POINTWISE SLANT SUBMERSIONS

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ABSTRACT. The purpose of this paper is to study pointwise slant submersions from almost Hermitian manifolds which extends slant submersion in a natural way. Several basic results in this point of view are proven in this paper.

1. Introduction

As a generalization of slant submanifolds, pointwise slant submanifolds were first defined by Etayo [10] under the name of quasi-slant submanifolds and such submanifolds have been recently studied in details by Chen and Garay [7]. Chen and Garay obtain simple characterizations, give a method how to construct such submanifolds in Euclidean space and investigate geometric and topological properties of pointwise slant submanifolds. Since slant submanifolds include holomorphic submanifolds and totally real submanifolds, the class of pointwise slant submanifolds is a general notion for the theory of submanifolds of almost Hermitian manifolds.

On the other hand, Riemannian submersions between Riemannian manifolds were studied by O'Neill [19] and Gray [12]. Riemannian submersions have several applications in mathematical physics. Indeed, Riemannian submersions have their applications in the Yang-Mills theory ([2, 24]), Kaluza-Klein theory ([3, 15]), supergravity and superstring theories ([16, 18]), etc. Later such submersions were considered between manifolds with differentiable structures, see: [11]. As an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases [23]. We now recall the notion of almost Hermitian submersion. Let M be a complex m-dimensional almost Hermitian manifold with Hermitian metric g_M and almost complex structure J_M and N be a complex n-dimensional almost Hermitian manifold with Hermitian metric g_N and almost complex structure J_N. A Riemannian submersion $F : M \longrightarrow N$ is called an

almost Hermitian submersion (or holomorphic submersion) if F is an almost complex mapping, i.e., $F_*J_M = J_N F_*$. The main result of this notion is that the vertical and horizontal distributions are J_M-invariant. On the other hand, Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds have been studied by many authors under the assumption that the vertical spaces of such submersions are invariant with respect to the complex structure. For instance, Escobales [9] studied Riemannian submersions from complex projective space onto a Riemannian manifold under the assumption that the fibers are connected, complex, totally geodesic submanifolds. One can see that this assumption implies that the vertical distribution is invariant. We note that almost Hermitian submersions have been extended to the almost contact manifolds [8], [13], locally conformal Kähler manifolds [17] and quaternion Kähler manifolds [14].

In [20], the second author introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds as follows. Let M be a complex m-dimensional almost Hermitian manifold with Hermitian metric g_M and almost complex structure J and N be a Riemannian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $F : M \longrightarrow N$ such that the integral manifold of the distribution $\ker F_*$ is anti-invariant with respect to J, i.e., $J(\ker F_*) \subseteq (\ker F_*)^\perp$. Then we say that F is an anti-invariant Riemannian submersion. Recently, the second author also introduced the notion of slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds [21] as follows: Let F be a Riemannian submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). If for any non-zero vector $X \in \Gamma(\ker F_*)$, the angle $\theta(X)$ between JX and the space $\ker F_*$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector X in $\ker F_*$, then we say that F is a slant submersion. In this case, the angle θ is called the slant angle of the slant submersion.

In this paper, we study pointwise slant submersions from almost Hermitian manifolds which include slant submersions. As a result of this, they also cover holomorphic submersions and anti-invariant submersions [20]. In Section 2, we recall basic facts for Riemannian submersions and almost Hermitian manifolds. In Section 3, we define pointwise slant Riemannian submersions and give a method for obtaining examples of such submersions. We also obtain a characterization of pointwise slant submersions and check the relation between slant submanifolds and pointwise slant submanifolds. We also investigate the harmonicity of such maps and find necessary and sufficient conditions for the fibers to be totally geodesic submanifolds. Then we give necessary and sufficient conditions for pointwise slant Riemannian submersions to be totally geodesic.

2. Preliminaries

Let $(\bar{M}, g, J)$ be an almost Hermitian manifold. This means that $\bar{M}$ admits a tensor field J of type $(1,1)$ on $\bar{M}$ such that

$$(2.1) \qquad J^2 = -I, \quad g(JX, JY) = g(X, Y), \quad X, Y \in \Gamma(T\bar{M}).$$

An almost Hermitian manifold $\bar{M}$ is called Kähler manifold if

$$(2.2) \qquad \left(\bar{\nabla}_X J\right) Y = 0, \quad X, Y \in \Gamma(T\bar{M}),$$

where $\bar{\nabla}$ is the operator of Levi-Civita covariant differentiation.

Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds, where $\dim M = m$, $\dim N = n$ and $m > n$. A Riemannian submersion $F : M \longrightarrow N$ is a map from M onto N satisfying the following axioms:

(S1) F has the maximal rank.
(S2) The differential F_* preserves the lengths of horizontal vectors.

For each $q \in N$, $F^{-1}(q)$ is an $(m - n)$-dimensional submanifold of M. The submanifolds $F^{-1}(q)$ are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is horizontal and F-related to a vector field X_* on N, i.e., $F_* X_p = X_{*F(p)}$ for all $p \in M$. Note that we denote the projection morphisms on the distributions $\ker F_*$ and $(\ker F_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

We recall the following lemma from O'Neil [19].

Lemma 2.1. *Let $F : M \longrightarrow N$ be a Riemannian submersion between Riemannian manifolds and X, Y be basic vector fields of M. Then*

(a) $g_M(X, Y) = g_N(X_*, Y_*) \circ F$.
(b) *the horizontal part $[X, Y]^{\mathcal{H}}$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i.e., $F_*([X, Y]) = [X_*, Y_*]$.*
(c) $[V, X]$ *is vertical for any vector field V of $\ker F_*$.*
(d) $(\nabla_X^M Y)^{\mathcal{H}}$ *is the basic vector field corresponding to $\nabla_{X_*}^N Y_*$.*

The geometry of Riemannian submersions is characterized by O'Neil's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$(2.3) \qquad \mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F,$$

$$(2.4) \qquad \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F,$$

where ∇ is the Levi-Civita connection of g_M. It is easy to see that a Riemannian submersion $F : M \longrightarrow N$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$(2.5) \qquad \mathcal{T}_U W = \mathcal{T}_W U, \quad U, W \in \Gamma(\ker F_*),$$

$$(2.6) \qquad \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y], \qquad X, Y \in \Gamma((\ker F_*)^\perp).$$

On the other hand, from (2.3) and (2.4), we have

$$(2.7) \qquad \nabla_V W \;=\; \mathcal{T}_V W + \hat{\nabla}_V W,$$

$$(2.8) \qquad \nabla_V X \;=\; \mathcal{H}\nabla_V X + \mathcal{T}_V X,$$

$$(2.9) \qquad \nabla_X V \;=\; \mathcal{A}_X V + \mathcal{V}\nabla_X V,$$

$$(2.10) \qquad \nabla_X Y \;=\; \mathcal{H}\nabla_X Y + \mathcal{A}_X Y$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. If X is basic, then $\mathcal{H}\nabla_V X = \mathcal{A}_X V$. Note that we will also use the symbols $\mathcal{V}$ and $\mathcal{H}$ to denote the distributions $\ker F_*$ and $(\ker F_*)^\perp$, respectively.

Finally, we recall the notion of harmonic maps between Riemannian manifolds. Let (M, g_M) and (N, g_N) be Riemannian manifolds and supposed that $F : M \longrightarrow N$ is a smooth map. Then the differential F_* of F can be viewed a section of the bundle $\mathrm{Hom}(TM, F^{-1}TN) \to M$, where $F^{-1}TN$ is the pullback bundle which has fibers $(F^{-1}TN)_p = T_{F(p)}N$, $p \in M$. $\mathrm{Hom}(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection ∇^F. Then the second fundamental form of F is given by

$$(2.11) \qquad (\nabla F_*)(X, Y) = \nabla_X^F (F_*(Y)) - F_*(\nabla_X^M Y)$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. We note that if F is a Riemannian submersion between Riemannian manifolds, then we have

$$(2.12) \qquad (\nabla F_*)(X, Y) = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. A smooth map $F : (M, g_M) \longrightarrow (N, g_N)$ is said to be harmonic if $trace(\nabla F_*) = 0$. On the other hand, the tension field of F is the section $\tau(F)$ of $\Gamma(F^{-1}TN)$ defined by

$$(2.13) \qquad \tau(F) = div F_* = \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i),$$

where $\{e_1, \ldots, e_m\}$ is the orthonormal frame on M. Then it follows that F is harmonic if and only if $\tau(F) = 0$ (for details, see [1]).

