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GENERALIZED FIBONACCI AND LUCAS NUMBERS OF THE FORM wx^2 AND $wx^2 \mp 1$

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Abstract. Let $P \geq 3$ be an integer and let (U_n) and (V_n) denote generalized Fibonacci and Lucas sequences defined by $U_0 = 0, U_1 = 1$; $V_0 = 2, V_1 = P$, and $U_{n+1} = PU_n - U_{n-1}$, $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. In this study, when P is odd, we solve the equations $V_n = kx^2$ and $V_n = 2kx^2$ with $k \mid P$ and $k > 1$. Then, when $k \mid P$ and $k > 1$, we solve some other equations such as $U_n = kx^2, U_n = 2kx^2, U_n = 3kx^2, V_n = kx^2 \mp 1, V_n = 2kx^2 \mp 1$, and $U_n = kx^2 \mp 1$. Moreover, when P is odd, we solve the equations $V_n = wx^2 + 1$ and $V_n = wx^2 - 1$ for $w = 2, 3, 6$. After that, we solve some Diophantine equations.

1. Introduction

Let P and Q be nonzero integers. Generalized Fibonacci sequence (U_n) and Lucas sequence (V_n) are defined by $U_0(P,Q) = 0, U_1(P,Q) = 1$; $V_0(P,Q) = 2, V_1(P,Q) = P$, and $U_{n+1}(P,Q) = PU_n(P,Q) + QU_{n-1}(P,Q)$, $V_{n+1}(P,Q) = PV_n(P,Q) + QV_{n-1}(P,Q)$ for $n \geq 1$. $U_n(P,Q)$ and $V_n(P,Q)$ are called n-th generalized Fibonacci number and n-th generalized Lucas number, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are defined as $U_{-n}(P,Q) = -(-Q)^{-n}U_n(P,Q)$ and $V_{-n}(P,Q) = (-Q)^{-n}V_n(P,Q)$, respectively.

Now assume that $P^2 + 4Q \neq 0$. Then it is well known that

$$U_n = U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = V_n(P,Q) = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = \frac{P+\sqrt{P^2+4Q}}{2}$ and $\beta = \frac{P-\sqrt{P^2+4Q}}{2}$, which are the roots of the characteristic equation $x^2 - Px - Q = 0$.

The above formulas are known as Binet's formulas. Since

$$U_n(-P,Q) = (-1)^{n-1}U_n(P,Q) \text{ and } V_n(-P,Q) = (-1)^n V_n(P,Q),$$

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it will be assumed that $P \geq 1$. Moreover, we will assume that $P^2 + 4Q > 0$. For $P = Q = 1$, we have classical Fibonacci and Lucas sequences (F_n) and (L_n). For $P = 2$ and $Q = 1$, we have Pell and Pell-Lucas sequences (P_n) and (Q_n). For more information about generalized Fibonacci and Lucas sequences one can consult [4, 10, 11, 12].

Generalized Fibonacci and Lucas numbers of the form kx^2 have been investigated since 1962. When P is odd and $Q = \mp 1$, by using elementary argument, many authors solved the equations $U_n = kx^2$ or $V_n = kx^2$ for specific integer values of k. The reader can consult [17] for a brief discussion of this subject. When P and Q are relatively prime odd integers, in [13], the authors solved $U_n = x^2, U_n = 2x^2, V_n = x^2, V_n = 2x^2$. Moreover, under the same assumption, in [15], the same authors solved $U_n = 3x^2$ and they solved $V_n = kx^2$ under some assumptions on k.

In [2], when P is odd, Cohn solved the equations $V_n = Px^2$ and $V_n = 2Px^2$ with $Q = \mp 1$. When P is odd, in [17], the authors solved the equation $V_n(P, 1) = kx^2$ for $k \mid P$ with $k > 1$. In this study, when P is odd, we will solve the equations $V_n(P, -1) = kx^2$ and $V_n(P, -1) = 2kx^2$ for $k \mid P$ with $k > 1$. Then, when $k \mid P$ with $k > 1$, we will solve some other equations such as $U_n(P, -1) = kx^2$, $U_n(P, -1) = 2kx^2$, $U_n(P, -1) = 3kx^2$, $V_n(P, -1) = kx^2 \mp 1$, $V_n(P, -1) = 2kx^2 \mp 1$, and $U_n(P, -1) = kx^2 \mp 1$. When P is odd, we will solve the equations $V_n(P, -1) = wx^2 + 1$ and $V_n(P, -1) = wx^2 - 1$ for $w = 2, 3, 6$. Thus we solve some Diophantine equations.

We will use the Jacobi symbol throughout this study. Our method is elementary and used by Cohn, Ribenboim and McDaniel in [2] and [15], respectively.

2. Preliminaries

From now on, sometimes, instead of $U_n(P, -1)$ and $V_n(P, -1)$, we will use U_n and V_n, respectively. Moreover, we will assume that $P \geq 3$. The following lemmas can be proved by induction.

