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A NOTE ON CUBICALLY HYPONORMAL WEIGHTED SHIFTS

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Abstract. In this paper, we show that any cubically hyponormal weighted shift with first two equal weights is flat. And we give an example of a weighted shift which is not cubically hyponormal but almost-cubically hyponormal.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator T is *polynomially hyponormal* if $p(T)$ is hyponormal for all (complex) polynomials p (cf. [6]). And an operator T in $\mathcal{L}(\mathcal{H})$ is *weakly n-hyponormal* if $p(T)$ is hyponormal for any polynomial p with degree n or less (cf. [8]). In particular, the weak 2-hyponormality (or weak 3-hyponormality) referred to as quadratic hyponormality (or cubic hyponormality, resp.) has been considered in detail in [4], [5] and [8]. It is well known that "subnormal $\Rightarrow$ polynomially hyponormal $\Rightarrow \cdots \Rightarrow$ weakly 3-hyponormal $\Rightarrow$ weakly 2-hyponormal $\Rightarrow$ hyponormal". However, one does not know about converse implications for $n \geq 3$ yet; see [2], [5], [8] for weak 2- and weak 3-hyponormalities.

A unilateral weighted shift is often used to study the bridges between subnormality and hyponormality. In [9] Stampfli proved that every subnormal weighted shift W_α with any two equal weights has flatness, i.e., if $\alpha_k = \alpha_{k+1}$ for some $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $\alpha_1 = \alpha_2 = \cdots$. In [2], R. Curto proved that the 2-hyponormal weighted shift W_α with any two equal weights has flatness. Also he obtained a weighted shift W_α with first two equal weights without being flat. Moreover, he raised a question: describe all quadratically hyponormal weighted shifts with first two equal weights, which was studied as several kinds of detections. In [1], Y. Choi proved that every polynomially hyponormal weighted

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shift W_α with any two equal weights has flatness. In particular, he showed that if W_α is a weighted shift with first two equal weights and $W_\alpha + sW_\alpha^n$ is hyponormal for all $s \in \mathbb{C}$ and all $n \in \mathbb{N}$, then W_α is flat. In this paper, we show that any cubically hyponormal weighted shift W_α with first two equal weights is flat (see Theorem 3.3 below). Also, we give an example of a weighted shift which is not cubically hyponormal but almost-cubically hyponormal (see Definition 2.1 below).

Some of the calculations in Section 3 were obtained through computer experiments using the software tool *Mathematica* [10].

2. Constructions of formulas

For $A, B \in \mathcal{L}(\mathcal{H})$, we write $[A, B] := AB - BA$. Let $\alpha := \{\alpha_i\}_{i=0}^\infty$ be a weight sequence in the positive real number set $\mathbb{R}_+$. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with an orthonormal basis $\{e_i\}_{i=0}^\infty$, is defined by $W_\alpha e_i = \alpha_i e_{i+1}$ for all $i \in \mathbb{N}_0$. Recall that a weighted shift W_α is *cubically hyponormal* if $W_\alpha + aW_\alpha^2 + bW_\alpha^3$ is hyponormal for any $a, b \in \mathbb{C}$ ([8]), i.e.,

$$D(a,b) := [(W_\alpha + aW_\alpha^2 + bW_\alpha^3)^*, W_\alpha + aW_\alpha^2 + bW_\alpha^3] \geq 0 \quad \text{for any } a, b \in \mathbb{C}.$$

Let P_n be the orthogonal projection onto the subspace $\vee_{i=0}^n \{e_i\}$. For $a, b \in \mathbb{C}$, we let

$$D_n(a,b) = P_n[(W_\alpha + aW_\alpha^2 + bW_\alpha^3)^*, W_\alpha + aW_\alpha^2 + bW_\alpha^3]P_n$$

$$= \begin{bmatrix} q_0 & r_0 & z_0 & 0 & & & & \\ \overline{r_0} & q_1 & r_1 & z_1 & 0 & & & \\ \overline{z_0} & \overline{r_1} & q_2 & r_2 & z_2 & 0 & & \\ 0 & \overline{z_1} & \overline{r_2} & q_3 & r_3 & z_3 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & z_{n-2} \\ & & & & \ddots & \ddots & \ddots & \ddots & r_{n-1} \\ & & & & & 0 & \overline{z_{n-2}} & \overline{r_{n-1}} & q_n \end{bmatrix},$$

