Bull. Korean Math. Soc. **51** (2014), No. 4, pp. 1063–1073
http://dx.doi.org/10.4134/BKMS.2014.51.4.1063

ON MINIMAL NON-$\mathcal{QNS}$-GROUPS

Zhangjia Han, Huaguo Shi, and Guiyun Chen

Abstract. A finite group G is called a $\mathcal{QNS}$-group if every minimal subgroup X of G is either quasinormal in G or self-normalizing. In this paper the authors classify the non-$\mathcal{QNS}$-groups whose proper subgroups are all $\mathcal{QNS}$-groups.

1. Introduction

Throughout this paper, all groups are finite. Given a group theoretical property $\mathcal{P}$, a $\mathcal{P}$-critical group or a minimal non-$\mathcal{P}$-group is a group which is not a $\mathcal{P}$-group but all of whose proper subgroups are $\mathcal{P}$-groups. There are many remarkable examples about minimal non-$\mathcal{P}$-groups: minimal non-abelian groups (Miller and Moreno [10]), minimal non-nilpotent groups (Schmidt), minimal non-supersoluble groups ([2]) and minimal non-p-nilpotent groups (Itô), minimal non-MSP-groups ([4]) and minimal non-NSN-groups ([5]). In [12], Sastry classified the minimal non-PN-groups.

Recall that a subgroup H is called quasinormal in a group G, if $HK = KH$ holds for every subgroup K of G and a group G is called a $\mathcal{QN}$-group if every minimal subgroup of G is quasinormal in G (see [14]). Clearly, a $\mathcal{QN}$-group is a generalization of PN-groups. In this paper, we consider a generalization of $\mathcal{QN}$-groups, which is called $\mathcal{QNS}$-groups.

Definition. A group G is called a $\mathcal{QNS}$-group if every minimal subgroup of G is either quasinormal in G or self-normalizing.

It is easy to see that a $\mathcal{QNS}$-group need not to be a $\mathcal{QN}$-group. An example is S_3, the symmetric group of degree 3.

In the first place, we investigate properties of $\mathcal{QNS}$-groups in general. Next, by applying the structure of $\mathcal{QNS}$-groups, we give the classification of minimal non-$\mathcal{QNS}$-groups.

Received August 16, 2013; Revised November 29, 2013.
2010 *Mathematics Subject Classification.* 20D35, 20E34.
Key words and phrases. minimal subgroups, quasinormal subgroups, self-normalizing subgroups, $\mathcal{QNS}$-groups, minimal non-$\mathcal{QNS}$-groups.
This work is supported by the National Scientific Foundation of China(No:11271301) and the National Scientific Foundation of China(No:11301426) and Scientific Research Foundation of SiChuan Provincial Education Department(No:14ZA0314).

Our main results are as follows:

Main Theorem. *Suppose that G is a non-$\mathcal{QNS}$-group, each of whose proper subgroup is a $\mathcal{QNS}$-group. Then G is solvable and one of the following statements is true*:

(1) $G = \langle a, x \mid a^{p^n} = b^p = x^p = 1,\ [x, a] = b, [a, b] = [b, x] = 1\rangle$.

(2) $G = \langle a, x \mid a^{p^n} = b^p = x^p = 1,\ [x, a] = b, [b, x] = 1, b^{-1}ab = a^{1+p^{n-1}}, x^{-1}a^p x = a^{p+p^{n-1}}\rangle$.

(3) $G = (\langle v_1\rangle \times \langle v_2\rangle) \rtimes \langle a\rangle$, *where* $v_1^q = v_2^q = a^p = 1$, $v_1^a = v_1^{m_1}$, $v_2^a = v_2^{m_2}, m_1 \not\equiv m_2 \ (mod\ q)$.

In the following (4)–(7), $P \in Syl_p(G)$, $Q \in Syl_q(G)$, $p < q$ *are distinct primes.*

(4) $G = PQ$, $Q \trianglelefteq G$, Q *is of prime order,* $|P| = p^2$. *Moreover* $|C_P(Q)| \le p$ *if* P *is an elementary abelian* p*-group or* $|C_P(Q)| = 1$ *if* P *is cyclic.*

(5) $G = PQ$, $P \trianglelefteq G$, P *is an ultraspecial 2-group of order* 2^{3s}, $\exp(P) = 4$ *and* $|Q|$ *is a prime dividing* $2^s + 1$. *Moreover,* $|C_P(Q)| > 1$.

(6) $G = PQ$, $P \trianglelefteq G$, P *is an elementary abelian* p*-group of rank* > 1, Q *is cyclic and* Q *acts irreducibly on* P.

(7) $G = PQ$, $Q \trianglelefteq G$, Q *is an elementary abelian* q*-group of rank* > 1, P *is cyclic and* P *acts irreducibly on* Q.

(8) $G = C_p \rtimes (C_q \rtimes C_r)$, *where* $p > q > r$ *are distinct primes and* $C_pC_r = C_p \times C_r$.

(9) $G = C_p \rtimes (C_q \times C_r)$, *where* $p > q, r$ *are distinct primes and* $Z(G) = 1$.

(10) $G = C_p \times (C_q \rtimes C_r)$, *where* p, q *and* r *are distinct primes and* $r < p$.