Lemma 2.1. *If n is a positive integer, then $V_{2n} \equiv \mp 2 \pmod{P^2}$ and $V_{2n+1} \equiv (2n+1)P(-1)^n \pmod{P^2}$.*

Lemma 2.2. *If n is a positive integer, then $U_{2n} \equiv n(-1)^{n+1}P \pmod{P^2}$ and $U_{2n+1} \equiv (-1)^n \pmod{P^2}$.*

The following lemma is given in [13] and [15].

Lemma 2.3. $\left(\frac{U_3}{V_{2^r}}\right) = 1$ *for $r \geq 1$.*

The following lemma is a consequence of a theorem given in [6].

Lemma 2.4. *All positive integer solutions of the equation $3x^2 - 2y^2 = 1$ are given by $(x, y) = (U_n(10, -1) - U_{n-1}(10, -1), U_n(10, -1) + U_{n-1}(10, -1))$ with $n \geq 1$.*

The following theorems are well known (see [3, 5, 8, 9]).

Theorem 2.5. *All positive integer solutions of the equation* $x^2-(P^2-4)y^2=4$ *are given by* $(x,y)=(V_n(P,-1),U_n(P,-1))$ *with* $n\geq 1$.

Theorem 2.6. *All positive integer solutions of the equation* $x^2-Pxy+y^2=1$ *are given by* $(x,y)=(U_n(P,-1),U_{n-1}(P,-1))$ *with* $n\geq 1$.

The proofs of the following two theorems are given in [16].

Theorem 2.7. *Let* $n\in\mathbb{N}\cup\{0\}$, m, $r\in\mathbb{Z}$ *and* m *be a nonzero integer. Then*

(2) $$U_{2mn+r}\equiv U_r \pmod{U_m}$$

and

(3) $$V_{2mn+r}\equiv V_r \pmod{U_m}.$$

Theorem 2.8. *Let* $n\in\mathbb{N}\cup\{0\}$ *and* m, $r\in\mathbb{Z}$. *Then*

(4) $$U_{2mn+r}\equiv (-1)^n U_r \pmod{V_m}$$

and

(5) $$V_{2mn+r}\equiv (-1)^n V_r \pmod{V_m}.$$

When P is odd, since $8\mid U_3$, using (3) we get

(6) $$V_{6q+r}\equiv V_r \pmod 8.$$

Thus

(7) $$4\nmid V_n.$$

Moreover, an induction method shows that

$$V_{2^r}\equiv 7 \pmod 8$$

and thus

(8) $$\left(\frac{2}{V_{2^r}}\right)=1$$

for $r\geq 1$.

When P is odd, it is seen that

(9) $$\left(\frac{-1}{V_{2^r}}\right)=-1$$

for $r\geq 1$.

Secondly, we give some identities concerning generalized Fibonacci and Lucas numbers:

$$U_{-n}=-U_n \text{ and } V_{-n}=V_n,$$

(10) $$U_{2n+1}-1=U_nV_{n+1},$$

(11) $$U_{2n}=U_nV_n,$$

(12) $$V_{2n}=V_n^2-2,$$

(13) $$U_{3n} = U_n((P^2-4)U_n^2+3) = U_n(V_n^2-1) = U_n(V_{2n}+1),$$

(14) $$V_{3n} = V_n(V_n^2-3) = V_n(V_{2n}-1),$$

(15) $$\text{If } P \text{ is odd, then } 2 \mid V_n \Leftrightarrow 2 \mid U_n \Leftrightarrow 3 \mid n,$$

(16) $$V_n^2 - (P^2-4)U_n^2 = 4.$$

Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geq 0$, and $d = (m, n)$. Then

(17) $$(U_n, U_m) = U_d,$$

(18) $$(U_m, V_n) = \begin{cases} V_d & \text{if } a > b, \\ 1 \text{ or } 2 & \text{if } a \leq b. \end{cases}$$

If P is odd and $r \geq 2$, then $V_{2^r} \equiv -1 \pmod{\frac{P^2-3}{2}}$ and thus

(19) $$\left(\frac{(P^2-3)/2}{V_{2^r}}\right) = \left(\frac{P^2-3}{V_{2^r}}\right) = 1.$$

(20) $$\text{If } r \geq 1, \text{ then } V_{2^r} \equiv \mp 2 \pmod{P}.$$

(21) $$\text{If } r \geq 2, \text{ then } V_{2^r} \equiv 2 \pmod{P}.$$

If $3 \nmid P$, then $3 \mid U_3$. Thus we get

(22) $$3 \mid U_n \Leftrightarrow 3 \mid n$$

by (2) and $V_{2^r} \equiv -1 \pmod 3$ and therefore

(23) $$\left(\frac{3}{V_{2^r}}\right) = 1$$

for $r \geq 1$.

If $3 \mid P$ and P is odd, then $V_{2^r} \equiv -1 \pmod 3$ for $r \geq 2$ and thus

(24) $$\left(\frac{3}{V_{2^r}}\right) = 1$$

for $r \geq 2$. Moreover, we have

(25) $$\left(\frac{P-1}{V_{2^r}}\right) = \left(\frac{P+1}{V_{2^r}}\right) = 1$$

for $r \geq 1$.

