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BOUNDED COMPOSITION OPERATORS FROM THE BERGMAN SPACE TO THE HARDY SPACE

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Abstract. Let ϕ be an analytic self map of the open unit disc D. In this paper, we study the composition operator C_ϕ from the Bergman space on D to the Hardy space on D.

1. Introduction

Let D be the open unit disc in the complex plane. L_a^2 and H^2 denote the Bergman space and the Hardy space on D, respectively. Then H^2 is contained in L_a^2. If H^∞ is a set of all bounded analytic functions, then H^∞ is contained in H^2. For an analytic self map ϕ of D, the composition operator C_ϕ is defined by $(C_\phi f)(z) = f(\phi(z))$ $(z \in D)$ for f in H, the set of all analytic functions on D. The Nevanlinna counting function of ϕ, is defined on $D\backslash\{\phi(0)\}$ by

$$N_\phi(w) = \textstyle\sum_{\phi(z)=w} \log \frac{1}{|z|}.$$

T. Nakazi [4, Theorem 4] gives a necessary and sufficient condition for an isometric operator C_ϕ from L_a^2 to H^2. That is, C_ϕ is isometric from L_a^2 to H^2 if and only if $N_\phi(w) = 2\int_{|w|}^1 \log \frac{r}{|w|} r dr$ for nearly all $w \in D\backslash\{0\}$.

W. Smith [6, Theorem 1.1] gives a necessary and sufficient condition for a bounded composition operator C_ϕ from L_a^2 to H^2. That is, C_ϕ is bounded from L_a^2 to H^2 if and only if $N_\phi(w) = O\ ([\log 1/|w|]^2)$ $(|w| \to 1)$. For given ϕ, we can use some times this result in order to show C_ϕ is bounded but it may not be easy to use it.

A function ϕ in H^∞ with $\|\phi\|_\infty = 1$ is called a Rudin's orthogonal function in H^2 if $\{\phi^n : n = 0, 1, 2, \ldots\}$ is a set of orthogonal functions in H^2. It should be also called a Choe's function because B. R. Choe told W. Rudin about such a function. An inner function which has zeros at the origin is a Rudin's orthogonal function. Hence the Möbius transform of a Rudin's (Choe's)

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orthogonal function is a generalization of an inner function. There exists a Rudin's (Choe's) orthogonal function which is not an inner function ([1], [7]).

In Section 2, we study isometric C_ϕ from L_a^2 to H^2. In Section 3, we study bounded C_ϕ from L_a^2 to H^2. In Section 4, we give few examples using a theorem of W. Smith. In Section 5, we study bounded C_ϕ from L_a^2 onto H^2.

2. Isometric composition operator from L_a^2 to H^2

Lemma 1. *Let ϕ be a Rudin's (Choe's) orthogonal function. Then, C_ϕ is isometric from L_a^2 to H^2 if and only if*

$$\int_0^{2\pi} |\phi|^{2j} d\theta/2\pi = \frac{1}{j+1} \ (j = 0, 1, 2, \ldots).$$

Proof. Suppose $f = \sum_{j=0}^\infty a_j z^j$ and $f \in H$. If C_ϕ is isometric, then

$$\sum_{j=0}^\infty \frac{1}{j+1} |a_j|^2 = \sum_{j=0}^\infty |a_j|^2 \int_0^{2\pi} |\phi|^{2j} d\theta/2\pi$$

because $\|f\|_{L_a^2} = \|f \circ \phi\|_{H^2}$. Since f is arbitrary in L_a^2, we can show

$$\int_0^{2\pi} |\phi|^{2j} d\theta/2\pi = \frac{1}{j+1} \ (j = 0, 1, 2, \ldots).$$

The converse is clear. □

Theorem 1. *Let ϕ be an analytic self map of the open unit disc. Then, C_ϕ is an isometric composition operator if and only if ϕ is a Rudin's (Choe's) orthogonal function and*

$$\int_0^{2\pi} |\phi|^{2j} d\theta/2\pi = \frac{1}{j+1} \ (j = 0, 1, 2, \ldots).$$

Proof. If C_ϕ is an isometric operator from L_a^2 to H^2, then Theorem 4 in [4] shows $N_\phi(w) = 2\int_{|w|}^1 \log \frac{r}{|w|} r dr$ for nearly all $w \in D\backslash\{0\}$. By Theorem 1 in [3] ϕ is a Rudin's (Choe's) orthogonal function. Now Lemma 1 shows the theorem. □

In Theorem 1, if C_ϕ is onto, then by Theorem 3 in [4] ϕ is inner. This contradicts Theorem 1. Hence there does not exist any isometric composition operator from L_a^2 onto H^2.

