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GENERATING SETS OF STRICTLY ORDER-PRESERVING TRANSFORMATION SEMIGROUPS ON A FINITE SET

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Abstract. Let O_n and PO_n denote the order-preserving transformation and the partial order-preserving transformation semigroups on the set $X_n = \{1, \dots, n\}$, respectively. Then the *strictly partial order-preserving transformation semigroup* SPO_n on the set X_n, under its natural order, is defined by $SPO_n = PO_n \setminus O_n$. In this paper we find necessary and sufficient conditions for any subset of $SPO(n,r)$ to be a (minimal) generating set of $SPO(n,r)$ for $2 \le r \le n-1$.

1. Introduction

The partial transformation semigroup $P_{\mathcal{X}}$ and the full transformation semigroup $T_{\mathcal{X}}$ on a set $\mathcal{X}$, the semigroups analogue of the symmetric group $S_{\mathcal{X}}$, have been much studied over the last fifty years, for both finite and infinite $\mathcal{X}$. Here we are concerned solely with the case where $\mathcal{X} = X_n = \{1, \dots, n\}$, and we write respectively P_n, T_n and S_n rather than P_{X_n}, T_{X_n} and S_{X_n}. Among recent contributions are [1, 2, 6, 10, 11]. The *domain*, *image*, *height* and *kernel* of $\alpha \in P_n$ are defined by

$$\begin{aligned} \operatorname{dom}(\alpha) &= \{x \in X_n : \text{ there exists } y \in X_n \text{ such that } x\alpha = y\}, \\ \operatorname{im}(\alpha) &= \{y \in X_n : \text{ there exists } x \in X_n \text{ such that } x\alpha = y\}, \\ \operatorname{h}(\alpha) &= |\operatorname{im}(\alpha)|, \\ \ker(\alpha) &= \{(x,y) \in X_n \times X_n : (x, y \in \operatorname{dom}(\alpha) \text{ and } x\alpha = y\alpha) \text{ or} \\ &\qquad (x, y \notin \operatorname{dom}(\alpha))\}, \end{aligned}$$

respectively. Notice that $\ker(\alpha)$ is an equivalence relation on X_n and the equivalence classes of $\ker(\alpha)$ are all of the pre-image sets of elements in $\operatorname{im}(\alpha)$ together with $X_n \setminus \operatorname{dom}(\alpha)$. Then, the set $\operatorname{kp}(\alpha) = \{y\alpha^{-1} : y \in \operatorname{im}(\alpha)\}$ is called the *kernel partition* of α and the ordered pair $\operatorname{ks}(\alpha) = \big(\operatorname{kp}(\alpha), X_n \setminus \operatorname{dom}(\alpha)\big)$

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is called the *kernel structure* of α. For any $\alpha, \beta \in P_n$ notice that

$$\begin{aligned} \mathrm{ks}\,(\alpha) = \mathrm{ks}\,(\beta) \quad &\Leftrightarrow \quad \mathrm{kp}\,(\alpha) = \mathrm{kp}\,(\beta) \\ &\Leftrightarrow \quad \ker(\alpha) = \ker(\beta) \text{ and } \mathrm{dom}\,(\alpha) = \mathrm{dom}\,(\beta). \end{aligned}$$

Moreover, for any α, β in P_n it is well known that $\mathrm{dom}\,(\alpha\beta) \subseteq \mathrm{dom}\,(\alpha)$, $\ker(\alpha) \subseteq \ker(\alpha\beta)$, $\mathrm{im}\,(\alpha\beta) \subseteq \mathrm{im}\,(\beta)$ and that $\alpha \in P_n$ is an idempotent if and only if $x\alpha = x$ for all $x \in \mathrm{im}\,(\alpha)$. We denote the set of all idempotents in any subset U of any semigroup by $E(U)$. (See [3, 7] for other terms in semigroup theory which are not explained here.)

