $Z(G) < C_G(U) \leq C_G(B) < G$, then $C_G(U) = C_G(B)$.

Therefore, the intersection of two proper element centralizers of an AC-group is the center of G . If G is an AC-group, then $\Delta(G)$ is a union of some complete graphs which all vertices are adjacent to the elements of $Z(G)$. So, $\Delta(G)$ is $n_1(C_G(x_1) \setminus Z(G)) \cup n_2(C_G(x_2) \setminus Z(G)) \cup \dots \cup n_r(C_G(x_r) \setminus Z(G))$ and also every elements of $Z(G)$ is adjacent to all elements of G , such that for each $i, 1 \leq i \leq r$, we have n_i isomorphic components with complete graph of size $|C_G(x_i) \setminus Z(G)|$. Hence, we prove the following theorem:

Theorem 5. *Let G be an AC-group, with the above notation*

$$\begin{aligned} \text{Aut}(\Delta(G)) \cong & ((S_{|C_G(x_1)|-|Z(G)|}) \wr S_{n_1}) \times ((S_{|C_G(x_2)|-|Z(G)|}) \wr S_{n_2}) \times \dots \\ & \times ((S_{|C_G(x_n)|-|Z(G)|}) \wr S_{n_r}) \times S_{|Z(G)|}. \end{aligned}$$

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