Our notations are all standard. For example, we denote by $A \rtimes B$ the semidirect product of A and B; C_n always denotes a cyclic group of order n and $\pi(G)$ denotes the set of all prime divisors of $|G|$. All unexplained notations can be found in [8] and [11].

2. Some preliminaries

In this section, we collect some lemmas which will be frequently used in the sequel.

Lemma 2.1 ([8, 7.2.2])**.** *Suppose that the Sylow* p*-subgroups of* G *are cyclic, where* p *is the smallest prime divisor of* $|G|$. *Then* G *has a normal* p*-complement.*

Lemma 2.2 (Maschke's Theorem, [8, 8.4.6])**.** *Suppose that the action of* A *on an elementary abelian group* G *is coprime and* H *is an* A*-invariant direct factor of* G. *Then* H *has an* A*-invariant complement in* G.

Lemma 2.3 ([9])**.** *Suppose that* p'*-group* H *acts on a* p*-group* G. *Let*

$$\Omega(G) = \begin{cases} \Omega_1(G) & p > 2, \\ \Omega_2(G) & p = 2. \end{cases}$$

If H *acts trivially on* $\Omega(G)$, *then* H *acts trivially on* G *as well.*

Lemma 2.4 ([13]). *If G is a minimal nonabelian simple group, i.e., nonabelian simple groups whose all proper subgroups solvable, then G is isomorphic to one of the following simple groups*:

(1) *$PSL(2,p)$, where p is a prime with $p > 3$ and $5 \nmid p^2 - 1$.*
(2) *$PSL(2,2^q)$, where q is a prime.*
(3) *$PSL(2,3^q)$, where q is a prime.*
(4) *$PSL(3,3)$.*
(5) *The Suzuki group $Sz(2^q)$, where q is an odd prime.*

The following lemma is an immediate consequence of [1, Theorem 2]:

Lemma 2.5. *Let P be a quasinormal p-subgroup of G. Then $O^p(G) \leq N_G(P)$.*

3. Finite $\mathcal{QNS}$-groups

In this section, we classify finite $\mathcal{QNS}$-groups. Note that finite $\mathcal{QN}$-groups are obviously $\mathcal{QNS}$-groups. Our results are the following theorem, which are similar to those of $\mathcal{QN}$-groups.

Theorem 3.1. *Let G be a $\mathcal{QNS}$-group. Then one of the following statements is true*:

(a) *G is a $\mathcal{QN}$-group.*
(b) *$G = N \rtimes C_p$ is a Frobenius group, where p is the smallest prime divisor of $|G|$ and every minimal subgroup of N is quasinormal in G.*

Proof. Suppose that G is not a $\mathcal{QN}$-group. We prove that G must be isomorphic to a group mentioned in (b) of the theorem.

We divide our proof into several steps.

(1) G is solvable.

Since G is not a $\mathcal{QN}$-group, there is at least one minimal subgroup X_0 in G such that X_0 is not quasinormal in G. By hypothesis, $N_G(X_0) = X_0$. Hence X_0 is a Sylow p-subgroup of G. If $p = 2$, then G is obviously solvable. $p \neq 2$, if the order of Sylow 2-subgroups of G is greater than 2, then any subgroup H of order 2 in G is quasinormal, and hence X_0H is a subgroup of G by hypothesis. Therefore we get by Lemma 2.1 that $N_G(X_0) \geq X_0H$, a contradiction, which implies that the order of any Sylow 2-subgroup of G is at most two. Thus G is solvable.

(2) There is a unique $p \in \pi(G)$ such that G has a non-quasinormal subgroup of order p.

Suppose that G has two non-quasinormal minimal subgroups X and Y which are of coprime order in G. Then by the proof of (1), X and Y are Sylow subgroups of G. Since G is solvable, we may assume that XY is a subgroup of G without loss of generality. Hence either X or Y must not self-normalize by Lemma 2.1, a contradiction.

(3) Conclusion established.

Let C_p be a non-quasinormal minimal subgroup of G. Then C_p is a Sylow p-subgroup of G. Let Y be any minimal subgroup of G such that the orders of

C_p and Y are coprime. Then Y is quasinormal in G by (2), and hence C_pY is a subgroup of G. Since $N_G(C_p) = C_p$, we have that $C_pY = Y \rtimes C_p$, and then p is the smallest prime divisor of $|G|$. Hence G has a normal p-complement N by Lemma 2.1. It follows that $G = N \rtimes C_p$. Again by $N_G(C_p) = C_p$, we have that $C_{C_p}(N) = 1$. Therefore G is a Frobenius groups with kernel N and complement C_p. Moreover, by (2), every minimal subgroup of N is quasinormal in G. This proves our theorem. □

4. Minimal non-$\mathcal{QNS}$-groups

It is easy to see that a subgroup H is quasinormal in G if and only if $HK = KH$ for every subgroup K of prime power order of G. We will use this fact freely in our following proof.

Obviously all minimal non-$\mathcal{QN}$-p-groups are minimal non-$\mathcal{QNS}$-groups. The classification of this kind of groups is given in [14], we list them as the following lemma.