The *order-preserving transformation semigroup* O_n and the *partial order-preserving transformation semigroup* PO_n on X_n, under its natural order, are defined by

$$\begin{aligned} O_n &= \{\alpha \in T_n \setminus S_n, : x \le y \Rightarrow x\alpha \le y\alpha \ (\forall x, y \in X_n)\}, \text{ and} \\ PO_n &= O_n \cup \{\alpha \in P_n \setminus T_n : x \le y \Rightarrow x\alpha \le y\alpha \ (\forall x, y \in \mathrm{dom}\,(\alpha))\}, \end{aligned}$$

respectively. Since $\mathrm{dom}\,(\alpha\beta) \subseteq \mathrm{dom}\,(\alpha)$, for all $\alpha, \beta \in PO_n$, $SPO_n = PO_n \setminus O_n$ is a subsemigroup of PO_n which is called the *strictly partial order-preserving transformation semigroup* on X_n, and

$$SPO(n, r) = \{\alpha \in SPO_n : |\mathrm{im}\,(\alpha)| \le r\}$$

is (under usual composition) a subsemigroup of SPO_n for $1 \le r \le n-1$. Notice that $SPO(n, n-1) = SPO_n$.

Let $\mathcal{A} = \{A_1, \ldots, A_k\}$ be a partition of a set $Y \subseteq X_n$. Then $\mathcal{A}$ is called an *ordered partition*, and we write $\mathcal{A} = (A_1, \ldots, A_k)$, if $x < y$ for all $x \in A_i$ and $y \in A_{i+1}$ $(1 \le i \le k-1)$ (the idea of ordering a family of sets appeared on p335 of [8]). Moreover, a set $\{a_1, \ldots, a_k\}$, such that $|\{a_1, \ldots, a_k\} \cap A_i| = 1$ for each $1 \le i \le k$, is called a *transversal* (or a *cross-section*) of $\mathcal{A}$. For $\alpha \in PO_n$ with height k, we have the order on the kernel classes $A_1, \ldots, A_k$ of $\mathrm{kp}\,(\alpha)$ as defined above and $\ker(\alpha) = \bigcup_{i=1}^{k+1} (A_i \times A_i)$, where $\emptyset \neq A_{k+1} = X_n \setminus \mathrm{dom}\,(\alpha)$. Without loos of generality, if $(A_1, \ldots, A_k)$ is an ordered partition of $\mathrm{dom}\,(\alpha)$, then $A_1\alpha < \cdots < A_k\alpha$, and moreover, α can be written in the following tabular forms:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k & A_{k+1} \\ A_1\alpha & A_2\alpha & \cdots & A_k\alpha & - \end{pmatrix} \text{ or } \alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_k \\ A_1\alpha & A_2\alpha & \cdots & A_k\alpha \end{pmatrix}.$$

For any α, β in $SPO(n, r)$, it is easy to show by using the definitions of the Green's equivalences that

$$\begin{aligned} &(\alpha, \beta) \in \mathcal{L} \quad \Leftrightarrow \quad \mathrm{im}\,(\alpha) = \mathrm{im}\,(\beta), & &(\alpha, \beta) \in \mathcal{R} \quad \Leftrightarrow \quad \mathrm{ks}\,(\alpha) = \mathrm{ks}\,(\beta), \\ &(\alpha, \beta) \in \mathcal{D} \quad \Leftrightarrow \quad \mathrm{h}\,(\alpha) = \mathrm{h}\,(\beta) \quad \text{and} & &(\alpha, \beta) \in \mathcal{H} \quad \Leftrightarrow \quad \alpha = \beta \end{aligned}$$

(see for the definitions of the Green's equivalences [7, pages 45–47]). For each r such that $1 \le r \le n-1$, we denote Green's $\mathcal{D}$-class of all elements in $SPO(n, r)$ of height k by D_k for $1 \le k \le r$.