3. Pointwise slant submersions

In this section, we are going to define pointwise slant submersions and investigate the geometry of such submersions.

Definition 3.1. Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). If, at each given point $p \in M$, the Wirtinger angle $\theta(X)$ between JX and the space $(\ker F_*)_p$ is independent of the choice of the nonzero vector $X \in (\ker F_*)$, then we say that F is a *pointwise slant submersion*. In this case, the angle θ can be regarded as a function on M, which is called the *slant function* of the pointwise slant submersion.

Definition 3.2. A point p in a pointwise slant submersion is called *totally real* if its slant function $\theta = \frac{\pi}{2}$ at p. Similarly, a point p is called a *complex point* if its slant function $\theta = 0$ at p.

Note. A pointwise slant submersion is called *slant* in the sense of [21] if its slant function θ is globally constant, i.e., θ is also independent of the choice of the point on M. In this case, the constant θ is called the *slant angle* of the slant submersion. A pointwise slant submersion F is called *totally real* if every point of M is a totally real point.

We now give an example for pointwise slant submersions.

Example 3.1. Let $(\mathbb{R}^4, g_0)$ be the standard Euclidean space with the standard metric g_0. Consider $\{J_0, J_1\}$ a pair of almost complex structures on $\mathbb{R}^4$ satisfying $J_0 J_1 = -J_1 J_0$, where

$$
\begin{aligned}
J_0(a, b, c, d) &= (-c, -d, a, b), \\
J_1(a, b, c, d) &= (-b, a, d, -c).
\end{aligned}
$$

For any real-valued function $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$, we define new almost complex structure J_f on $\mathbb{R}^4$ by

$$
J_f = (\cos f) J_0 + (\sin f) J_1.
$$

Then $\mathbb{R}^4_f = (\mathbb{R}^4, J_f, g_0)$ is an almost Hermitian manifold.
Consider a Riemannian submersion $F : \mathbb{R}^4_f \longrightarrow \mathbb{R}^2$ by $F(x_1, x_2, x_3, x_4) = (\frac{x_1 - x_2}{\sqrt{2}}, \frac{x_3 - x_4}{\sqrt{2}})$. Then F is a pointwise slant submersion with the slant function f.

Note. In fact, as Chen and Garay emphasized in [7] for pointwise slant submanifolds, if a Riemannian submersion is invariant [22] or holomorphic [23] with respect to J_1, then it will always produce a pointwise slant submersion with the slant function f. This shows that there are many pointwise slant submersions.

Now, we assume that F is a Pointwise slant Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto Riemannian manifold (N, g_N). Then for $V \in \Gamma(\ker F_*)$, we have

$$
(3.1) \qquad\qquad\qquad JV = \varphi V + \omega V,
$$

where φV (ωV, resp.) is vertical (horizontal, resp.) of JV. Also for $X \in \Gamma((\ker F_*)^\perp)$, we have

$$
(3.2) \qquad\qquad\qquad JX = \mathcal{B}X + \mathcal{C}X,
$$

where $\mathcal{B}X$ ($\mathcal{C}X$, resp.) is vertical (horizontal, resp.) of JX. We denote the orthogonal complementary distribution to $\omega(\ker F_*)$ in $(\ker F_*)^\perp$ by μ. That is,

$$
(\ker F_*)^\perp = \omega(\ker F_*) \oplus \mu.
$$

It is easy to see that μ is an invariant subbundle of $(\ker F_*)^\perp$ with respect to J. Using (2.7), (2.8), (3.1) and (3.2), we obtain

$$(3.3) \qquad (\nabla_V \omega)\, W \;\; = \;\; \mathcal{C}\mathcal{T}_V W - \mathcal{T}_V \varphi W,$$

$$(3.4) \qquad (\nabla_V \varphi)\, W \;\; = \;\; \mathcal{B}\mathcal{T}_V W - \mathcal{T}_V \omega W,$$

where ∇ is the Levi-Civita connection on M and

$$(\nabla_V \omega)\, W \;\; := \;\; \mathcal{H}\nabla_V \omega W - \omega \hat{\nabla}_V W,$$
$$(\nabla_V \varphi)\, W \;\; := \;\; \hat{\nabla}_V \varphi W - \varphi \hat{\nabla}_V W$$

for $V, W \in \Gamma(\ker F_*)$. We say that ω is parallel with respect to the Levi-Civita connection ∇ on $\ker F_*$ if its covariant derivative with respect to ∇ vanishes, i.e., $(\nabla_V \omega)\, W = 0$ for $V, W \in \Gamma(\ker F_*)$.