Lemma 4.1. *Let G be a minimal non-$\mathcal{QN}$-p-group. Then one of the following statements is true*:

(a) $G = \langle a, x | a^{p^n} = b^p = x^p = 1,\ [x, a] = b, [a, b] = [b, x] = 1\rangle$.

(b) $G = \langle a, x | a^{p^n} = b^p = x^p = 1,\ [x, a] = b, [b, x] = 1, b^{-1}ab = a^{1+p^{n-1}}, x^{-1}a^p x = a^{p+p^{n-1}}\rangle$.

So, it only remains to classify finite minimal non-$\mathcal{QNS}$-groups which are not minimal non-$\mathcal{QN}$-p-groups. In the first place, we study some basic properties of minimal non-$\mathcal{QNS}$-groups.

Proposition 4.2. *Let G be a minimal non-$\mathcal{QNS}$-group. Then G is solvable.*

Proof. Suppose that G is not solvable. By Theorem 3.1, every proper subgroup of G is solvable and hence $G/\Phi(G)$ is a minimal non-abelian simple group, where $\Phi(G)$ is the Frattini subgroup of G. Let H be the 2-complement of $\Phi(G)$. Then $H \trianglelefteq G$ and H is nilpotent. We have

(1) Every minimal subgroup of $\Phi(G)$ is normal in G.

Suppose that there exists a prime $p \in \pi(G)$ such that $O^p(G)$ is a proper subgroup of G. Then by the minimality of G, we know that $O^p(G)$ is solvable and hence G is solvable, a contradiction. So, we have that $O^p(G) = G$ for each $p \in \pi(G)$. Let A be a proper subgroup of G. Then $A\Phi(G)$ is a proper subgroup of G as well and hence every minimal subgroup X of $\Phi(G)$ is quasinormal in $A\Phi(G)$. Thus $XA = AX$. It follows that X is quasinormal in G. Now by Lemma 2.5, we have that $X \trianglelefteq G$.

(2) $H \leq Z(G)$.

Indeed, let $P \in Syl_p(H)$, where p is a prime in $\pi(H)$. Then $P \trianglelefteq G$. By (1), every subgroup X of order p in P is normal in G. Hence $G/C_G(X) = N_G(X)/C_G(X) \lesssim \mathrm{Aut}(X) \cong C_{p-1}$. If $C_G(X)$ is a proper subgroup G, then $C_G(X)$ is solvable and G is hence solvable, a contradiction. Thus $C_G(X) = G$, i.e., $X \leq Z(G)$. It follows that every subgroup of P of order p lies in the center $Z(G)$. Let $S_2 \in Syl_2(G)$ and $K = S_2P$. Apply Ito's Lemma, we see that K is p-nilpotent and so K is nilpotent. Then we have that $S_2 \leq C_G(P) \trianglelefteq G$. Using the simplicity of $G/\Phi(G)$, we conclude that $H \leq Z(G)$. Thus (2) holds.

(3) $H = 1$.

Set $K = G/S_0$, where $S_0 \in Syl_2(\Phi(G))$. Then by (2) $K/Z(K) \cong G/\Phi(G)$ and K is a quasisimple group with the center of odd order. Hence in order to prove $H = 1$, i.e., $Z(K) = 1$, it will suffice to show that the Schur multiplier of each of the minimal simple groups is a 2-group. Indeed, this is true by checking the table on the Schur multipliers of the known simple groups (see [3, p. 302]).

(4) Every subgroup of order $2^m p$ (p an odd prime) of $\overline{G} = G/\Phi(G)$ is 2-nilpotent.

By (3), $\Phi(G)$ is a 2-group. Assume $L/\Phi(G)$ is a proper subgroup of order $2^m p$ of $\overline{G}$. Then L is a proper subgroup of order $2^n p$ of G for some natural number n. Let $P \in Syl_p(L)$. Then $|P| = p$ and hence P is quasinormal in L. Since P is subnormal in L, we have that P is normal in L, that is, L is 2-nilpotent. Thus $L/\Phi(G)$ is 2-nilpotent.

(5) Final contradiction.

We know that $\overline{G}$ is isomorphic to one of the simple groups mentioned in Lemma 2.4. Suppose that $\overline{G} \cong PSL(2,p), PSL(2,3^q)$ or $PSL(3,3)$. Indeed, each of $PSL(2,p), PSL(2,3^q)$ and $PSL(3,3)$ contains a subgroup which is isomorphic to A_4, the alternating group of degree 4, by (4) we conclude that $\overline{G}$ cannot be any one of $PSL(2,p), PSL(2,3^q)$ and $PSL(3,3)$. Suppose that $\overline{G} \cong PSL(2,2^q)$ or $Sz(2^q)$. Then $\overline{G}$ is a Zassenhaus group of odd degree and the stabilizer of a point is a Frobenius group with kernel a 2-group. So $\overline{G}$ cannot be any one of $PSL(2,2^q)$ and $Sz(2^q)$ as well. Thus the proof is complete. □

By Proposition 4.2, we always assume in the following that G is a solvable minimal non-$\mathcal{QNS}$-group.