where

$$\begin{aligned} q_n &:= (\alpha_n^2 - \alpha_{n-1}^2) + (\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-2}^2\alpha_{n-1}^2)\,|a|^2 \\ &\quad + (\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2)\,|b|^2, \\ r_n &:= \alpha_n(\alpha_{n+1}^2 - \alpha_{n-1}^2)\bar{a} + \alpha_n(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-1}^2\alpha_{n-2}^2)a\bar{b}, \\ z_n &:= \alpha_n\alpha_{n+1}(\alpha_{n+2}^2 - \alpha_{n-1}^2)\bar{b}. \end{aligned}$$

Then it is obvious that W_α is cubically hyponormal if and only if the pentadiagonal matrix $D_n(a,b) \geq 0$ for any $a, b \in \mathbb{C}$ and any $n \in \mathbb{N}$.

Definition 2.1. (i) A weighted shift W_α is *semi-cubically hyponormal with type I* if $W_\alpha + sW_\alpha^3$ is hyponormal for any $s \in \mathbb{C}$.

(ii) A weighted shift W_α is *semi-cubically hyponormal with type II* if $W_\alpha^2 + sW_\alpha^3$ is hyponormal for any $s \in \mathbb{C}$.

(iii) A weighted shift W_α is *almost-cubically hyponormal* if it is semi-cubically hyponormal with types I and II.

We first construct formulas on type I (see [7]). Let n be a fixed positive integer. For $s \in \mathbb{C}$, we let

$$D_m^{[n]}(s) = P_m\left[(W_\alpha + sW_\alpha^n)^*, W_\alpha + sW_\alpha^n\right]P_m$$

$$= \begin{bmatrix} q_{n,0} & 0 & \cdots & 0 & z_{n,0} & 0 & & \\ 0 & q_{n,1} & \ddots & \ddots & 0 & z_{n,1} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & q_{n,n-2} & 0 & \ddots & \ddots & z_{n,m-n+1} \\ \bar{z}_{n,0} & 0 & \ddots & 0 & q_{n,n-1} & 0 & \ddots & 0 \\ 0 & \bar{z}_{n,1} & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & 0 & \ddots & \ddots & q_{n,m-1} & 0 \\ & & 0 & \bar{z}_{n,m-n+1} & 0 & \cdots & 0 & q_{n,m} \end{bmatrix},$$

where

$$\begin{aligned} q_{n,k} &= u_{n,k} + v_{n,k}\,|s|^2, \\ z_{n,k} &= \sqrt{w_{n,k}}\,\bar{s}, \\ u_{n,k} &= \alpha_k^2 - \alpha_{k-1}^2, \\ v_{n,k} &= \alpha_k^2\alpha_{k+1}^2\cdots\alpha_{k+n-1}^2 - \alpha_{k-n}^2\alpha_{k-2}^2\cdots\alpha_{k-1}^2, \\ w_{n,k} &= \alpha_k^2\alpha_{k+1}^2\cdots\alpha_{k+n-2}^2(\alpha_{k+n-1}^2 - \alpha_{k-1}^2)^2 \end{aligned}$$

with $\alpha_{-1} = \alpha_{-2} = \cdots = \alpha_{-n} = 0$ (for our convenience). Then it is obvious that W_α is semi-weakly n-hyponormal if and only if $D_m^{[n]}(s) \geq 0$ for all $s \in \mathbb{C}$, $m \in \mathbb{N}$. To detect $D_m^{[n]}(s) \geq 0$ for all $s \in \mathbb{C}$, we usually use the Nested Determinant Test ([3]). Now we consider $d_j^{[n]} := d_j^{[n]}(s) = \det D_j^{[n]}(s)$ for $1 \leq j \leq m$.

By changing the standard basis of $\mathbb{C}^{n+1}$, we have the following lemma.