Let $\Pi = (V(\Pi), \overrightarrow{E}(\Pi))$ be a digraph. For two vertices $u, v \in V(\Pi)$ we say u is *connected to* v in Π if there exists a directed path from u to v, that is, either $(u,v) \in \overrightarrow{E}(\Pi)$ or $(u,w_1),\ldots,(w_i,w_{i+1}),\ldots,(w_n,v) \in \overrightarrow{E}(\Pi)$ for some $w_1,\ldots,w_n \in V(\Pi)$. Moreover, we say Π is *strongly connected* if, for any two vertices $u,v \in V(\Pi)$, u is connected to v in Π. Let X be a non-empty subset of Green's $\mathcal{D}$-class D_r of $SPO(n,r)$ for $2 \le r \le n-1$. Then we define a digraph Γ_X as follows:

- the vertex set of Γ_X, denoted by $V = V(\Gamma_X)$, is X; and
- the directed edge set of Γ_X, denoted by $\overrightarrow{E} = \overrightarrow{E}(\Gamma_X)$, is

$$\overrightarrow{E} = \{(\alpha,\beta) \in V \times V : \alpha\beta \in D_r\} .$$

Let S be any semigroup, and let A be any non-empty subset of S. Then the subsemigroup generated by A, that is, the smallest subsemigroup of S containing A, is denoted by $\langle A\rangle$. The *rank* of a finitely generated semigroup S, a semigroup generated by its a finite subset, is defined by

$$\operatorname{rank}(S) = \min\{\,|A| : \langle A\rangle = S\}.$$

Gomes and Howie pointed out that the semigroup SPO_n is not idempotent generated and the rank of SPO_n is $2n-2$ in [5]. Moreover, for $2 \le r \le n-2$, Garba proved in [4] that the subsemigroup $SPO(n,r)$ is generated by idempotents of height r, and the rank of $SPO(n,r)$ is $\sum_{k=r}^{n-1}\binom{n}{k}\binom{k-1}{r-1}$.

The main goal of this paper is to find necessary and sufficient conditions for any subset of $SPO(n,r)$ to be a (minimal) generating set of $SPO(n,r)$ for $2 \le r \le n-1$.

2. Generating sets of SPO_n

For convenience we state and prove probably a well known proposition.

Proposition 1. *For $n,k \ge 2$ and $1 \le r \le n-1$, let $\alpha,\beta,\alpha_1,\ldots,\alpha_k \in D_r$ in $SPO(n,r)$. Then,*

(i) *$\alpha\beta \in D_r$ if and only if $\operatorname{im}(\alpha\beta) = \operatorname{im}(\beta)$. In other words, $\operatorname{im}(\alpha)$ is a transversal of the kernel partition $\operatorname{kp}(\beta)$ of β.*

(ii) *$\alpha_1\cdots\alpha_k \in D_r$ if and only if $\alpha_i\alpha_{i+1} \in D_r$ for each $1 \le i \le k-1$.*

Proof. The proof of (i) is clear. We prove (ii):

(⇒) This part of the proof is also clear.

(⇐) Suppose that $\alpha_i\alpha_{i+1} \in D_r$ for each $1 \le i \le k-1$. We use the inductive hypothesis on k to complete the proof.

For $k=2$ the claim is obviously true. Suppose that the claim holds for $k-1 \ge 2$. If $\operatorname{im}(\alpha_{k-1}) = \{y_1,\ldots,y_r\}$, then $\operatorname{im}(\alpha_1\cdots\alpha_{k-1}) = \{y_1,\ldots,y_r\}$ since $\operatorname{im}(\alpha_1\cdots\alpha_{k-1}) \subseteq \operatorname{im}(\alpha_{k-1})$ and $\alpha_1\cdots\alpha_{k-1},\, \alpha_{k-1} \in D_r$. Thus it follows from (i) that

$$\operatorname{im}(\alpha_1\cdots\alpha_{k-1}\alpha_k) = \{y_1\alpha_k,\ldots,y_r\alpha_k\} = \operatorname{im}(\alpha_{k-1}\alpha_k) = \operatorname{im}(\alpha_k),$$

and so $\alpha_1\cdots\alpha_{k-1}\alpha_k \in D_r$, as required. □

For $0 \le s \le r \le n$ recall the set

$$[r,s] = \{\alpha \in PO_n : |\mathrm{dom}\,(\alpha)| = r \ \text{ and } \ |\mathrm{im}\,(\alpha)| = s\}$$