Proposition 4.3. *Let G be a minimal non-$\mathcal{QNS}$-group. Then $|\pi(G)| \leq 3$.*

Proof. Suppose that $|\pi(G)| > 3$. Let $\{P_1, P_2, \dots, P_k, \dots, P_r\}$, $r > 3$ be a Sylow system of G, where $P_i \in Syl_{p_i}(G)$, $i = 1, 2, \dots, r$. Since G is not a $\mathcal{QNS}$-group by hypothesis, G has at least one minimal subgroup X_0 such that X_0 is neither quasinormal in G nor self-normalizing. Without loss of generality we may assume that $X_0 \leq P_1$. Then there is a subgroup $Y \leq P_k (k \neq 1)$ such that $X_0Y \neq YX_0$. Let $G_1 = P_1P_k$. Then G_1 is a $\mathcal{QNS}$-group by hypothesis. Since X_0 is not quasinormal in G_1, we have $P_1 = X_0$ and $G_1 = P_k \rtimes X_0$ is a Frobenius group. On the other hand, since $N_G(X_0) > X_0$, there is a $P_i \in Syl_{p_i}(G)$, $i \neq k$ such that $N_{P_i}(X_0) > 1$. Let $G_2 = P_iX_0$. Then G_2 is a $\mathcal{QNS}$-group with $N_{G_2}(X_0) > X_0$. Hence X_0 is quasinormal in G_2. Now $G_3 = P_kP_iX_0$ is a proper subgroup of G since $|\pi(G)| > 3$, and therefore is a $\mathcal{QNS}$-group. However, X_0 is neither quasinormal in G_3 nor self-normalizing, a contradiction. Thus $|\pi(G)| \leq 3$. □

The following theorem classifies all minimal non-$\mathcal{QNS}$-groups whose order having just two prime divisors.

Theorem 4.4. *Suppose that G is a minimal non-$\mathcal{QNS}$-group with $|G| = p^a q^b$, where $a > 0, b > 0, p < q$ are distinct primes and $P \in Syl_p(G)$ and $Q \in Syl_q(G)$. Then one of the following statements is true*:

(i) *$G = PQ$, $Q \trianglelefteq G$, Q is of prime order, $|P| = p^2$. Moreover $|C_P(Q)| \leq p$ if P is an elementary abelian p-group or $|C_P(Q)| = 1$ if P is cyclic.*

(ii) *$G = (\langle v_1\rangle \times \langle v_2\rangle) \rtimes \langle a\rangle$, where $v_1^q = v_2^q = a^p = 1$, $v_1^a = v_1^{m_1}$, $v_2^a = v_2^{m_2}$, $m_1 \not\equiv m_2 \pmod q$.*

(iii) *$G = PQ$, $P \trianglelefteq G$, P is an ultraspecial 2-group of order 2^{3s}, $\exp(P) = 4$ and $|Q|$ is a prime dividing $2^s + 1$. Moreover, $|C_P(Q)| > 1$.*

(iv) *$G = PQ$, $P \trianglelefteq G$, P is an elementary abelian p-group of rank > 1, Q is cyclic and Q acts irreducibly on P.*

(v) *$G = PQ$, $Q \trianglelefteq G$, Q is an elementary abelian q-group, P is cyclic and P acts irreducibly on Q.*

Proof. Let $G = PQ$ with $P \in Syl_p(G)$ and $Q \in Syl_q(G)$, where $p < q$. We divide our proof into two cases.

Case 1. G is supersolvable.

Assume that $|Q| = q$. Since G is not a $\mathcal{QNS}$-group, there exists a minimal subgroup $X_0 \leq P$ such that X_0 is neither quasinormal in G nor self-normalizing. If $|P| > p^2$, let P^* be any maximal subgroup of P containing X_0. Then P^*Q is a $\mathcal{QNS}$-group. Hence X_0 is quasinormal in P^*Q by Theorem 3.1 and thus X_0 is quasinormal in G, a contradiction. Therefore we have $|P| = p^2$. As $X_0 \nleq C_P(Q)$, we have $|C_P(Q)| \leq p$ if P is an elementary abelian p-group and $|C_P(Q)| = 1$ if P is cyclic. It follows that G is of type (i)

Suppose that $|Q| > q$. If Q is cyclic, then the minimal subgroup Q_0 is normal in G and $|P| > p$. Since PQ_0 is a proper $\mathcal{QNS}$-subgroup of G, we get by Theorem 3.1 that each minimal subgroup P_0 of P is quasinormal in PQ_0. Hence $Q_0 \leq C_G(P_0)$ by Lemma 2.5. Thus P_0 is quasinormal in G by Lemma 2.3, a contradiction. Therefore Q is non-cyclic.

Let $V = \Omega_1(Q)$. Since Q is a $\mathcal{QN}$-group, V is an elementary abelian q-group and so by the supersolvability of G and Lemma 2.2, we have $V = \langle v_1\rangle \times (\langle v_2\rangle \times \cdots \times \langle v_n\rangle)$, where $\langle v_1\rangle \trianglelefteq G$ and $\langle v_2\rangle \times \cdots \times \langle v_n\rangle$ is P-invariant. Now $P(\langle v_2\rangle \times \cdots \times \langle v_n\rangle)$ is a proper subgroup of G and so is a $\mathcal{QNS}$-group. If $|P| > p$, then every minimal subgroup of P is quasinormal in $P(\langle v_2\rangle \times \cdots \times \langle v_n\rangle)$ and in $P\langle v_1\rangle$ by Theorem 3.1. Hence for every minimal subgroup P_0 of P, we have $V \leq C_G(P_0)$ by Lemma 2.5. Thus P_0 acts trivially on V and therefore P_0 acts trivially on Q by Lemma 2.3, which implies that G is a $\mathcal{QNS}$-group, a contradiction. Hence we may assume that $|P| = p$.