3. Bounded composition operator from L_a^2 to H^2

In the following theorem, (1) is known in [6] and (2) is known in [2].

Theorem 2. *Let ϕ be an analytic self map of the open unit disc.*

(1) *C_ϕ is bounded from L_a^2 into H^2 if and only if*

$$N_\phi(z) = O((\log \frac{1}{|z|})^2) \ \textit{as}\ |z| \to 1.$$

(2) *If ϕ has radial limits of modulus one on a set of positive measure, then C_ϕ does not map L^2_a into H^2.*

Lemma 2. *Let ϕ be a Rudin's (Choe's) orthogonal function. Then C_ϕ is bounded from L^2_a to H^2 if and only if $\int_0^{2\pi} |\phi|^{2j} d\theta/2\pi \le \frac{\gamma}{j+1}$ $(j = 0, 1, 2, \ldots)$ for some finite constant $\gamma > 0$.*

Proof. It is clear by the proof of Lemma 1. □

Theorem 3. *Let ϕ be a polynomial of a Rudin's (Choe's) orthogonal function ϕ_0 with $\|\phi\|_\infty = 1$. If C_ϕ is bounded from L^2_a into H^2, then*

$$\sum_{j=0}^{\infty} \left| \begin{pmatrix} -\frac{1}{2} \\ j \end{pmatrix} \right|^2 \int_0^{2\pi} |\phi_0|^{2j} d\theta/2\pi < \infty.$$

In order to prove the Theorem 3 we need to prove two lemmas.

Lemma 3. *If p is a polynomial with $\|p\|_\infty = 1$, then $1 - p(z) = \prod_{j=1}^{n}(z - a_j)g(z)$, where $|a_j| = 1$ $(1 \le j \le n)$ and $|g(z)| > 0$ on $\bar{D}$.*

Proof. Obvious. □

Lemma 4. *For a with $|a| = 1, (z - a)^{-1/2}$ belongs to L^2_a but does not belong to H^2.*

Proof. It is enough to show that $(1 - z)^{-1/2} \in L^2_a$ but $(1 - z)^{-1/2} \notin H^2$. This is a result of Wallis formula. □

The proof of Theorem 3. Suppose $\|\phi\|_\infty = 1$. Let ϕ_0 be a Rudin's (Choe's) orthogonal function and p a polynomial and $\phi(z) = p(\phi_0(z))$ where $\|p\|_\infty = 1$. Suppose C_ϕ maps L^2_a into H^2. Then by the hypothesis and Lemma 4, $(1 - \phi)^{-1/2}$ belongs to H^2. By Lemma 3

$$1 - \phi(z) = \prod_{j=1}^{n} (\phi_0(z) - a_j) g(\phi_0(z)).$$

Hence $(\phi_0(z) - a_j)^{-1/2} \in H^2$ and so $(1 - \phi_0(z))^{-1/2} \in H^2$. Since ϕ_0 is a Rudin's (Choe's) orthogonal function,

$$\|(1 - \phi_0)^{-1/2}\|_2^2 = \sum_{j=0}^{\infty} \left| \begin{pmatrix} -\frac{1}{2} \\ j \end{pmatrix} \right|^2 \int_0^{2\pi} |\phi_0|^{2j} d\theta/2\pi < \infty.$$

□

By Theorem 3, if ϕ is a polynomial of an inner function ϕ_0 with zero at the origin, then C_ϕ is not bounded by Lemma 4. In general, it is clear that C_ϕ is bounded when $\|\phi\|_\infty < 1$. By (1) of Theorem 2, if ϕ is an inner function, then C_ϕ is not bounded.