Lemma 2.2 ([7, Lemma 2.1])**.** *Let $n \geq 2$ be a fixed positive integer. For $m \in \mathbb{N}$, if two non-negative integers k and j satisfy $m = (n-1)k + j$, i.e., $m \equiv j \pmod{n-1}$, then $D_m^{[n]}(s)$ is unitarily equivalent to* $[\oplus_{i=0}^{j} D_{k,1}^{[n]}(i,j)] \oplus$

$[\oplus_{i=j+1}^{n-2} D_{k,2}^{[n]}(i,j)]$, *where*
(2.1)

$$D_{k,1}^{[n]}(i,j) = \begin{bmatrix} q_{n,j-i} & z_{n,j-i} & 0 & & \\ \bar{z}_{n,j-i} & q_{n,(n-1)+j-i} & \ddots & \ddots & \\ 0 & \bar{z}_{n,(n-1)+j-i} & \ddots & \ddots & 0 \\ & \ddots & \ddots & q_{n,(k-1)(n-1)+j-i} & z_{n,(k-1)(n-1)+j-i} \\ & & 0 & \bar{z}_{n,(k-1)(n-1)+j-i} & q_{n,k(n-1)+j-i} \end{bmatrix}$$

for $0 \le i \le j$, *and*
(2.2)

$$D_{k,2}^{[n]}(i,j) = \begin{bmatrix} q_{n,(n-1)+j-i} & z_{n,(n-1)+j-i} & 0 & & \\ \bar{z}_{n,(n-1)+j-i} & q_{n,2(n-1)+j-i} & \ddots & & \\ 0 & \bar{z}_{n,2(n-1)+j-i} & \ddots & \ddots & 0 \\ & \ddots & \ddots & q_{n,(k-1)(n-1)+j-i} & z_{n,(k-1)(n-1)+j-i} \\ & & 0 & \bar{z}_{n,(k-1)(n-1)+j-i} & q_{n,k(n-1)+j-i} \end{bmatrix}$$

for $j+1 \le i \le n-2$. *Therefore, we have that*

$$\det D_m^{[n]}(s) = \prod_{i=0}^{j} \det D_{k,1}^{[n]}(i,j) \cdot \prod_{i=j+1}^{n-2} \det D_{k,2}^{[n]}(i,j). \tag{2.3}$$

Moreover, a weighted shift W_α *is semi-weakly* n*-hyponormal if and only if* $D_{k,1}^{[n]}(i,j) \ge 0$ $(0 \le i \le j)$ *and* $D_{k,2}^{[n]}(i,j) \ge 0$ $(j+1 \le i \le n-2)$.

As a special example of Lemma 2.2, we consider here the case of $n = 3$. This case is $j = 0$ or 1 obviously. Lemma 2.2 implies that the matrix $D_m^{[3]}(s)$ induces the following two cases:

$$\det D_m^{[3]}(s) = \begin{cases} \det D_{2k}^{[3]}(s) = \det D_{k,1}^{[n]}(0,0) \cdot \det D_{k,2}^{[n]}(1,0) & \text{if } j = 0, \\ \det D_{2k+1}^{[3]}(s) = \det D_{k,1}^{[n]}(0,1) \cdot \det D_{k,1}^{[n]}(1,1) & \text{if } j = 1. \end{cases}$$

Since

$$D_{k,1}^{[n]}(0,0) = D_{k,1}^{[n]}(1,1) = \begin{bmatrix} q_{3,0} & z_{3,0} & 0 & \\ \bar{z}_{3,0} & q_{3,2} & z_{3,2} & \\ 0 & \bar{z}_{3,2} & \ddots & \ddots \\ & & \ddots & q_{3,2k} \end{bmatrix}$$

and $D_{k,2}^{[n]}(1,0)$ is a submatrix of $D_{k,1}^{[n]}(0,1)$, if the following two matrices

$$D_{k,1}^{[3]}(0,0) = \begin{bmatrix} q_{3,0} & z_{3,0} & 0 & \\ \bar{z}_{3,0} & q_{3,2} & z_{3,2} & \\ 0 & \bar{z}_{3,2} & \ddots & \ddots \\ & & \ddots & q_{3,2k} \end{bmatrix}, \ D_{k,1}^{[3]}(0,1) = \begin{bmatrix} q_{3,1} & z_{3,1} & 0 & \\ \bar{z}_{3,1} & q_{3,3} & z_{3,3} & \\ 0 & \bar{z}_{3,3} & \ddots & \\ & & & q_{3,2k+1} \end{bmatrix}$$

are all positive, then W_α is semi-cubically hyponormal with type I.