Identities in between (11)–(16) and (17)–(18) can be found in [12, 15, 16] and [7, 14, 15], respectively. The proofs of the others are easy and will be omitted.

3. Main theorems

From now on, we will assume that n is a positive integer and P is odd.

Lemma 3.1. *Let $m > 1$ be odd. Then $V_{2m} + 1 = 2x^2$ has no solutions.*

Proof. Assume that $V_{2m} + 1 = 2x^2$ for some integer x. Let $2m = 2(4q \mp 1) = 2(2^r a \mp 1)$ with a odd and $r \geq 2$. Thus

$$2x^2 = V_{2m} + 1 \equiv 1 - V_2 \equiv -(P^2 - 3) \pmod{V_{2^r}}$$

by (5), which implies that

$$x^2 \equiv -\left(\frac{P^2 - 3}{2}\right) \pmod{V_{2^r}}.$$

This shows that

$$\left(\frac{(P^2-3)/2}{V_{2^r}}\right) = -1 \tag{26}$$

which is impossible by (19). □

Theorem 3.2. *If $V_n = kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1$.*

Proof. Assume that $V_n = kx^2$ for some $k \mid P$ with $k > 1$. Then by Lemma 2.1, it is seen that n is odd. Let $n = 6q + r$ with $r \in \{1, 3, 5\}$. Then by (6), $V_n = V_{6q+r} \equiv V_r \pmod 8$ and therefore $V_n \equiv V_1, V_3, V_5 \pmod 8$. It is seen that $V_n \equiv P, 6P \pmod 8$. Then $kx^2 \equiv P, 6P \pmod 8$. Let $P = kM$. Thus, we get $kMx^2 \equiv PM, 6PM \pmod 8$, which implies that $Px^2 \equiv PM, 6PM \pmod 8$. This shows that $x^2 \equiv M, 6M \pmod 8$ since P is odd. Therefore $M \equiv 1 \pmod 8$ since M is odd. Now assume that $n > 1$. Then $n = 4q \mp 1 = 2^{r+1}a \mp 1$ with a odd and $r \geq 1$. Since $V_n = kx^2$, we get

$$kx^2 = V_n \equiv -V_{\mp 1} \equiv -P \pmod{V_{2^r}}$$

by (5). This shows that

$$x^2 \equiv -M \pmod{V_{2^r}},$$

which implies that

$$\left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = 1. \tag{27}$$

By using the fact that $M \equiv 1 \pmod 8$, we get

$$\left(\frac{M}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{M}\right) = \left(\frac{\mp 2}{M}\right) = 1$$

by (20). Since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (9), we get $\left(\frac{-1}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -1$, which contradicts (27). Therefore $n = 1$. □

We can give the following corollary by using Theorems 2.5 and 3.2.

Corollary 3.3. *The only positive integer solution of the equation* $P^2x^4-(P^2-4)y^2=4$ *is* $(x,y)=(1,1)$.

Theorem 3.4. *If* $V_n=2kx^2$ *for some* $k\mid P$ *with* $k>1$, *then* $n=3$.

Proof. Assume that $V_n=2kx^2$ for some $k\mid P$ with $k>1$. Then $3\mid n$ and n is odd by (15) and Lemma 2.1. Let $n=3m$ with m odd. Then $2kx^2=V_n=V_{3m}=V_m(V_m^2-3)$ by (14) and thus $(V_m/k)(V_m^2-3)=2x^2$. Since $(V_m,V_m^2-3)=1$ or 3, we get $V_m^2-3=wa^2$ for some $w\in\{1,2,3,6\}$. Since $V_m^2-3=V_{2m}-1$, it is seen that $V_{2m}-1=wa^2$. Assume that $m>1$. Then $m=4q\mp1=2^ra\mp1$ with a odd and $r\geq 2$. Thus,

$$wa^2=V_{2m}-1\equiv -1-V_{\mp 2}\equiv -1-(P^2-2)\equiv -(P^2-1)\pmod{V_{2^r}}.$$

This shows that

$$\left(\frac{w}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-1}{V_{2^r}}\right).$$

By using (8), (23), and (24), it can be seen that $\left(\frac{w}{V_{2^r}}\right)=1$ for $w=2,3,6$. Moreover, $\left(\frac{-1}{V_{2^r}}\right)=-1$ and $\left(\frac{P^2-1}{V_{2^r}}\right)=1$ by (9) and Lemma 2.3, respectively. Thus, we get

$$1=\left(\frac{w}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-1}{V_{2^r}}\right)=-1,$$

which is impossible. Therefore $m=1$ and thus, $n=3$. □

We can give the following corollary easily.