Suppose that $PV < G$. If P is quasinormal in PV, then $V \leq N_G(P)$ by Lemma 2.5 and hence $P \leq C_G(V)$. By Lemma 2.3, P acts trivially on Q, a contradiction. Thus P is self-normalizing in PV, that is, $N_{PV}(P) = P$, which implies that $N_G(P) = P$. By hypothesis, there exists a minimal subgroup Q_0

of Q such that Q_0 is neither quasinormal in G nor self-normalizing in G. If $Q_0^G < Q$, then $Q_0^G P$ is a $\mathcal{QNS}$-group and hence Q_0 is quasinormal in $Q_0^G P$. Therefore Q_0 is quasinormal in G, a contradiction. Therefore we get that $Q_0^G = Q$, which implies that $Q = Q_0^G = V$, a contradiction as well. Hence we obtain that $PV = G$. In this case $Q = V = \langle v_1\rangle \times (\langle v_2\rangle \times \cdots \times \langle v_n\rangle)$ is an elementary abelian q-group. Since $P\langle v_2\rangle \times \cdots \times \langle v_n\rangle$ is a $\mathcal{QNS}$-group , we have that $P \leq N_G(\langle v_i\rangle)$ by Lemma 2.5, where $i = 2, \ldots, n$. Therefore we get $\langle v_i\rangle$ is normal in G for $i = 2, \ldots, n$. Hence we obtain that $\langle v_i\rangle \trianglelefteq G$ for $i = 1, \ldots, n$.

An element a is said to act on V by scalars if there exists an integer m such that $a^{-1}va = v^m$ for all v in V. We claim that P can not act by scalars on V. Assume that P acts by scalars on V. Then every subgroup of V is normal in G. Hence, every subgroup of order q is normal in G. Set $P = \langle a\rangle$, and let m be an integer satisfying $a^{-1}va = v^m$ for all v in V. If $m = 1$, then a centralizes $V = Q$. Thus $P \trianglelefteq G$. If $m \neq 1 \pmod q$, then $C_Q(a) = C_V(a) = 1$. It follows that $N_G(P) = P$. Therefore we have proven that every minimal subgroup of G is either normal in G or self-normalizing. By definition then, G is a $\mathcal{QNS}$-group, a contradiction. This contradiction concludes our claim. Hence $n \geq 2$ and we may choose v_1 and v_2 so that $a^{-1}v_1a = v_1^{m_1}$ and $a^{-1}v_2a = v_2^{m_2}$, where $m_1 \not\equiv m_2 \pmod q$. Then $\langle v_1v_2\rangle$ is not a quasinormal subgroup of $\langle a\rangle\langle v_1, v_2\rangle$. Hence $\langle a\rangle\langle v_1, v_2\rangle$ is not a $\mathcal{QNS}$-group, and so $G = \langle a\rangle\langle v_1, v_2\rangle$, $P = \langle a\rangle$ is of prime order and $Q = \langle v_1, v_2\rangle$ is of order q^2. That is, G is of type (ii).

Case 2. G is non-supersolvable.

Let $F(G)$ be the Fitting subgroup of G. Then $F(G) = O_p(G) \times O_q(G)$.

Suppose in the first place that $G = O_p(G)Y = PY$ for some $Y < G$ with $|Y| = q$.

Since G is not a $\mathcal{QNS}$-group, there exists a minimal subgroup X_0 in G such that X_0 is neither quasinormal in G nor self-normalizing. If $X_0 = Y$, then $C_P(Y) > 1$. If $\Omega_1(P) = P$, then there exists two Y-invariant proper subgroups A and B of P such that $P = A \times B$ by Lemma 2.2. By hypothesis, both AY and BY are $\mathcal{QNS}$-groups. By Theorem 3.1, every minimal subgroup of A and B is quasinormal G. Thus G is nilpotent, a contradiction. Hence $\Omega_1(P) < P$. Now $\Omega_1(P)Y$ is a $\mathcal{QNS}$-group, and hence $C_P(Y) \geq \Omega_1(P)$ since $C_P(Y) > 1$. If $p > 2$, then Y acts trivially on P by Lemma 2.3, a contradiction. Hence, we have that $p = 2$. By the same argument as above we get that $\exp(P) = 4$ and P is non-abelian. Let P' is a Y-invariant proper subgroup of P. If $P'Y < G$, then $[P', Y] = 1$. Hence Y acts irreducibly on $P/\Phi(P)$ and $[P, Y] = P$. By [6, Theorem 1.3], we know that P is an ultraspecial 2-group of order 2^{3s}, and $|Q|$ is a prime dividing $2^s + 1$. That is, G is of type (iii).