The proof of the following result is the same as slant immersions (see [7]), therefore we omit its proof.

Theorem 3.1. *Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then F is a pointwise slant submersion if and only if there exists a real-valued function θ defined on $\ker F_*$ such that*

$$\varphi^2 = - \left(\cos^2 \theta \right) I.$$

A pointwise slant submersion is said to be proper if it has neither complex points nor totally real points.

Corollary 3.2. *Let F be a pointwise proper slant submersion from an almost Hermitian manifold (M^m, g_M, J) onto a Riemannian manifold (N^n, g_N) with the slant function θ. If $e_1, e_2, \ldots, e_{m-n}$ are locally orthonormal basis for $\ker F_*$, then*

$$\{\csc \theta \omega e_1, \csc \theta \omega e_2, \ldots, \csc \theta \omega e_{m-n}\}$$

is a local orthonormal basis of $\omega(\ker F_*)$.

The following result is a consequence from Theorem 3.2 and Corollary 3.3.

Corollary 3.3. *Let F be a pointwise proper slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). Then the distributions μ and $\ker F_* \oplus \omega(\ker F_*)$ are even dimensional.*

The following proposition gives a characterization of pointwise slant submersions

Proposition 3.4. *F is a pointwise slant submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) if and only if $\varphi : \ker F_* \longrightarrow \ker F_*$ preserves orthogonality, φ carries each pair of orthogonal vectors into orthogonal vectors.*

Proof. Denote by $S(\ker F_*)$ the unit kernel bundle of $\ker F_*$. Then consider the function $\theta : S(\ker F_*) \longrightarrow R$ on the unit kernel bundle $S(\ker F_*)$. With respect to the induced metric, $S_p(\ker F_*)$ is the unit sphere in $\ker F_{*p}$ centered at o.

At a given point $p \in M$, we have

$$g_M(\varphi X, \varphi X) = \cos^2 \theta(X)$$

for any unit vector $X \in \ker F_{*p}$. For each unit vector Y tangent to $\ker F_{*p}$ at $X \in \ker F_{*p}$ $(X \perp Y)$, using $g_M(\varphi X, (\nabla_Y \varphi)X) = 0$, we have

$$2g_M(\varphi X, \varphi Y) = 2g_M(\varphi X, \varphi(\nabla_Y X)) = 2g_M(\varphi X, \nabla_Y(\varphi X) - (\nabla_Y \varphi)X)$$
$$= 2g_M(\varphi X, \nabla_Y(\varphi X)) = Y g_M(\varphi X, \varphi X) = -(Y\theta)\sin 2\theta(X).$$

In the long run, φ carries each pair of orthogonal vector in $\ker F_{*p}$ into a pair of orthogonal vectors in $\ker F_{*p}$ if and only if the slant function θ is independent of $X \in \ker F_{*p}$. $\square$

Proposition 3.5. *Let F be a pointwise slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). F is slant submersion if and only if*

$$\mathcal{T}_X \omega \varphi X = \mathcal{T}_{\varphi X} \omega X$$

for $X \in \Gamma(\ker F_*)$.