We construct formulas on type II. For $s \in \mathbb{C}$, we have

$$M_n(s) := P_n \left[\left(W_\alpha^2 + sW_\alpha^3\right)^*, \left(W_\alpha^2 + sW_\alpha^3\right) \right] P_n$$

$$= \begin{bmatrix} \omega_0 & \bar{\phi}_0 & 0 & & & & \\ \phi_0 & \omega_1 & \bar{\phi}_1 & 0 & & & \\ 0 & \phi_1 & \omega_2 & \bar{\phi}_2 & 0 & & \\ & 0 & \phi_2 & \omega_3 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \ddots & \bar{\phi}_{n-1} \\ & & & & 0 & \phi_{n-1} & \omega_n \end{bmatrix},$$

where

$\omega_k := \xi_k + |s|^2 \eta_k, \quad \phi_k := s\sqrt{\delta_k}, \quad \xi_k := \alpha_k^2 \alpha_{k+1}^2 - \alpha_{k-2}^2 \alpha_{k-1}^2,$

$\eta_k := \alpha_k^2 \alpha_{k+1}^2 \alpha_{k+2}^2 - \alpha_{k-3}^2 \alpha_{k-2}^2 \alpha_{k-1}^2, \ \delta_k := \alpha_k^2 (\alpha_{k+1}^2 \alpha_{k+2}^2 - \alpha_{k-2}^2 \alpha_{k-1}^2)^2 \ (k \geq 0)$

and $\alpha_{-1} = \alpha_{-2} = \alpha_{-3} := 0$. Put $\sigma_n := \sigma_n(\cdot) = \det(M_n(\cdot))$. Then

$$\sigma_0 = \omega_0, \ \sigma_1 = \omega_0\omega_1 - |\phi_0|^2, \ \sigma_{n+2} = \omega_{n+2}\sigma_{n+1} - |\phi_{n+1}|^2 \sigma_n \ \ (n \geq 0)$$

and σ_n is actually a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin's expansion $\sigma_n(t) := \sum_{i=0}^{n+1} c(n,i)\, t^i$. It follows from the similar method in [2] that

(i) $c(0,0) = \xi_0, \quad c(0,1) = \eta_0$;
(ii) $c(1,0) = \xi_1\xi_0, \quad c(1,1) = \xi_1\eta_0 + \xi_0\eta_1, \quad c(1,2) = \eta_1\eta_0$;
(iii) $c(n+2,i) = \xi_{n+2}c(n+1,i) + \eta_{n+2}c(n+1,i-1) - \delta_{n+1}c(n,i-1)$ $(n \geq 0, \ \ 0 \leq i \leq n+3)$;
(iv) $c(n,n+1) = \eta_0\eta_1\cdots\eta_n > 0$ and $c(n,0) = \xi_0\xi_1\cdots\xi_n > 0$ for all $n \in \mathbb{N}_0$.

3. Main results

It is well known that if a weighted shift W_α is quadratically hyponormal with first two equal weights, it is not necessarily flat ([2]). For example, if $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \ldots$, then the associated weighted shift W_α is quadratically hyponormal. And the author in [1] showed the following:

Proposition 3.1 ([1, Theorem 1]). *Let W_α be a quadratically hyponormal weighted shift with $\alpha = \{\alpha_n\}_{n=0}^\infty$. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then W_α is flat, i.e., $\alpha_1 = \alpha_2 = \cdots$.*

First, we give the following lemma.