Corollary 3.5. *The equation* $4P^2x^4-(P^2-4)y^2=4$ *has no integer solutions.*

Proof. Assume that $4P^2x^4-(P^2-4)y^2=4$ for some positive integer x and y. Then by Theorems 2.5 and 3.4, we get $V_3=2Px^2$. This shows that $2Px^2=P(P^2-3)$, which implies that $P^2-2x^2=3$. But this is impossible. □

Now we give some known theorems from [13], which will be useful for solving the equations $U_n=kx^2, U_n=2kx^2$, and $U_n=3kx^2$, where $k\mid P$ with $k>1$. We use a theorem from [1] while solving $V_n=2x^2$.

Theorem 3.6. *If* $V_n=x^2$ *for some integer* x, *then* $n=1$. *If* $V_n=2x^2$ *for some integer* x, *then* $n=3, P=3,27$.

Theorem 3.7. *If* $U_n=x^2$ *for some integer* x, *then* $n=1$ *or* $n=2$, $P=\square$ *or* $n=6, P=3$. *If* $U_n=2x^2$ *for some integer* x, *then* $n=3$.

Theorem 3.8. *Let* $P\geq 3$ *be odd. If* $U_n=kx^2$ *for some* $k\mid P$ *with* $k>1$, *then* $n=2$ *or* $n=6$ *and* $3\mid P$.

Proof. Assume that $U_n=kx^2$ for some $k\mid P$ with $k>1$. Then by Lemma 2.2, n is even and therefore $n=2m$ for some positive integer m. Thus, $kx^2=U_n=$

$U_{2m} = U_m V_m$. Assume that m is even. Then we get $(U_m/k)V_m = x^2$. In this case, since $(U_m, V_m) = 1$ or 2, by (18) either

$$U_m = ku^2 \text{ and } V_m = v^2, \tag{28}$$

or

$$U_m = 2ku^2 \text{ and } V_m = 2v^2 \tag{29}$$

for some integers u and v. Since m is even, the identities (28) and (29) are impossible by Theorem 3.6. Now assume that m is odd. In this case, either

$$U_m = u^2 \text{ and } V_m = kv^2, \tag{30}$$

or

$$U_m = 2u^2 \text{ and } V_m = 2kv^2 \tag{31}$$

for some integers u and v. If (30) is satisfied, then $m = 1$ by Theorem 3.7 and therefore $n = 2$. Assume that (31) is satisfied. Then $m = 3$ by Theorems 3.4 and 3.7. Thus $n = 6$. In which case, it can be seen that if $U_6 = U_3V_3 = kx^2$ for some $k \mid P$ with $k > 1$, then $3 \mid P$. □

Corollary 3.9. *The equation* $P^2x^4 - P^2x^2y + y^2 = 1$ *has only the positive solution* $(x, y) = (1, 1)$.

Proof. Assume that $P^2x^4 - P^2x^2y + y^2 = 1$ for some positive integers x and y. Then $Px^2 = U_2$ or $Px^2 = U_6$ by Theorems 2.6 and 3.8. It can be seen that $Px^2 = U_6$ is impossible. Therefore $Px^2 = U_2$ and thus $y = U_1 = 1$. This shows that $(x, y) = (1, 1)$. □

Theorem 3.10. *If* $k \mid P$ *with* $k > 1$, *then* $U_n = 2kx^2$ *has no solutions.*

Proof. Assume that $U_n = 2kx^2$. Then $n = 6m$ for some positive integer m by Lemma 2.2 and (15). Thus, we get $U_n = U_{6m} = U_{3m}V_{3m} = 2kx^2$, which implies that $(U_{3m}/k)(V_{3m}/2) = x^2$ or $(V_{3m}/2k)U_{3m} = x^2$. Since $(U_{3m}, V_{3m}) = 2$ and $4 \nmid V_{3m}$ by (7), we get either

$$U_{3m} = ku^2 \text{ and } V_{3m} = 2v^2 \tag{32}$$

or

$$V_{3m} = 2ku^2 \text{ and } U_{3m} = v^2 \tag{33}$$

for some integers u and v. But (32) is impossible by Theorems 3.6 and 3.8. Moreover (33) is impossible by Theorems 3.4 and 3.7. □

Corollary 3.11. *The equations* $4P^2x^4 - 2P^2x^2y + y^2 = 1$ *and* $x^2 - 4P^2(P^2 - 4)y^4 = 4$ *have no integer solutions.*

Theorem 3.12. *Let* $k \mid P$ *with* $k > 1$. *If* $3 \nmid P$, *then* $U_n = 3kx^2$ *has no solutions.*

Proof. Assume that $U_n = 3kx^2$. Since $3 \nmid P$, we get $n = 3m$ for some even integer m by (22) and Lemma 2.2. Thus, $3kx^2 = U_{3m} = U_m(V_{2m} + 1)$ by (13), which implies that $(U_m/k)(V_{2m} + 1) = 3x^2$. Since $(U_m/k, V_{2m} + 1) = 1$ or 3 by (13), it follows that $V_{2m} + 1 = wu^2$ for some $w \in \{1, 3\}$. Since m is even, $m = 2^r a$ with a odd and $r \geq 1$. Thus $wu^2 = V_{2m}+1 \equiv 1-V_0 \pmod{V_{2^r}}$ by (5), which implies that $wu^2 \equiv -1 \pmod{V_{2^r}}$. This is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (9) and $\left(\frac{3}{V_{2^r}}\right) = 1$ by (23). □