Proof. Let F be a pointwise slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) with the slant function θ. For any unit vector field $X \in \Gamma(\ker F_*)$, we may put

$$\varphi X = (\cos \theta) X^*,$$

where X^* is a unit vector field orthogonal of X. Then for $Y \in \Gamma(\ker F_*)$, we have

$$\nabla_Y(JX) = \nabla_Y((\cos\theta) X^*) + \nabla_Y(\omega X)$$
(3.5) $$= -(\sin\theta)(Y\theta) X^* + (\cos\theta)\left(\mathcal{T}_Y X^* + \hat{\nabla}_Y X^*\right)$$
$$+ \mathcal{H}\nabla_Y(\omega X) + \mathcal{T}_Y(\omega X).$$

On the other hand, we also have

$$\nabla_Y(JX) = J(\nabla_Y X)$$
(3.6) $$= \mathcal{B}\mathcal{T}_Y X + \mathcal{C}\mathcal{T}_Y X + \varphi\hat{\nabla}_Y X + \omega\hat{\nabla}_Y X.$$

Comparing the vertical components of (3.5) and (3.6)

(3.7) $$-(\sin\theta)(Y\theta) X^* + (\cos\theta)\hat{\nabla}_Y X^* + \mathcal{T}_Y(\omega X) = \mathcal{B}\mathcal{T}_Y X + \varphi\hat{\nabla}_Y X.$$

Therefore, by taking the inner product of (3.7) with X^*, we have

$$-(\sin\theta)(Y\theta) + g_M(\mathcal{T}_{X^*}\omega X, Y) = g_M(Y, \mathcal{T}_X \omega X^*).$$

Consequently, the pointwise slant sumbersion is slant if and only if

$$\mathcal{T}_{X^*}\omega X = \mathcal{T}_X \omega X^*.$$ $\square$

Corollary 3.6. *Let F be a pointwise slant submersion from a Kähler manifold (M, g_M, J) onto a Riemannian manifold (N, g_N). If F has totally geodesic fibers, it is slant.*

Proof. If F has totally geodesic fibers, $\mathcal{T}$ vanishes identically. Thus the proof follows from Proposition 3.6. $\qquad\square$

Let F be a Remannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N). Recall that a Riemannian submersion is called a Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_X Y = g_M(X, Y) H$$

for $X, Y \in \Gamma(\ker F_*)$, where H is the mean curvature vector field of the fiber.

Corollary 3.7. *Let F be a pointwise proper slant submersion with totally umbilical fibers from a Kähler manifold (M^{2n}, g_M, J) onto a Riemannian manifold (N^n, g_N). If F has no totally geodesic fibers, then it is always non-slant.*

Proof. Assume that F is a pointwise proper slant submersion with totally umbilical fibers. Then we have

$$\mathcal{T}_X Y = g_M(X, Y) H$$

for $X, Y \in \Gamma(\ker F_*)$. Therefore, we have

$$(3.8) \qquad g_M(\mathcal{T}_X \omega \varphi X, Y) = -g_M(X, Y) g_M(\omega \varphi X, H),$$
$$(3.9) \qquad g_M(\mathcal{T}_{\varphi X} \omega X, Y) = -g_M(\varphi X, Y) g_M(\omega X, H).$$

If the submersion is slant, then Proposition 3.6 and (3.8)-(3.9) imply that

$$g_M(\omega \varphi X, H) X = g_M(\omega X, H) \varphi X.$$

Since $g_M(X, \varphi X) = 0$,

$$g_M(\omega \varphi X, H) = g_M(\omega X, H) = 0,$$

which implies $H \in \Gamma(\mu)$. On the other hand, we note that $\dim(\ker F_*) = 2n - n = n$. Thus using Corollary 3.3, we have $\dim((\ker F_*) \oplus \omega(\ker F_*)) = 2(2n - n) = 2n$. Therefore, we get $\dim(\mu) = 2n - 2n = 0$, which contradicts that H vanishes identically. $\qquad\square$

We now investigate certain properties of pointwise slant submersions. The following theorem proposes a new condition for Riemannian submersions to be harmonic. We first recall that if $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ is a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2), then the adjoint map $^*F_*$ of F_* is characterized by $g_1(x, {}^*F_{*p_1} y) = g_2(F_{*p_1} x, y)$ for $x \in T_{p_1} M_1$, $y \in T_{F(p_1)} M_2$ and $p_1 \in M_1$. Considering F_*^h at each $p_1 \in M_1$ from a linear transformation

$$F_{*p_1}^h : ((\ker F_*)^\perp(p_1), g_{1_{p_1}((\ker F_*)^\perp(p_1))}) \longrightarrow (range F_*(p_2), g_{2_{p_2(range F_*)(p_2)}}),$$