Assume that $X_0 < P$. If $X_0^G < P$, then $X_0^G Y$ is a $\mathcal{QNS}$-group and hence X_0 is quasinormal in $X_0^G Y$. Therefore X_0 is quasinormal in G, a contradiction. Hence we get that $X_0^G = P$, which implies that $P = X_0^G = \Omega_1(P)$ is an elementary group. If P is Y-reducible, then there exists two Y-invariant proper subgroups A and B of P such that $P = A\times B$. By hypothesis, both AY and BY are $\mathcal{QNS}$-groups. By Theorem 3.1, every minimal subgroup of A is in normal

AY, which implies that every minimal subgroup of A lies in the center of AY. Similarly, we have that every minimal subgroup of B lies in the center of BY. Thus we obtain $G = P \times Y$, a contradiction, which means P is Y-irreducible. That is, G is of type (iv).

Secondly we suppose that $O_p(G)Y < G$ for each $Y < G$ with $|Y| = q$. In this case $O_p(G)Y$ is a $\mathcal{QNS}$-group, hence Y is quasinormal in $O_p(G)Y$ by Theorem 3.1 and so Y centralizes $O_p(G)$ by Lemma 2.5.

Subcase 1. There is a minimal subgroup Y of order q satisfying $Y \nleq O_q(G)$.

Suppose first that $F(G)Y < G$. Then by what has been said as above, $Y \leq C_G(O_p(G))$. If $C_G(O_p(G)) < G$, then Y is quasinormal in $C_G(O_p(G))$, and hence is subnormal in $C_G(O_p(G))$, which implies that Y is subnormal in G. Therefore $Y \leq O_q(G)$, a contradiction. Hence we may assume that $C_G(O_p(G)) = G$, and then $O_p(G) \leq Z(G)$. By the same argument, we can get that $\Omega_1(O_q(G)) \leq Z(G)$.

Since G is solvable, there is a normal maximal subgroup M of G such that $|G : M| = r$ is a prime. If $Y \leq M$, then we have by Theorem 3.1 that Y is quasinormal in M and hence Y is subnormal in M. Thus we get $Y \leq O_q(G)$, a contradiction. Therefore $G = MY$. Since M is a $\mathcal{QNS}$-group, $\Omega_1(O_q(G)) \leq \Omega_1(Q \cap M) \leq \Omega_1(O_q(M)) \leq \Omega_1(O_q(G)) \leq Z(G)$, that is, every subgroup of order q in M is normal in M. By [7, IV, 5.5], M is q-nilpotent and hence $P = O_p(G) \leq Z(G)$. Thus $G = PQ$ is nilpotent, a contradiction.

Suppose now that $F(G)Y = G$. In this case $G = O_p(G) \times (O_q(G)Y) = O_p(G) \times Q$ is a nilpotent group, a contradiction.

Subcase 2. Every minimal subgroup Y of order q lies in $O_q(G)$.

By hypothesis G is not a $\mathcal{QNS}$-group. Assume that G contains a minimal subgroup X_0 such that X_0 is neither quasinormal in G nor self-normalizing.

(1) Suppose that $|X_0| = p$. We claim now that $|P| > p$. Assume that $|P| = p$. Then $Q \trianglelefteq G$ by Lemma 2.1. If $\Omega_1(Q)X_0 = G$, then Q is an elementary abelian q-group. Since $N_Q(X_0) > 1$, we know that Q is X_0-reducible by Lemma 2.2. Then there exists two X_0-invariant proper subgroups A and B of Q such that $Q = A \times B$. By hypothesis, both AX_0 and BX_0 are $\mathcal{QNS}$-groups. By Theorem 3.1, every minimal subgroup of A and B is quasinormal G. Hence we have that all minimal subgroups of A and B are all normal in G by Lemma 2.5, which implies that G is supersolvable, a contradiction. If $\Omega_1(Q)X_0 < G$, then $\Omega_1(Q)X_0$ is a $\mathcal{QNS}$-group by hypothesis and hence $\Omega_1(Q)X_0 = \Omega_1(Q) \times X_0$ since X_0 is not self-normalizing. Therefore we obtain that G is nilpotent by Lemma 2.3, a contradiction too. Thus $|P| > p$.

Suppose that $L = PO_q(G) < G$. Then by Theorem 3.1, every minimal subgroup of Q is quasinormal in L and hence quasinormal in G. Let Y be any minimal subgroup of Q. Then PY is a proper subgroup of G and hence is a $\mathcal{QN}$-group by hypothesis and every minimal subgroup of order q in PY is normal in PY by Theorem 3.1. It follows that X acts trivially on $\Omega_1(O_q(G))$ for every minimal subgroup X of P and therefore X acts trivially on $O_q(G)$ by Lemma 2.3, that is, $X \leq C_G(O_q(G)) \trianglelefteq G$. If $C_G(O_q(G)) = G$, then $O_q(G) \leq Z(G)$.

Now [7, IV, 5.5] tells us that G is q-nilpotent and hence $P = O_p(G)$. If $C_G(O_q(G)) < G$, then X is subnormal in G. Therefore $X \leq O_p(G)$. Thus we obtain that $X \leq O_p(G)$ for each $X < G$ with $|X| = p$.