(defined in [5, p. 276]). It is indicated in [5] that the top Green's $\mathcal{D}$-class $D_{n-1} = [n-1, n-1]$ in SPO_n does not generate SPO_n, and that SPO_n is not idempotent generated.

Suppose that A is a (minimal) generating set of SPO_n. Since there exist n different Green's $\mathcal{L}$-classes and n different Green's $\mathcal{R}$-classes in D_{n-1}, A must contain at least n elements from D_{n-1}. Next notice that a typical element $\alpha \in [n-1, n-2] \subseteq D_{n-2}$ has the form:

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_{n-1} \\ b_1 & \cdots & b_{n-1} \end{pmatrix} \in [n-1, n-2],$$

where $a_1 < \cdots < a_{n-1}$, $b_1 \le \cdots \le b_{n-1}$ and all but one of the inequalities between the b's are strict. Then α is called *of kernel type* i if $b_i = b_{i+1}$, and we write $K(\alpha) = i$, and so the possible values for $K(\alpha)$ are $1, 2, \ldots, n-2$. It is shown in [5] that $A \cap [n-1, n-2]$ must contain at least one element of each of the $n-2$ possible kernel types in $[n-1, n-2]$. For each $i = 2, \ldots, n$ let

$$\alpha_i : X_n \setminus \{i\} \to X_n \setminus \{i-1\} \ \text{ and } \ \alpha_1 : X_n \setminus \{1\} \to X_n \setminus \{n\} \tag{1}$$

be the unique order-preserving (bijective) transformations. That is, let α_1 denote the unique order-preserving (bijective) transformation from $X_n \setminus \{1\}$ onto $X_n \setminus \{n\}$ and, for each $i \in \{2, \ldots, n\}$, let α_i denote the unique order-preserving (bijective) transformation from $X_n \setminus \{i\}$ onto $X_n \setminus \{i-1\}$. For each $i = 1, \ldots, n-2$, let

$$\beta_i : X_n \setminus \{n\} \to X_n \tag{2}$$

be the order-preserving transformation defined by

$$j\beta_i = \begin{cases} i+1 & \text{if } j = i \\ j & \text{otherwise.} \end{cases}$$

Then it is clear that $\{\alpha_1, \ldots, \alpha_n\} \subseteq [n-1, n-1] = D_{n-1}$ and $\{\beta_1, \ldots, \beta_{n-2}\} \subseteq [n-1, n-2]$ and it is showed in [5] that

$$\mathcal{Z} = \{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_{n-2}\} \tag{3}$$

is a minimal generating set of SPO_n. Thus $\mathrm{rank}\,(SPO_n) = 2n-2$. Although $SPO_n = SPO(n, n-1)$ is not idempotent generated, it is shown in [4] that $SPO(n,r)$ is generated by the idempotents in D_r, and that

$$\mathrm{rank}\,(SPO(n,r)) = \sum_{k=r}^{n-1} \binom{n}{k} \binom{k-1}{r-1}$$

for $2 \le r \le n-2$.

Notice that if $\alpha, \beta \in [r,r] \subseteq SPO(n,r)$ for $1 \le r \le n-1$, then it follows from Proposition 1(i) that $\alpha\beta \in D_r$ if and only if $\mathrm{im}\,(\alpha) = \mathrm{dom}\,(\beta)$.