Corollary 3.13. *The equations* $9P^2x^4 - 3P^2x^2y + y^2 = 1$ *and* $x^2 - 9P^2(P^2 - 4)y^4 = 4$ *have no integer solutions.*

Theorem 3.14. *If* $k \mid P$ *with* $k > 1$*, then the equation* $V_n = kx^2 + 1$ *has no solutions.*

Proof. Assume that $V_n = kx^2 + 1$ for some integer x. Then by Lemma 2.1, n is even. Let $n = 2m$. If m is even, then we get $n = 4u$, which implies that $V_n \equiv 2 \pmod{U_2}$ by (3). Therefore $kx^2 + 1 \equiv 2 \pmod{P}$, which is impossible since $k \mid P$ and $k > 1$. Thus, m is odd. Since $V_{2m} = V_m^2 - 2$ by (12), we get $kx^2 + 1 = V_m^2 - 2$, which implies that $k \mid 3$ by Lemma 2.1 and therefore $k = 3$. Since $k = 3$, we see that $3 \mid P$. Assume that $m > 1$. Then $m = 4q \mp 1 = 2^r a \mp 1$ with a odd and $r \geq 2$. So, $3x^2 + 1 = V_n = V_{2m} \equiv -V_{\mp 2} \pmod{V_{2^r}}$ by (5). This shows that $3x^2 \equiv -(P^2 - 1) \pmod{V_{2^r}}$. That is, $3x^2 \equiv -U_3 \pmod{V_{2^r}}$. Thus, by using Lemma 2.3, (9), and (24), we get

$$1 = \left(\frac{3}{V_{2^r}}\right) = \left(\frac{-U_3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{U_3}{V_{2^r}}\right) = -1, \tag{34}$$

which is impossible. Therefore $m = 1$ and thus, $n = 2$. So, $3x^2 + 1 = V_2 = P^2 - 2$, which is impossible. □

Corollary 3.15. *The equation* $(Px^2 + 1)^2 - (P^2 - 4)y^2 = 4$ *has no solutions.*

Theorem 3.16. *If* $k \mid P$ *with* $k > 1$*, then the equation* $U_n = kx^2 + 1$ *has only the solution* $n = 1$*.*

Proof. Assume that $U_n = kx^2 + 1$ for some integer x. Then by Lemma 2.2, n is odd. Let $n = 2m + 1$. Assume that $m > 0$. By using (10), we get $kx^2 = U_{2m+1} - 1 = U_mV_{m+1}$. Assume that m is odd. Then by (18), $(U_m, V_{m+1}) = 1$. Thus $U_m = k_1u^2$ and $V_{m+1} = k_2v^2$ with $k_1k_2 = k$. This is impossible by Theorems 3.2 and 3.8, since $k_1 > 1$ or $k_2 > 1$. Now assume that m is even. Then $(U_m, V_{m+1}) = P$ by (18). Thus it follows that $U_m = k_1Pu^2$ and $V_{m+1} = k_2Pv^2$ with $k_1k_2 = k$. This is impossible by Theorem 3.2. Therefore $m = 0$ and thus $n = 1$. □

Corollary 3.17. *The equations* $(Px^2 + 1)^2 - P(Px^2 + 1)y + y^2 = 1$ *and* $x^2 - (P^2 - 4)(Py^2 + 1)^2 = 4$ *have no positive integer solutions.*

By using the fact that $U_{2m+1} + 1 = U_{m+1}V_m$, we can give the following theorem easily.

Theorem 3.18. *If* $k \mid P$ *with* $k > 1$, *then the equation* $U_n = kx^2 - 1$ *has only the solution* $n = 3$.

If $P = 3$, then $V_4 = 47 = 3x^2 - 1$ has solution $x = 4$. Now we give the following theorem.

Theorem 3.19. *If* $k \mid P$ *with* $k > 1$, *then the equation* $V_n = kx^2 - 1$ *has only the solution* $(n, P, k, x) = (4, 3, 3, 4)$.