Lemma 3.2. *Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ be any cubic real polynomial. Then $f(x) \geq 0$ for all $x \geq 0$ if and only if it holds one of the following cases*:
(1) $a_0 > 0, \ a_1 > 0, \ a_2 \geq 0, \ a_3 \geq 0$;
(2) $a_0 > 0, \ a_3 \geq 0$ *and* $a_1^2 - 3a_0a_2 \leq 0$;

(3) $a_0 > 0,\ a_3 \geq 0,\ a_2 < 0,\ 27a_0^2a_3 + 2a_1^3 - 9a_0a_1a_2 \geq 0$ *and*

$$4a_0a_2^3 + 4a_1^3a_3 + 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 \geq 0;$$

(4) $a_0 > 0,\ a_3 \geq 0,\ a_1 \leq 0,\ a_2 \geq 0,\ a_1^2 - 3a_0a_2 > 0,\ 27a_0^2a_3 + 2a_1^3 - 9a_0a_1a_2 \geq 0$ *and*

$$4a_0a_2^3 + 4a_1^3a_3 + 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 \geq 0.$$

Proof. It is easy to see that one of the necessary conditions is $a_0 > 0,\ a_3 \geq 0$. From a calculation, we have $f'(x) = 3a_0x^2 + 2a_1x + a_2$. If $a_1^2 - 3a_0a_2 \leq 0$, using $a_0 > 0$, then $f'(x) \geq 0$ for all $x \geq 0$, which implies that $f(x) \geq 0$ for all $x \geq 0$. This is the case of (2).

If $a_0 > 0, a_3 \geq 0$ and $a_1 > 0, a_2 \geq 0$, then it is the case of (1).

Now we suppose that $a_1^2 - 3a_0a_2 > 0,\ a_0 > 0$ and $a_3 \geq 0$. Denote $x_2 := \frac{-a_1 + \sqrt{a_1^2 - 3a_0a_2}}{3a_0}$ be the largest root of the equation $f'(x) = 0$. We consider two cases of $x_2 < 0$ and $x_2 \geq 0$. For the case of $x_2 < 0$, it follows from $a_0 > 0$ that $f'(x) \geq 0$ for all $x \geq 0$. Using the condition $a_3 \geq 0$, we have $f(x) \geq 0$ $(x \geq 0)$. If $x_2 \geq 0$, i.e., $a_2 < 0$ or $a_1 \leq 0, a_2 \geq 0$, which is the case (3) or (4) resp., then from Fermat's theorem, $f(x_2)$ is the local minimum of $f(x)$ for $x \geq 0$. From simple computation, we have

$$f(x_2) = \frac{1}{27} \frac{27a_0^2a_3 + 2a_1^3 - 9a_0a_1a_2 - 2\left(a_1^2 - 3a_0a_2\right)^{3/2}}{a_0^2}.$$

Hence $f(x_2) \geq 0$ is equivalent to $27a_0^2a_3 + 2a_1^3 - 9a_0a_1a_2 \geq 0$, and

$$2\left(a_1^2 - 3a_0a_2\right)^{3/2} \leq 27a_0^2a_3 + 2a_1^3 - 9a_0a_1a_2,$$

that is,

$$4a_0a_2^3 + 4a_1^3a_3 + 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 \geq 0.$$

Thus we have the lemma. □

The following result shows that the cubically hyponormal weighted shift with first two equal weights has flatness. Hence, the nonsubnormal but cubically hyponormal weighted shift operator should have strictly increasing weights.

Theorem 3.3. *If a weighted shift W_α is cubically hyponormal with $\alpha_0 = \alpha_1$, then W_α is flat.*

Proof. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is nondecreasing, and without loss of generality, let $\alpha_0 = \alpha_1 = 1$. We claim that $\alpha_2 = 1$.

Suppose that $\alpha_2 > 1$. Since W_α is cubically hyponormal, we must have

$$\begin{aligned}
&f(a,b) := \det D_2(a,b) \\
&= \begin{vmatrix} 1 + a^2 + \alpha_2^2 b^2 & a + \alpha_2^2 ab & \alpha_2^2 b \\ a + \alpha_2^2 ab & \alpha_2^2 a^2 + \alpha_2^2\alpha_3^2 b^2 & a\left(\alpha_2^2 - 1\right) + ab\alpha_2^2\alpha_3^2 \\ \alpha_2^2 b & a(\alpha_2^2 - 1) + ab\alpha_2^2\alpha_3^2 & \alpha_2^2 - 1 + a^2(\alpha_2^2\alpha_3^2 - 1) + b^2\alpha_2^2\alpha_3^2\alpha_4^2 \end{vmatrix} \\
&= a_0a^6 + a_1a^4 + a_2a^2 + a_3 \geq 0,
\end{aligned}$$