Theorem 2. *Let A be a subset of SPO_n, and let $B = A \cap [n-1, n-1]$ and $C = A \cap [n-1, n-2]$. Then A is a generating set of SPO_n if and only if*

(i) *there exists $\lambda_i \in B$ such that $\operatorname{dom}(\lambda_i) = X_n \setminus \{i\}$ for each $i = 1, 2, \ldots, n$,*
(ii) *there exist $\gamma_1 \in B$ such that $\operatorname{im}(\gamma_1) = X_n \setminus \{n\}$ and $\gamma_i \in B$ such that $\operatorname{im}(\gamma_i) = X_n \setminus \{i-1\}$ for each $i = 2, 3, \ldots, n$,*
(iii) *λ_i is connected to γ_i in the digraph Γ_B for each $i = 1, 2, \ldots, n$; and*
(iv) *there exists at least one element of each of the $n-2$ possible kernel types in C.*

Proof. ($\Rightarrow$) Suppose that A is a generating set of SPO_n. Let α_i, with $i = 1, 2, \ldots, n$, be the elements defined in (1). Since A is a generating set, there exist $\lambda_{(i,1)}, \ldots, \lambda_{(i,k)} \in A$ such that

$$\alpha_i = \lambda_{(i,1)} \cdots \lambda_{(i,k)}$$

for each $i \in \{1, \ldots, n\}$. Since $\ker(\lambda_{(i,1)}) \subseteq \ker(\alpha_i)$, $\operatorname{dom}(\alpha_i) \subseteq \operatorname{dom}(\lambda_{(i,1)})$, $\operatorname{im}(\alpha_i) \subseteq \operatorname{im}(\lambda_{(i,k)})$ and $\alpha_i \in [n-1, n-1]$, it follows that $\operatorname{dom}(\alpha_i) = X_n \setminus \{i\} = \operatorname{dom}(\lambda_{(i,1)})$ for each $i = 1, 2, \ldots, n$ and that $\operatorname{im}(\alpha_1) = X_n \setminus \{n\} = \operatorname{im}(\lambda_{(1,k)})$ and $\operatorname{im}(\alpha_i) = X_n \setminus \{i-1\} = \operatorname{im}(\lambda_{(i,k)})$ for each $i = 2, 3, \ldots, n$. Then it is clear that $\lambda_{(i,1)}, \ldots, \lambda_{(i,k)} \in B$. Let $\lambda_i = \lambda_{(i,1)}$ and $\gamma_i = \lambda_{(i,k)}$. Hence, the first two conditions hold. Moreover, it follows from Proposition 1(ii) that $\lambda_{(i,j)}\lambda_{(i,j+1)} \in [n-1, n-1]$ for each $1 \leq j \leq k-1$, and so there exists a directed edge from $\lambda_{(i,j)}$ to $\lambda_{(i,j+1)}$ in Γ_B for each $1 \leq j \leq k-1$. Thus, λ_i is connected to γ_i in the digraph Γ_B. Therefore, the third condition holds as well. Since the last condition follows from the result in [5, p. 280], the first part of the proof is complete.

($\Leftarrow$) For this part of the proof it is enough to show that the generating set $\mathcal{Z}$ given in (3) is a subset of $\langle A \rangle$.

For any element α_i defined in (1) it follows from the conditions that there exist $\lambda_i, \gamma_i \in B$ such that $\operatorname{dom}(\lambda_i) = \operatorname{dom}(\alpha_i)$, $\operatorname{im}(\gamma_i) = \operatorname{im}(\alpha_i)$ and λ_i is connected to γ_i in the digraph Γ_B. Thus there exists a directed path from λ_i to γ_i, say

$$\lambda_i = \sigma_1 \to \sigma_2 \to \cdots \to \sigma_{k-1} \to \sigma_k = \gamma_i,$$