Proof. Assume that $V_n = kx^2 - 1$ for some integer x. Then by Lemma 2.1, n is even. Let $n = 2m$. Then $kx^2 - 1 = V_n = V_{2m} = V_m^2 - 2$ by (12). Assume that m is odd. Then $P \mid V_m$ by Lemma 2.1, which implies that $k \mid 1$, a contradiction. Therefore m is even and thus $n = 4u$, which shows that $kx^2 - 1 = V_n \equiv 2 \pmod{U_2}$ by (3). This shows that $k = 3$ and therefore $3 \mid P$. In this case we have $n = 4u = 2(2^r a)$ for some odd integer a with $r \geq 1$ and thus, $3x^2 - 1 \equiv -2 \pmod{V_{2^r}}$ by (5). This shows that $3x^2 \equiv -1 \pmod{V_{2^r}}$. Therefore

$$\left(\frac{3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right) = -1,$$

which is impossible if $r \geq 2$ by (24). Therefore $r = 1$. Then $n = 4u$ with u odd. Thus, $3x^2 - 1 \equiv V_{4u} = V_{2u}^2 - 2$, which implies that $V_{2u}^2 - 1 = 3x^2$. That is, $(V_{2u} - 1)(V_{2u} + 1) = 3x^2$. Let $(V_{2u} - 1, V_{2u} + 1) = 1$. Then we have either

$$V_{2u} - 1 = a^2 \text{ and } V_{2u} + 1 = 3b^2 \tag{35}$$

or

$$V_{2u} - 1 = 3a^2 \text{ and } V_{2u} + 1 = b^2 \tag{36}$$

for some integers a and b. It can be seen that (35) and (36) are impossible. Let $(V_{2u} - 1, V_{2u} + 1) = 2$. Then we have

$$V_{2u} - 1 = 2a^2 \text{ and } V_{2u} + 1 = 6b^2 \tag{37}$$

or

$$V_{2u} - 1 = 6a^2 \text{ and } V_{2u} + 1 = 2b^2 \tag{38}$$

for some integers a and b. It can be shown that (37) is impossible. If $u > 1$, then (38) is impossible by Lemma 3.1. Therefore $u = 1$ and so $n = 4$. Thus $V_4 = 3x^2 - 1$ and therefore $V_2^2 - 2 = 3x^2 - 1$. This implies that $(P^2 - 2)^2 - 3x^2 = 1$. Since all positive integer solutions of the equation $u^2 - 3v^2 = 1$ are given by $(u, v) = (V_n(4, -1)/2, U_n(4, -1))$ with $n \geq 1$, we get

$$P^2 - 2 = \frac{V_n(4, -1)}{2}$$

for some natural number n. Thus, $V_n(4, -1) = 2P^2 - 4$, which shows that n is even. Let $n = 2m$. Then $2P^2 - 4 = V_{2m}(4, -1) = V_m^2(4, -1) - 2$. For the time being, we use V_n instead of $V_n(4, -1)$. If m is even, then $m = 2t$ and so $(V_t^2-2)^2-2=2P^2-4$. This shows that $V_t^4-4V_t^2+6=2P^2$, which is impossible since V_t is even. Therefore m is odd. Assume that $m>1$. Then $n=2m=2(4q\mp 1)=2(2^r a\mp 1)$ with a odd and $r\geq 2$. Thus,

$$2P^2-4=V_n\equiv -V_2\equiv -14 \pmod{V_{2^r}},$$

which implies that

$$P^2\equiv -5 \pmod{V_{2^r}/2}. \tag{39}$$

A simple computation shows that $V_{2^r}/2\equiv 1 \pmod 4$ and $V_{2^r}/2\equiv 2 \pmod 5$. Thus, we obtain

$$\left(\frac{-5}{V_{2^r}/2}\right)=\left(\frac{5}{V_{2^r}/2}\right)=\left(\frac{V_{2^r}/2}{5}\right)=\left(\frac{2}{5}\right)=-1,$$

which is impossible by (39). Therefore $m=1$ and thus $n=2$. This implies that $2P^2-4=V_n(4,-1)=V_2(4,-1)=14$. Hence we get $P=3$. Since $(P^2-2)^2-3x^2=1$, it follows that $x=4$. This completes the proof. □

We can give the following corollary easily.

Corollary 3.20. *The only positive integer solutions of the equation* $(Px^2-1)^2-(P^2-4)y^2=4$ *is given by* $(P,x,y)=(3,4,21)$.

Theorem 3.21. *The equation* $V_n=3x^2-1$ *has the solutions*

$$(n,P,x)=(4,3,4),(n,P,x)=(1,3a^2-1,a)$$

with a *even or* $(n,P,x)=(2,V_{2t}(4,-1)/2,U_{2t}(4,-1))$ *with* $t\geq 1$.

Proof. If $3\mid P$, then by Theorem 3.19, we get $(n,P,k,x)=(4,3,3,4)$. Assume that $3\nmid P$. Let $n>1$ be odd. Then $n=4q\mp 1=2^{r+1}b\mp 1$ with b odd and $r\geq 1$. Thus,

$$3x^2=V_n+1\equiv -V_1+1\equiv -(P-1) \pmod{V_{2^r}}$$

by (5). By using (23), (9), and (25), it is seen that

$$1=\left(\frac{3}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P-1}{V_{2^r}}\right)=-1,$$

which is impossible. Therefore $n=1$. And so, $P=3a^2-1$ with a even. Now let n be even. Then $n=2m$ for some positive integer m. Assume that $m>1$ and m is odd. Then $n=2m=2(4q\mp 1)=2(2^r a\mp 1)$ with a odd and $r\geq 2$. Thus,

$$3x^2=1+V_n\equiv 1-V_2\equiv -(P^2-3) \pmod{V_{2^r}},$$

by (5). By using (23), (9), and (19), we get

$$1=\left(\frac{3}{V_{2^r}}\right)=\left(\frac{-1}{V_{2^r}}\right)\left(\frac{P^2-3}{V_{2^r}}\right)=-1,$$

which is a contradiction. Therefore $m=1$ and so $n=2$. Thus $3x^2-1=V_2=P^2-2$ and this implies that $P^2-3x^2=1$. Therefore $P=V_{2t}(4,-1)/2$ with