for any $a, b \in \mathbb{R}$, where

$a_0 = \alpha_2^2 \left(\alpha_2^2\alpha_3^2 - 1\right)$,

$a_1 = \alpha_2^2\alpha_3^2 \left(\alpha_2^2\alpha_4^2 - 1\right) b^2 - 2\alpha_2^2 \left(-\alpha_3^2 + 2\alpha_2^2\alpha_3^2 - 1\right) b + \alpha_2^2\alpha_3^2 \left(\alpha_2^2 - 1\right)$,

$a_2 = \alpha_2^4\alpha_3^2 \left(\alpha_3^2\alpha_4^2 - 1\right) b^4 - 2\alpha_2^4\alpha_3^2 \left(\alpha_4^2 - 1\right) b^3 + \alpha_2^2 \left(-2\alpha_3^2 + 3\alpha_2^2\alpha_3^2 - \alpha_3^2\alpha_4^2 + \alpha_2^2\alpha_3^2\alpha_4^2 - 1\right) b^2 - 2\alpha_2^2\alpha_3^2 \left(\alpha_2^2 - 1\right) b$,

$a_3 = \left(\alpha_2^6\alpha_3^4\alpha_4^2\right) b^6 + \alpha_2^4\alpha_3^2 \left(\alpha_3^2\alpha_4^2 - 1\right) b^4 + \alpha_2^2\alpha_3^2 \left(\alpha_2^2 - 1\right) b^2$.

Since $a_0 > 0,\ a_3 \geq 0$, by Lemma 3.2, if $f(a, b) = a_0a^6 + a_1a^4 + a_2a^2 + a_3 \geq 0$, for any $a, b \in \mathbb{R}$, then either one of the followings holds:

(i) $a_2 \geq 0$. But $a_2 < 0$, if b is positive infinitesimal, so it is a contradiction.

(ii) Since

$$\Delta_1 := a_1^2 - 3a_0a_2 = (\cdot)\, b^4 + (\cdot)\, b^3 + (\cdot)\, b^2 + (\cdot)\, b + \alpha_2^4\alpha_3^4 \left(\alpha_2^2 - 1\right)^2,$$

thus $\Delta_1 \leq 0$ implies that $\alpha_2^4\alpha_3^4 \left(\alpha_2^2 - 1\right)^2 \leq 0$ by taking $b = 0$, which induces a contradiction.

(iii) Since

$$\begin{aligned}\Delta_2 &:= 4a_0a_2^3 + 4a_1^3a_3 + 27a_0^2a_3^2 - a_1^2a_2^2 - 18a_0a_1a_2a_3 \\ &= (\cdot)\, b^{12} + (\cdot)\, b^{11} + (\cdot)\, b^{10} + \cdots + (\cdot)\, b^4 + 4\alpha_2^8\alpha_3^8\alpha_4^2 \left(\alpha_2 - 1\right)^4 \left(\alpha_2 + 1\right)^4 b^3,\end{aligned}$$

if b is negative infinitesimal, then $\Delta_2 < 0$. It is also a contradiction. Hence we must have $\alpha_2 = 1$. Since cubic hyponormality implies quadratic hyponormality, by Proposition 3.1, we know that W_α is flat. □

By Proposition 3.1 and Theorem 3.3, we have the following results.

Corollary 3.4. *Let W_α be a cubically hyponormal weighted shift with $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then W_α is flat.*

Corollary 3.5. *Let W_α be a weighted shift with a weight sequence α, where*

$$\alpha : \alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \text{and} \quad \alpha_k = \sqrt{\frac{k+1}{k+2}}, \quad k = 2, 3, \ldots. \tag{3.1}$$

Then W_α is not cubically hyponormal.