we will denote the adjoint of F_*^h by $^*F^h{}_{*p_1}$. Let $^*F_{*p_1}$ be the adjoint of F_{*p_1} : $(T_{p_1}M_1, g_{1_{p_1}}) \longrightarrow (T_{p_2}M_2, g_{2_{p_2}})$. Then the linear transformation

$$(^*F_{*p_1})^h : range F_*(p_2) \longrightarrow (\ker F_*)^\perp(p_1)$$

defined by $(^*F_{*p_1})^h y = {}^*F_{*p_1} y$, where $y \in \Gamma(range F_{*p_1}), p_2 = F(p_1)$, is an isomorphism and $(F_{*p_1}^h)^{-1} = (^*F_{*p_1})^h = {}^*(F_{*p_1}^h)$.

Theorem 3.8. *Let F be a pointwise slant submersion from Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) and have no complex points. Then F is harmonic if and only if*

$$trace^*F_*((\nabla F_*)((.), \omega\varphi(.))) - trace\,\omega\mathcal{T}_{(.)}\omega(.) + trace\mathcal{C}\,^*F_*(\nabla F_*)((.), \omega(.)) = 0.$$

Proof. From (2.7), (2.1), (2.2), (3.1) and (3.2) we have

$$g_1(\mathcal{T}_U U, X) = -g_1(\nabla_U J\varphi U, X) + g_1(\nabla_U \omega U, JX)$$

for $U \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. Then Theorem 3.1, (3.1) and (2.8) imply

$$\begin{aligned} g_1(\mathcal{T}_U U, X) = & -g_1(\sin 2\theta U(\theta)U, X) + g_1(\cos^2 \theta \nabla_U U, X) \\ & -g_1(\nabla_U \omega\varphi U, X) + g_1(\mathcal{T}_U \omega U, \mathcal{B}X) \\ & +g_1(\nabla_U \omega U, \mathcal{C}X). \end{aligned}$$

Using (2.11) we arrive at

$$\begin{aligned} \sin^2 \theta g_1(\mathcal{T}_U U, X) = & g_2((\nabla F_*)(U, \omega\varphi U), F_*(X)) + g_1(\mathcal{T}_U \omega U, \mathcal{B}X) \\ & -g_2((\nabla F_*)(U, \omega U), F_*(\mathcal{C}X)). \end{aligned}$$

Then from (3.2) we get

$$\begin{aligned} \sin^2 \theta g_1(\mathcal{T}_U U, X) = & g_1(^*F_*((\nabla F_*)(U, \omega\varphi U)), X) - g_1(\omega\mathcal{T}_U \omega U, X) \\ & + g_1(\mathcal{C}\,^*F_*((\nabla F_*)(U, \omega U)), X). \end{aligned}$$

Thus the proof follows from a fact that a Riemannian submersion is harmonic if and only if its fibers are minimal submanifolds in M_1. $\square$

Next result gives a necessary and sufficient condition for fibers of a pointwise slant submersion to be totally geodesic submanifolds.

Theorem 3.9. *Let F be a pointwise slant submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the fibers are totally geodesic submanifolds in M_1 if and only if*

$$\begin{aligned} g_2(\nabla_{X'}^2 F_*(\omega U), F_*(\omega V)) = & -\sin^2 \theta g_1([U, X], V) + \sin 2\theta\, X(\theta)g_1(U, V) \\ & + g_1(\mathcal{A}_X \omega\varphi U, V) - g_1(\mathcal{A}_X \omega U, \varphi V), \end{aligned}$$

where X and X' are F-related vector fields and ∇^2 is the Levi-Civita connection on M_2.