If $O_p(G)Q < G$, then X_0 is quasinormal in $O_p(G)Q$ and hence is quasinormal in G, a contradiction. Therefore we may assume that $O_p(G)Q = G$. If $\Omega_1(O_p(G))Q < G$, then X_0 is quasinormal in $\Omega_1(O_p(G))Q$ and hence is quasinormal in G, a contradiction too. Thus we get $\Omega_1(O_p(G))Q = G$, and $P = \Omega_1(O_p(G))$ is a normal elementary abelian p-subgroup of G. Let $1 \neq y \in Q$ be an element with minimal order such that $\langle y\rangle$ cannot normalize X_0. Then $P\langle y\rangle$ is clearly not a $\mathcal{QNS}$-group. Hence $G = P\langle y\rangle = PQ$. If P is Q-reducible, then there exists two Q-invariant proper subgroups A and B of P such that $P = A \times B$. By hypothesis, both AQ and BQ are $\mathcal{QNS}$-groups. By Theorem 3.1, every minimal subgroup of A is in normal AQ, which implies that every minimal subgroup of A lies in the center of AQ. Similarly, we have that every minimal subgroup of B lies in the center of BQ. Thus we obtain $G = P \times Q$, a contradiction, which means P is Q-irreducible. Since $L = PO_q(G) < G$, we have that $|Q| > q$. That is, G is of type (iv).

Suppose that $L = PO_q(G) = G$. Then Q is normal in G. Let $M = P\Omega_1(Q)$. If M is a proper subgroup of G, then every minimal subgroup of M is quasinormal in M by Theorem 3.1. It follows that X_0 acts trivially on $\Omega_1(Q)$ by Lemma 2.5 and therefore X_0 acts trivially on Q by Lemma 2.3, which implies that X_0 is quasinormal in G, a contradiction. Hence $M = P\Omega_1(Q) = G$ and so Q is an elementary abelian q-group. Let P^* be a maximal subgroup of P containing X_0. Then QP^* is a $\mathcal{QNS}$-group by hypothesis. If $|P^*| > p$, then we have X_0 is quasinormal in QP^* by Theorem 3.1 and so X_0 is quasinormal in G, a contradiction. Hence $|P^*| = p$ and so $|P| = p^2$. If P is an elementary abelian p-group, let $P = \langle a\rangle \times \langle b\rangle$. Then $\langle a\rangle Q$ and $\langle b\rangle Q$ are all $\mathcal{QNS}$-groups. Hence every minimal subgroup of Q is quasinormal in G by Theorem 3.1, which implies that G is supsolvable, a contradiction. Thus P is cyclic of order p^2. By the same argument as above we know that the action of P on Q is irreducible. In addition, since G is non-supersolvable, we have $|Q| > q$. It follows that G is of type (v).

(2) Suppose that $|X_0| = q$. Let $R = P\Omega_1(O_q(G))$. Then $X_0 \leq R$ and hence $R = G$ by the choice of X_0. In particular, $Q = \Omega_1(O_q(G))$ is normal in G and Q is an elementary abelian q-group. Since $X_0 \leq Q$ is not quasinormal in G, there exists an element $y \in P$ such that $y^{-1}X_0y \neq X_0$. Obviously $\langle y\rangle Q$ is not a $\mathcal{QNS}$-group, hence $G = \langle y\rangle Q$. In particular, P is cyclic. Now we claim that P acts irreducibly on Q. Indeed, since G is non-supersolvable, some G-chief factor Q_0 of Q has order more than q. Then by Lemma 2.2, we may assume Q_0 is a minimal normal subgroup of G contained in Q. If PQ_0 is a $\mathcal{QNS}$-group, then every minimal subgroup of Q_0 is quasinormal in PQ_0 and so quasinormal in G since Q is a $\mathcal{QNS}$-group, a contradiction. Hence $Q_0 = Q$ and our claim holds. It follows that G is of type (v).

The proof of the theorem is now complete. □

The following theorem classifies all minimal non-$\mathcal{QNS}$-groups whose order having just three prime divisors.

Theorem 4.5. *Suppose that G is a minimal non-$\mathcal{QNS}$-group with $|\pi(G)| = 3$. Then one of the following statements is true*:

(i) $G = C_p \rtimes (C_q \rtimes C_r)$, *where $p > q > r$ are distinct primes and $C_pC_r = C_p \times C_r$.*

(ii) $G = C_p \rtimes (C_q \times C_r)$, *where $p > q, r$ are distinct primes and $Z(G) = 1$.*

(iii) $G = C_p \times (C_q \rtimes C_r)$, *where p, q and r are distinct primes and $r < p$.*

Proof. Since G is solvable by Proposition 4.2, we may assume that $G = P_1P_2P_3$, where $P_i \in Syl_{p_i}(G)$, $i = 1, 2, 3$. By hypothesis G is not a $\mathcal{QNS}$-group, we may assume that G contains a minimal subgroup $X \leq P_1$ such that X is neither quasinormal in G nor self-normalizing without loss of generality. As P_1P_2 and P_1P_3 are proper subgroups of G, both P_1P_2 and P_1P_3 are $\mathcal{QNS}$-groups. Since X is not quasinormal in G, we know that either P_2 or P_3 cannot normalize X. Assume that P_2 can not normalize X without loss of generality, then $P_1P_2 = P_2 \rtimes P_1$. It follows that $X = P_1$ by Theorem 3.1. On the other hand, since X is not self-normalizing, we have $XP_3 = X \rtimes P_3$ or $XP_3 = X \times P_3$.