where $\sigma_1, \ldots, \sigma_k \in B$, and hence, $\sigma_j\sigma_{j+1} \in [n-1, n-1]$ for each $1 \leq j \leq k-1$. It follows from Proposition 1(ii) that $\delta = \sigma_1 \cdots \sigma_k \in [n-1, n-1]$. Thus we have $\operatorname{im}(\delta) = \operatorname{im}(\sigma_k) = \operatorname{im}(\gamma_i) = \operatorname{im}(\alpha_i)$ and $\operatorname{dom}(\delta) = \operatorname{dom}(\sigma_1) = \operatorname{dom}(\lambda_i) = \operatorname{dom}(\alpha_i)$, and so $\alpha_i = \delta \in \langle B \rangle \subseteq \langle A \rangle$ since α_i, $\delta \in [n-1, n-1]$ are order-preserving bijections.

It follows from the last condition that, for each $i = 1, 2, \ldots, n-2$, we may choose and fix an element with the kernel type i in C and say θ_i. Thus, we have

$$\theta_i = \begin{pmatrix} a_1 & \cdots & a_{i-1} & a_i & a_{i+1} & a_{i+2} & \cdots & a_{n-1} \\ b_1 & \cdots & b_{i-1} & b_i & b_i & b_{i+1} & \cdots & b_{n-2} \end{pmatrix} \in C$$

for some $a_1 < \cdots < a_{n-1}$ and $b_1 < \cdots < b_{n-2}$ in X_n. For any element β_i $(1 \leq i \leq n-2)$ defined in (2), consider two strictly partial order-preserving transformations

$$\begin{aligned} \varphi_i &= \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix} \in [n-1, n-1] \text{ and} \\ \psi_i &= \begin{pmatrix} b_1 & \cdots & b_{i-1} & b_i & \cdots & b_{n-2} \\ 1 & \cdots & i-1 & i+1 & \cdots & n-1 \end{pmatrix} \in [n-2, n-2]. \end{aligned}$$

From [5, Lemmas 3.4 and 3.12] we have that $[r, r] \subseteq \langle B \rangle$ for all $r = 1, \ldots, n-1$, and so $\varphi_i, \psi_i \in \langle B \rangle$ $(1 \leq i \leq n-2)$. Thus it follows from the fact

$$\beta_i = \varphi_i \theta_i \psi_i$$

that $\beta_i \in \langle A \rangle$ for all $i = 1, \ldots, n-2$. Therefore, $\mathscr{Z}$ defined in (3) is a subset of $\langle A \rangle$, and so $SPO_n = \langle A \rangle$. □

Since a generating set of SPO_n must contain at least n elements from $[n-1, n-1]$ and at least $n-2$ elements from $[n-1, n-2]$, we have the following corollary from Theorem 2:

Corollary 3. *Let B be a subset of $[n-1, n-1]$ with cardinality n and let C be a subset of $[n-1, n-2]$ with cardinality $n-2$. Then $B \cup C$ is a minimal generating set of SPO_n if and only if*

(i) *there exists exactly one element $\lambda_i \in B$ such that $\operatorname{dom}(\lambda_i) = X_n \setminus \{i\}$ for each $i = 1, \ldots, n$,*
(ii) *there exist exactly one element $\gamma_1 \in B$ such that $\operatorname{im}(\gamma_1) = X_n \setminus \{n\}$ and exactly one element $\gamma_i \in B$ such that $\operatorname{im}(\gamma_i) = X_n \setminus \{i-1\}$ for each $i = 2, 3, \ldots, n$,*
(iii) *λ_i is connected to γ_i in the digraph Γ_B for $i = 1, 2, \ldots, n$; and*
(iv) *there exists exactly one element of each of the $n-2$ possible kernel types in C.*

For example, consider SPO_3, $B = \{\sigma_1, \sigma_2, \sigma_3\} \subseteq [n-1, n-1]$ with cardinality 3, and $C = \{\theta\} \subseteq [n-1, n-2]$ with cardinality 1 where