4. Some special case

In this section, we study whether C_ϕ is not bounded from L_a^2 to H^2 when $\phi = (1+q)/2$ and q is inner. If $q(0) = 0$, then by Theorem 3 C_ϕ is not bounded because q is a Rudin's (Choe's) orthogonal function. Hence we have to study in case $q(0) \neq 0$. Our main tools are (1) of Theorem 2 and the following Lemma 5.

Lemma 5. *For nearly all w in D,*

$$N_\phi(w) = \int_0^{2\pi} \log \left| \frac{w - \phi(e^{i\theta})}{1 - \bar{w}\phi(e^{i\theta})} \right| d\theta/2\pi - \log \left| \frac{w - \phi(0)}{1 - \bar{w}\phi(0)} \right|.$$

Proof. This is well known (see [3]). □

Lemma 6.

$$\limsup_{\substack{|w|\to 1 \\ |w|\le 1 \\ |2w-1|\le 1}} \frac{-\log|2w-1|^2}{(\log|w|)^2} = \infty.$$

Proof. Put $w = r(x+iy)$, $0 \le r < 1$ and $x^2 + y^2 = 1$. Then $r \le x$ when $|2w-1| \le 1$. Hence

$$\limsup_{|w|\to 1, |2w-1|\le 1} \frac{-\log|2w-1|^2}{(\log|w|)^2} = \limsup_{r\to 1, r\le x} \frac{-\log(4r^2 - 4rx + 1)}{(\log r)^2} = \infty.$$

In fact, put $r = 1-t$, $x = 1 - \frac{1}{2}t$ and $t \to 0$. □

Lemma 7.

$$\lim_{|w|\to 1} \frac{\log|1+\bar{a}w|^2 - \log|w+a|^2}{(\log|w|)^2} = \infty.$$

Proof. Put $w = re^{i\alpha}$ and $a = be^{i\beta}$ where $r = |w|$ and $b = |a|$, then $1 + \bar{a}w = 1 + bre^{i(\alpha-\beta)}$ and $w + a = (re^{i(\alpha-\beta)} + b)e^{i\beta}$. Hence we may assume $w = re^{i\alpha} = r(x+iy)$ and $a = b$. Then

$$\begin{aligned}
&\lim_{r\to 1} \frac{\log(1 + a^2r^2 + 2arx) - \log(r^2 + a^2 + 2arx)}{(\log r)^2} \\
&= \lim_{r\to 1} \left\{ \frac{2a^2r + 2ax}{1 + a^2r^2 + 2arx} - \frac{2r + 2ax}{r^2 + a^2 + 2arx} \right\} \frac{2r}{\log r} \\
&= \lim_{r\to 1} \frac{(2a^3x - 2ax)r^2 + (2a^4 - 2)r + 2a^3x - 2ax}{(1 + a^2r^2 + 2arx)(r^2 + a^2 + 2arx)} \times \frac{2r}{\log r} = \infty.
\end{aligned}$$

□

Corollary 4. *If $\phi = (1+q^n)/2$, n is a positive integer and $q = (z-a)/(1-\bar{a}z)$ with $|a| < 1$, then C_ϕ is not bounded.*

Proof. By Lemma 5, for nearly all $re^{i\alpha}$

$$N_\phi(re^{i\alpha}) = \int_0^{2\pi} \log \left| (2re^{i\alpha} - 1) - \left(\frac{z-a}{1-\bar{a}z} \right)^n \right| d\theta/2\pi - \log|2re^{i\alpha} - 1 - (-a)^n|.$$

Put $s_1, s_2, \ldots, s_n$ are distinct nth roots of s and $s = 2re^{i\alpha} - 1$ when $|s| < 1$. Moreover put $b_j = \frac{s_j + a}{1 + \bar{a}s_j}$ $(1 \leq j \leq n)$. Since $|b_j| < 1$ $(1 \leq j \leq n)$, by Jensen's formula

$$\int_0^{2\pi} \log \left| (2re^{i\alpha} - 1) - \left(\frac{z-a}{1-\bar{a}z} \right)^n \right| d\theta/2\pi = \sum_{j=1}^n \log \frac{1}{|b_j|} + \log |(2re^{i\alpha} - 1) - (-a)^n|.$$