Case 1. $XP_3 = X \rtimes P_3$. Then $P_3 = Z$ is of prime order, and $p_2 > p_1 > p_3$ by Theorem 3.1. Hence $P_2P_3 = P_2 \rtimes P_3$ or $P_2P_3 = P_2 \times P_3$.

If $P_2P_3 = P_2 \rtimes P_3$, then $G = P_2 \rtimes (X \rtimes Z)$. Choose a minimal subgroup Y of P_2 and let $T = Y \rtimes (X \rtimes Z)$. If T is a proper subgroup of G, then T is a $\mathcal{QNS}$-group. However as we know X is neither quasinormal in T nor self-normalizing, a contradiction. On the other hand, $X \rtimes Z \cong N_G(Y)/C_G(Y) \lesssim \mathrm{Aut}(Y)$ is a cyclic group, a contradiction.

By the similar way, we can get that if $P_2P_3 = P_2 \times P_3$, then G is of type (i).

Case 2. $XP_3 = X \times P_3$. If P_2P_3 is a $\mathcal{QNS}$-group and $P_2P_3 = P_2 \rtimes P_3$, then $P_3 = Z$ is of prime order, and $p_2 > p_1, p_3$ by Theorem 3.1. In this case, $G = P_2 \rtimes (X \times Z)$. Choose a minimal subgroup Y of P_2 and let $U = Y \rtimes (X \times Z)$. If U is a proper subgroup of G, then U is a $\mathcal{QNS}$-group. However X and Z are neither quasinormal in U nor self-normalizing, a contradiction. Hence $U = G$ and G is of type (ii).

If P_2P_3 is a $\mathcal{QNS}$-group and $P_2P_3 = P_3 \rtimes P_2$, then $P_2 = Y$ is of prime order, and $p_3 > p_2 > p_1$ by Theorem 3.1. By the same way as in Case 1, we can get G is of type (i).

If P_2P_3 is a $\mathcal{QN}$-group, choose a minimal subgroup Y of P_2 and a minimal subgroup Z of P_3. Let $W = Z \times (Y \rtimes X)$. If W is a proper subgroup of G, then W is a $\mathcal{QNS}$-group. However X is neither quasinormal in W nor self-normalizing, a contradiction. Hence $G = W$. That is, G is of type (iii).

The proof of the theorem is now complete. □

Proof of Main Theorem. It follows from Lemma 4.1, Proposition 4.2, Theorems 4.4 and 4.5. □

Acknowledgement. The authors would like to acknowledge the referee with deep gratitude for his(her) suggestions revising the paper.

References

[1] W. E. Deskins, *On quasinormal subgroups of finite groups*, Math. Z. **82** (1963), 125–132.

[2] K. Doerk, *Minimal nicht überauflösbare endliche Gruppen*, Math. Z. **91** (1966), 198–205.

[3] D. Gorenstein, *Finite Simple Groups*, New York, Plenum Press, 1982.

[4] P. Guo and X. Zhang, *On minimal non-MSP-groups*, Ukrainian Math. J. **63** (2012), no. 9, 1458–1463.

[5] Z. Han, G. Chen, and H. Shi, *On minimal non-NSN-groups*, J. Korean Math. Soc. **50** (2013), no. 3, 579–589.

[6] T. Hawkes, *On the automorphism group of a 2-group*, Proc. London Math. Soc. **26** (1973), 207–225.

[7] B. Huppert, *Endliche Gruppen I*, New York, Berlin, Springer-Verlag, 1967.

[8] H. Kurzweil and B. Stellmacher, *The Theory of Finite Groups: An Introduction*, New York, Springer-Verlag, 2004.

[9] T. J. Laffey, *A lemma on finite p-group and some consequences*, Proc. Cambridge Philos. Soc. **75** (1974), 133–137.

[10] G. A. Miller and H. C. Moreno, *Non-abelian groups in which every subgroup is abelian*, Trans. Amer. Math. Soc. **4** (1903), no. 4, 398–404.

[11] D. J. S. Robinson, *A Course in the Theory of Groups*, New York, Springer, 1993.

[12] N. Sastry, *On minimal non-PN-groups*, J. Algebra **65** (1980), no. 1, 104–109.

[13] J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable. I*, Bull. Amer. Math. Soc. **74** (1968), 383–437.

[14] L. Wang, *The Influence of Permutable Properties of Subgroups on the Structure of Finite Groups*, Doctoral Dissertation of Zhongshan University, 2006.

Zhangjia Han
School of Applied Mathematics
Chengdu University of Information Technology
Chengdu, Sichuan 610225, P. R. China
E-mail address: hzjmm11@163.com

Huaguo Shi
Sichuan Vocational and Technical College
Sichuan 629000, P. R. China
E-mail address: shihuaguo@126.com

Guiyun Chen
School of Mathematics and Statistics
Southwest University
Beibei, Chongqing 400715, P. R. China
E-mail address: gychen1963@163.com