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & 3 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & - & 2 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & - \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & - \end{pmatrix}.$$

First of all it is easy to see that B satisfies the first two conditions and C satisfies the last condition of Corollary 3. Moreover, since $\operatorname{im}(\sigma_i)$ is a transversal of the kernel partition $\operatorname{kp}(\sigma_{i+1})$, there exists a directed edge from σ_i to σ_{i+1} for each $1 \leq i \leq 3$ (where $\sigma_4 = \sigma_1$) in the digraph Γ_B. Thus Γ_B is a Hamiltonian digraph. It follows from their definitions that Hamiltonian digraphs are strongly connected (see [9, pages 88 and 148]) the third condition of Corollary 3 is satisfied as well, and so $A = B \cup C$ is a minimal generating set of SPO_3.

Notice that if $A \subseteq SPO_n$ is a (minimal) generating set of SPO_n, then we may use the paths in Γ_B to write each α_i (for $1 \leq i \leq n$) defined in (1) as a product of elements from B, where $B = A \cap [n-1, n-1]$. For example, consider the minimal generating set $A = \{\sigma_1, \sigma_2, \sigma_3, \theta\}$ of SPO_3 given above and consider the transformation

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ - & 1 & 2 \end{pmatrix}$$

defined in (1). Since $B = \{\sigma_1, \sigma_2, \sigma_3\}$, $\operatorname{dom}(\alpha_1) = \operatorname{dom}(\sigma_1)$, $\operatorname{im}(\alpha_1) = \operatorname{im}(\sigma_2)$ and

$$\sigma_1 \to \sigma_2$$

is a path in Γ_B, it follows that $\operatorname{dom}(\sigma_1\sigma_2) = \operatorname{dom}(\alpha_1)$ and $\operatorname{im}(\sigma_1\sigma_2) = \operatorname{im}(\alpha_1)$, that is, $\sigma_1\sigma_2 = \alpha_1$.

3. Generating sets of $SPO(n, r)$

Now we consider the subsemigroups $SPO(n, r)$ for all $2 \leq r \leq n-2$. Since $\alpha \in D_k$ $(1 \leq k \leq r)$ can not be written as a product of elements with height smaller than k, and since $SPO(n, r)$ is generated by its idempotents of height r, it is enough to consider only the subsets of D_r as a generating set of $SPO(n, r)$. Moreover, a subset X of D_r is a generating set of $SPO(n, r)$ if and only if $E(D_r) \subseteq \langle X \rangle$ for $2 \leq r \leq n-2$.

Theorem 4. *Let X be a subset of Green's $\mathcal{D}$-class D_r in $SPO(n, r)$ for $2 \leq r \leq n-2$. Then X is a generating set of $SPO(n, r)$ if and only if, for each idempotent ξ in $E(D_r)$, there exist $\alpha, \beta \in X$ such that*

(i) $\operatorname{ks}(\alpha) = \operatorname{ks}(\xi)$,

(ii) $\operatorname{im}(\beta) = \operatorname{im}(\xi)$, *and*

(iii) *α is connected to β in the digraph Γ_X.*

Proof. The proof is similar to the proof of Theorem 2. But it is much easier since each $SPO(n, r)$ (for $2 \leq r \leq n-2$) is idempotent generated. □

Similarly, since $\operatorname{rank}(SPO(n, r)) = \sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}$ for $2 \leq r \leq n-2$, we have the following corollary:

Corollary 5. *For $2 \leq r \leq n-2$ let X be a subset of Green's $\mathcal{D}$-class D_r with cardinality $\sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}$. Then X is a minimal generating set of $SPO(n, r)$ if and only if, for each idempotent $\xi \in E(D_r)$, there exist $\alpha, \beta \in X$ such that* $\operatorname{ks}(\alpha) = \operatorname{ks}(\xi)$, $\operatorname{im}(\beta) = \operatorname{im}(\xi)$ *and α is connected to β in Γ_X.* □

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