$t \geq 1$. Assume that m is even. Then $n = 4u = 2^{r+1}b$ with b odd and $r \geq 1$. This implies by (5) that

$$3x^2 = 1 + V_n \equiv 1 - V_0 \equiv -1 \pmod{V_{2^r}},$$

which is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$ and $\left(\frac{3}{V_{2^r}}\right) = 1$ by (9) and (23). □

Corollary 3.22. *The equation* $(3x^2-1)^2-(P^2-4)y^2 = 4$ *has integer solutions only when* $P = 3, P = 3a^2 - 1$ *with* a *even or* $P = V_{2t}(4,-1)/2$ *with* $t \geq 1$.

Theorem 3.23. *Let* $k \mid P$ *with* $k > 1$. *Then the equation* $V_n = 2kx^2 - 1$ *has no solutions.*

Proof. Assume that $V_n = 2kx^2-1$ for some integer x. Then by Lemma 2.1, n is even. Let $n = 2m$. Then $2kx^2-1 = V_n = V_{2m} = V_m^2-2$ by (12). Assume that m is odd. Then $P \mid V_m$ by Lemma 2.1, which implies that $k \mid 1$, a contradiction. Therefore m is even and so $n = 4u$, which shows that $2kx^2 - 1 = V_n \equiv 2 \pmod{U_2}$ by (3). This implies that $k = 3$ and therefore $3 \mid P$. In this case, we have $n = 4u = 2^{r+1}a$ with a odd and $r \geq 1$. Thus $6x^2 - 1 = V_n \equiv -2 \pmod{V_{2^r}}$. This shows that $6x^2 \equiv -1 \pmod{V_{2^r}}$. Therefore

$$\left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right) = -1,$$

which is impossible if $r \geq 2$ by (8) and (24). Therefore $r = 1$. Then $n = 4u$ with u odd. Thus, $6x^2 - 1 = V_{4u} = V_{2u}^2 - 2$, which implies that $V_{2u}^2 - 1 = 6x^2$. That is, $(V_{2u} - 1)(V_{2u} + 1) = 6x^2$. Since $(V_{2u} - 1, V_{2u} + 1) = 1$ or 2, we have one of the following cases:

$$V_{2u} - 1 = 2a^2 \text{ and } V_{2u} + 1 = 3b^2, \tag{40}$$

$$V_{2u} - 1 = a^2 \text{ and } V_{2u} + 1 = 6b^2, \tag{41}$$

$$V_{2u} - 1 = 6a^2 \text{ and } V_{2u} + 1 = b^2, \tag{42}$$

$$V_{2u} - 1 = 3a^2 \text{ and } V_{2u} + 1 = 2b^2. \tag{43}$$

A simple argument shows that (40), (41), and (42) are impossible. Assume that (43) is satisfied. Then $3a^2 = V_{2u} - 1 = V_u^2 - 3$ and so $a^2 + 1 = 3(V_{2u}/3)^2$, which is impossible. □

Corollary 3.24. *The equation* $(2Px^2 - 1)^2 - (P^2 - 4)y^2 = 4$ *has no integer solutions.*

Theorem 3.25. *The equation* $V_n = 6x^2 - 1$ *has the solutions*

$$(n, P, x) = (2, V_t(10, -1)/2, U_t(10, -1))$$

with $t \geq 1$ *or* $(n, P, x) = (1, 6a^2 - 1, a)$ *with* a *integer.*

Proof. If $3 \mid P$, then the proof follows from Theorem 3.23. Assume that $3 \nmid P$. Let $n > 1$ be odd. Then $n = 4q \mp 1 = 2^{r+1}b \mp 1$ with b odd and $r \geq 1$. Thus we get

$$6x^2 = V_n + 1 \equiv -V_1 + 1 \equiv -(P-1) \pmod{V_{2^r}}$$

by (5). By using (8), (23), (9), and (25), it is seen that

$$1 = \left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{P-1}{V_{2^r}}\right) = -1,$$

which is impossible. Therefore $n = 1$. And so, $P = 6a^2 - 1$ with a integer. Let n be even, i.e., $n = 2m$. Assume that $m > 1$ is odd. Then $m = 4q \mp 1 = 2^r b \mp 1$ with b odd and $r \geq 2$ and thus,

$$6x^2 = V_n + 1 \equiv -V_2 + 1 \equiv -(P^2 - 3) \pmod{V_{2^r}}$$

by (5). Similarly, by using (8), (9), (23), and (19), we get a contradiction. Therefore $m = 1$ and so $n = 2$. If m is even, we get $n = 2^{r+1}b$ with b odd and $r \geq 1$ and thus

$$6x^2 = V_n + 1 \equiv -V_0 + 1 \equiv -1 \pmod{V_{2^r}},$$

which is a contradiction by (8), (9), and (23). Therefore $n = 2$, which implies that $P^2 - 6x^2 = 1$. Therefore $P = V_t(10, -1)/2$ and $x = U_t(10, -1)$ with $t \geq 1$. Therefore $P = V_t(10, -1)/2$ is a solution. □