Proof. Since $\mathcal{T}$ is skew symmetric with respect to g_1 and ∇ is a torsion free connection on M_1, using (2.8) we have

$$g_1(\mathcal{T}_U V, X) = -g_1([U, X], V) - g_1(\nabla_X U, V)$$

for $U, V \in \Gamma(\mathcal{V})$ and $X \in \Gamma(\mathcal{H})$. Then from (3.1), (3.2), (2.1) and (2.2) we obtain

$$\begin{aligned} g_1(\mathcal{T}_U V, X) = & -g_1([U, X], V) + g_1(\nabla_X \varphi^2 U + \omega\varphi U, V) \\ & - g_1(\nabla_X \omega U, \varphi V) - g_1(\nabla_X \omega U, \omega V). \end{aligned}$$

Thus Theorem 3.1, (2.10) and (2.12) imply

$$\begin{aligned} g_1(\mathcal{T}_U V, X) = & -g_1([U, X], V) + \sin 2\theta\, X(\theta) g_1(U, V) - \cos^2\theta g_1(\nabla_X U, V) \\ & + g_1(\mathcal{A}_X \omega\varphi U, V) - g_1(\mathcal{A}_X \omega U, \varphi V) \\ & - g_2(\nabla^2_{X'} F_*(\omega U), F_*(\omega V)). \end{aligned}$$

Since ∇ is a torsion free connection, using skew symmetry property of $\mathcal{T}$ and (2.7) we get

$$\begin{aligned} \sin^2\theta g_1(\mathcal{T}_U V, X) = & -\sin^2\theta g_1([U, X], V) + \sin 2\theta\, X(\theta) g_1(U, V) \\ & + g_1(\mathcal{A}_X \omega\varphi U, V) - g_1(\mathcal{A}_X \omega U, \varphi V) \\ & - g_2(\nabla^2_{X'} F_*(\omega U), F_*(\omega V)) \end{aligned}$$

which gives the assertion. $\qquad\square$

Finally we give a result for pointwise slant submersions to be totally geodesic maps. We recall that a differentiable map F between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths.

Theorem 3.10. *Let F be a pointwise slant submersion from Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then F is a totally geodesic map if and only if*

$$\begin{aligned} g_2(\nabla^2_{X'} F_*(\omega U), F_*(\omega V)) = & -\sin^2\theta g_1([U, X], V) + \sin 2\theta\, X(\theta) g_1(U, V) \\ & + g_1(\mathcal{A}_X \omega\varphi U, V) - g_1(\mathcal{A}_X \omega U, \varphi V) \end{aligned}$$

(3.10)

and

(3.11)

$$g_1(\mathcal{A}_X \omega U, \mathcal{B}Y) = -\left[-g_2(\nabla^F_X F_*(\omega\varphi U), F_*(Y)) + g_2(\nabla^F_X F_*(\omega U), F_*(\mathcal{C}Y)) \right],$$

where X and X' are F-related vector fields and ∇^F is the pull-back connection along F.

Proof. By definition, it follows that F is totally geodesic if and only if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(\mathcal{H})$, $(\nabla F_*)(X, V) = 0$ for $V \in \Gamma(\mathcal{V})$ and $(\nabla F_*)(U, V) = 0$ for $U \in \Gamma(\mathcal{V})$. From (2.12), it follows that $(\nabla F_*)(X, Y) = 0$. Also from Theorem 3.9, it follows that $(\nabla F_*)(U, V) = 0$ if and only if (3.10) is satisfied. On the other hand, since F is a Riemannian submersion, using (2.11) we have

$$g_2((\nabla F_*)(X, U), F_*(Y)) = -g_1(\nabla_X U, Y).$$

Then from Theorem 3.1, (2.1), (2.2), (3.1), (3.2) and (2.10) we obtain

$$\begin{aligned}
g_2((\nabla F_*)(X, U), F_*(Y)) = & -\sin 2\theta X(\theta) g_1(U, Y) - \cos^2 \theta g_1(\nabla_X U, Y) \\
& + g_1(\nabla_X \omega\varphi U, Y) - g_1(\mathcal{A}_X \omega U, \mathcal{B}Y) \\
& - g_1(\nabla_X \omega U, \mathcal{C}Y).
\end{aligned}$$

Thus using (2.11) and (2.12) we get

$$\begin{aligned}
& \sin^2 \theta g_2((\nabla F_*)(X, U), F_*(Y)) \\
= & -\left[-g_2(\nabla_X^F F_*(\omega\varphi U), F_*(Y)) + g_2(\nabla_X^F F_*(\omega U), F_*(\mathcal{C}Y)) \right] \\
& - g_1(\mathcal{A}_X \omega U, \mathcal{B}Y)
\end{aligned}$$

which completes proof. $\qquad\qquad\square$

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