Proposition 3.6. *Let $W_{\alpha(x,x)}$ be a weighted shift with $\alpha(x, x)$, where*

$$\alpha(x, x) : \alpha_0 = \alpha_1 = \sqrt{x}, \quad \text{and} \quad \alpha_k = \sqrt{\frac{k+1}{k+2}}, \quad k = 2, 3, \ldots. \tag{3.2}$$

If $\frac{1}{19}\left(14 - \sqrt{6}\right) \leq x \leq \sqrt{5} - \frac{3}{2}$, then $W_{\alpha(x,x)}$ is semi-cubically hyponormal with type I. In particular, $W_{\alpha\left(\frac{2}{3},\frac{2}{3}\right)}$ is semi-cubically hyponormal with type I.

Proof. See [7, Corollary 3.7]. □

Proposition 3.7. *Let $W_{\alpha(x)}$ be a weighted shift with $\alpha(x)$, where*

$$\alpha(x) : \alpha_0 = \sqrt{x} \text{ and } \alpha_k = \sqrt{\frac{k+1}{k+2}}, \quad k = 1, 2, \ldots. \tag{3.3}$$

If $0 < x < \frac{9}{10}$, then $W_{\alpha(x)}$ is semi-cubically hyponormal with type II.

Proposition 3.7 can be proved by some computations as following steps.

Lemma 3.8. *$\xi_{n+1}\eta_n = \delta_n$ for $n \geq 4$.*

Proof. From simple computations, we can have

$$\xi_n = \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-2}^2\alpha_{n-1}^2 = \frac{4}{(n+1)(n+3)} \quad (n \geq 3)$$

$$\eta_n = \alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2 = \frac{9}{(n+1)(n+4)} \quad (n \geq 4)$$

$$\delta_n = \alpha_n^2\left(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-2}^2\alpha_{n-1}^2\right)^2 = \frac{36}{(n+1)(n+2)(n+4)^2} \quad (n \geq 3)$$

which induces this lemma. □

Lemma 3.9. *For $n \geq 5$ and $i \geq 1$, we have*

$c(n,i) = \eta_n c(n-1,i-1) + (\xi_n \cdot \cdots \cdot \xi_5)\, h_i$ *with* $h_i := \xi_4 c(3,i) - \delta_3 c(2,i-1)$.

Proof. Using Lemma 3.8, we have the result by similar to the proof of the claim in [2, page 64]. □

Now we consider the sequence $\alpha(x)$ as in (3.3), we can obtain

$$\sigma_0(t) = \frac{x}{6}(3t+4), \qquad \sigma_1(t) = \frac{x}{15}\left(4t+3t^2+5\right),$$

$$\sigma_2(t) = \frac{1}{90}x\left(9t^3 + (12-12x)\,t^2 + (15-16x)\,t + (18-20x)\right),$$

$$\sigma_3(t) = -\frac{x}{3780}\Big[(189x-216)t^4 + (36x-99)t^3 + (111x-108)t^2 + (130x-117)t + 140x - 126\Big],$$

$$\sigma_4(t) = -\frac{x}{151\,200}\Big[(1701x-1944)\,t^5 + (1188x-1323)\,t^4 + (423x-684)\,t^3 + (690x-621)\,t^2 + (620x-558)\,t + 640x - 576\Big],$$

$$\sigma_5(t) = -\frac{x}{907\,200}\Big[(1701x-1944)\,t^6 + (1188x-1323)\,t^5 + (855x-900)t^4 + (402x-477)t^3 + (380x-342)t^2 + (320x-288)t + 320x - 288\Big],$$

and

$$h_1 = \frac{2x}{4725}(10x-9), \; h_2 = \frac{x}{3150}(10x-9), \; h_3 = \frac{x}{525}(2x-1),$$

$$h_4 = -\frac{x}{350}(2x-1), \; h_i = 0 \quad \text{for all } i \geq 5.$$

Proof of Proposition 3.7. For $0 < x < \frac{9}{10}$, it is easy to see that each of the coefficients of $\sigma_i(t)$ $(i = 0, 1, 2, 3, 4, 5)$ is positive, furthermore we have

$$\begin{aligned}
c(6,2) &= -\frac{17}{5953\,500}x(10x-9) > 0,\\
c(6,3) &= -\frac{1}{31\,752\,000}x(1070x-1219) > 0,\\
c(6,4) &= -\frac{1}{21\,168\,000}x(1846x-1751) > 0,\\
c(7,3) &= -\frac{1}{174\,636\,000}x(334x-371) > 0,\\
c(7,4) &= -\frac{1}{931\,392\,000}x(4618x-4361) > 0,\\
c(8,4) &= -\frac{1}{698\,544\,000}x(154x-145) > 0.
\end{aligned}$$