Corollary 3.26. *The equation* $(6x^2 - 1)^2 - (P^2 - 4)y^2 = 4$ *has integer solutions only when* $P = 6a^2 - 1$ *with* a *integer or* $P = V_m(10, -1)/2$ *with* $m \geq 1$.

Theorem 3.27. *Let* $k \mid P$ *with* $k > 1$. *Then the equation* $V_n = 2kx^2 + 1$ *has the solutions* $(n, P, x) = (2, 3(U_m(10, -1) - U_{m-1}(10, -1)), U_m(10, -1) + U_{m-1}(10, -1))$ *with* $m \geq 1$.

Proof. Assume that $V_n = 2kx^2 + 1$. Then n is even by Lemma 2.1. Let $n = 2m$. If m is even, then we get $2kx^2 + 1 = V_n \equiv 2 \pmod{U_2}$, which is impossible. Therefore m is odd. Since $2kx^2 + 1 = V_m^2 - 2$, it follows that $k \mid 3$ and therefore $k = 3$. Assume that $m > 1$. Then $m = 4q \mp 1 = 2^r a \mp 1$ with a odd and $r \geq 2$. Thus, we get

$$6x^2 = V_n - 1 = V_{2m} - 1 \equiv -V_2 - 1 \equiv -(P^2 - 1) \pmod{V_{2^r}},$$

which is impossible by (8), (9), (23), and Lemma 2.3. Then $m = 1$ and therefore $n = 2$. Thus $6x^2 + 1 = V_2 = P^2 - 2$. That is, $P^2 - 6x^2 = 3$. Let $P = 3a$. Then $3a^2 - 2x^2 = 1$. By Lemma 2.4, we get

$$(a, x) = (U_m(10, -1) - U_{m-1}(10, -1)), U_m(10, -1) + U_{m-1}(10, -1))$$

and therefore

$$(n, P, x) = (2, 3(U_m(10, -1) - U_{m-1}(10, -1)), U_m(10, -1) + U_{m-1}(10, -1))$$

with $m \geq 1$. □

Corollary 3.28. *The equation* $(2Px^2+1)^2-(P^2-4)y^2=4$ *has positive integer solutions only when* $P=3(U_m(10,-1)-U_{m-1}(10,-1))$ *with* $m\geq 1$.

We can give the following theorems easily.

Theorem 3.29. *The equation* $V_n=3x^2+1$ *has no solutions.*

Theorem 3.30. *The equation* $V_n=6x^2+1$ *has the solutions*

$$(n,P,x)=(2,3(U_m(10,-1)-U_{m-1}(10,-1)),U_m(10,-1)+U_{m-1}(10,-1))$$

with $m\geq 1$.

Theorem 3.31. *The equation* $V_n=2x^2-1$ *has the solutions*

$$(n,P,x)=(2,V_m(6,-1)/2,2U_m(6,-1))$$

with $m\geq 1$ *or* $(n,P,x)=(1,2a^2-1,a)$ *with a integer.*

Theorem 3.32. *The equation* $V_n=2x^2+1$ *has the solution* $(n,P,x)=(1,2a^2+1,a)$ *with a integer.*

Theorem 3.33. *The equation* $V_n=x^2+1$ *has the solution* $(n,P,x)=(1,4a^2+1,2a)$ *with a integer.*

Theorem 3.34. *The equation* $V_n=x^2-1$ *has the solution* $(n,P,x)=(1,4a^2-1,2a)$ *with a integer.*

Corollary 3.35. *The equation* $(3x^2+1)^2-(P^2-4)y^2=4$ *has no integer solutions.*

Corollary 3.36. *The equation* $(6x^2+1)^2-(P^2-4)y^2=4$ *has positive integer solutions only when* $P=3(U_m(10,-1)-U_{m-1}(10,-1))$ *with* $m\geq 1$.

Corollary 3.37. *The equation* $(2x^2-1)^2-(P^2-4)y^2=4$ *has positive integer solutions only when* $P=2a^2-1$ *with a integer or* $P=V_m(6,-1)/2$ *with* $m\geq 1$.

Corollary 3.38. *The equation* $(2x^2+1)^2-(P^2-4)y^2=4$ *has positive integer solutions only when* $P=2a^2+1$, *in which case the only solution is* $(x,y)=(a,1)$ *with a integer.*

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