Let

$$\begin{aligned}
\rho_1 &= -\frac{2}{4725}\frac{x(5n+11)}{(n+1)(n+3)(n+4)},\\
\rho_2 &= -\frac{2}{1575}\frac{x(11n+2)}{n(n+1)(n+2)(n+3)(n+4)},\\
\rho_3 &= -\frac{2}{525}\frac{x(266n-526)}{n(n-1)(n+1)^2(n+2)(n+3)(n+4)},\\
\rho_4 &= -\frac{2}{175}\frac{x(926n-1864)}{n^2(n-1)(n+1)^2(n-2)(n+2)(n+3)(n+4)},
\end{aligned}$$

we can see that $\rho_i < 0$ $(i = 1, 2, 3, 4)$ for $n \geq 3$. Since $0 < x < \frac{9}{10}$, $\frac{265n-371}{266n-526} > \frac{9}{10}$ and $\frac{859n-1652}{926n-1864} > \frac{9}{10}$, using Lemma 3.9, we have

$$\begin{aligned}
c(n,1) &= \rho_1(10x-9)(\xi_{n-1}\cdot\cdots\cdot\xi_5) > 0 \quad \text{for } n \geq 6,\\
c(n,2) &= \rho_2(10x-9)(\xi_{n-2}\cdot\cdots\cdot\xi_5) > 0 \quad \text{for } n \geq 7,\\
c(n,3) &= \rho_3\left(x - \frac{265n-371}{266n-526}\right)(\xi_{n-3}\cdot\cdots\cdot\xi_5) > 0 \quad \text{for } n \geq 8,\\
c(n,4) &= \rho_4\left(x - \frac{859n-1652}{926n-1864}\right)(\xi_{n-4}\cdot\cdots\cdot\xi_5) > 0 \quad \text{for } n \geq 9.
\end{aligned}$$

This concludes the proof. □

Finally, by Corollary 3.5, Proposition 3.6 and Proposition 3.7, we obtain:

Theorem 3.10. *Let W_α be a weighted shift with a sequence α as in* (3.1). *Then W_α is almost-cubically hyponormal but not cubically hyponormal.*

References

[1] Y. Choi, *A propagation of quadratically hyponormal weighted shifts*, Bull. Korean Math. Soc. **37** (2000), no. 2, 347–352.

[2] R. Curto, *Quadratically hyponormal weighted shifts*, Integ. Equ. Oper. Theory **13** (1990), no. 1, 49–66.
[3] R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integ. Equ. Oper. Theory **17** (1993), no. 2, 202–246.
[4] ______, *Recursively generated weighted shifts and the subnormal completion problem. II*, Integ. Equ. Oper. Theory **18** (1994), no. 4, 369–426.
[5] R. Curto and I. Jung, *Quadratically hyponormal weighted shifts with two equal weights*, Integ. Equ. Oper. Theory **37** (2000), no. 2, 208–231.
[6] R. Curto and M. Putinar, *Nearly subnormal operators and moment problems*, J. Funct. Anal. **115** (1993), no. 2, 480–497.
[7] Y. Do, G. Exner, I. Jung, and C. Li, *On semi-weakly n-hyponormal weighted shifts*, Integ. Equ. Oper. Theory **73** (2012), no. 1, 93–106.
[8] I. Jung and S. Park, *Cubically hyponormal weighted shifts and their examples*, J. Math. Anal. Appl. **247** (2000), no. 2, 557–569.
[9] J. Stampfli, *Which weighted shifts are subnormal*, Pacific J. Math. **17** (1966), 367–379.
[10] Wolfram Research, Inc. *Mathematica*, Version 8.0, Wolfram Research, Champaign, 2010.

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