Hence for nearly all $re^{i\alpha}$

$$N_\phi(re^{i\alpha}) = \begin{cases} -\sum_{j=1}^n \log |b_j| & (|2re^{i\alpha} - 1| < 1) \\ 0 & (|2re^{i\alpha} - 1| \geq 1). \end{cases}$$

By Lemma 7,

$$\begin{aligned} \limsup_{r \to 1, |s|<1} \frac{N_\phi(re^{i\alpha})}{(\log r)^2} &= -\frac{1}{2} \liminf_{r \to 1, |s|<1} \sum_{j=1}^n \frac{\log \left| \frac{s_j + a}{1 + \bar{a}s_j} \right|^2}{(\log r)^2} \\ &\geq -\frac{1}{2} \sum_{j=1}^n \limsup_{r \to 1, |s|<1} \frac{\log \left| \frac{s_j + a}{1 + \bar{a}s_j} \right|^2}{(\log r)^2} \\ &\geq -\frac{1}{2} \sum_{j=1}^n \limsup_{r \to 1, |s|<1} \frac{\log \left| \frac{s_j + a}{1 + \bar{a}s_j} \right|^2}{(\log |s_j|)^2} \frac{(\log |s_j|)^2}{(\log |s|)^2} \frac{(\log |s|)^2}{(\log r)^2} \\ &= \infty \end{aligned}$$

because $\limsup_{r \to 1, |s|<1} (\log |s|)^2/(\log r)^2 = \infty$ by Lemma 6 and

$$(\log |s_j|)^2/(\log |s|)^2 = 1/n^2 \ (1 \leq j \leq n).$$

Hence (1) of Theorem 2 shows that C_ϕ is not bounded. □

Now we are interested in the case

$$q = \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \quad (z \in D,\ |a| < 1 \text{ and } |b| < 1).$$

We would like to prove if $\phi = (1+q)/2$, then C_ϕ is not bounded. Unfortunately we could not prove it except two special cases.

Lemma 8. *Let x and y be complex numbers with $|x| < 1$ and $|y| < 1$. Fix x and as $|y| \to 1$, $\log |y|/(\log |y| + \log |\frac{x-y}{1-yx}|)$ converges to a finite constant $\gamma(x)$.*

Proof. As $|y| \to 1$

$$\begin{aligned} &\log |y| / \left(\log |y| + \log \left| \frac{x-y}{1-yx} \right| \right) \\ &= 1/ \left(1 + \log \left| \frac{x-y}{1-yx} \right| / \log |y| \right) \approx \frac{1 - |\frac{x-y}{1-yx}|^2}{1 - |y|} = \frac{1-|x|^2}{|1-yx|}, \end{aligned}$$

where $A(x) \approx B(x)$ means const$\cdot A(x) \le B(x) \le$ const$\cdot B(x)$. $\square$

Corollary 5. *If $\phi = (1+q)/2$ and $q = \frac{z-a}{1-\bar{a}z} \cdot \frac{z+a}{1+\bar{a}z}$ with $|a| < 1$, then C_ϕ is not bounded.*

Proof. As in Corollary 4, for nearly all $re^{i\alpha}$

$$N_\phi(re^{i\alpha}) = \int_0^{2\pi} \log \left| (2re^{i\alpha} - 1) - \frac{z-a}{1-\bar{a}z} \cdot \frac{z+a}{1-\bar{a}z} \right| d\theta/2\pi - \log|2re^{i\alpha} - 1 + a^2|.$$

Put $s = 2re^{i\alpha} - 1$ when $|s| < 1$. Moreover $b_j^2 = \frac{s+a^2}{1+\bar{a}^2 s}$ $(j = 1, 2)$ where b_1, b_2 are distinct roots of $\frac{s+a^2}{1+\bar{a}^2 s}$. Since $|b_j| < 1$ $(j = 1, 2)$, by Jensen's formula for nearly all $re^{i\alpha}$

$$N_\phi(re^{i\alpha}) = \begin{cases} -\sum_{j=1}^{2} \log|b_j| & (|2re^{i\alpha} - 1| < 1) \\ 0 & (|2re^{i\alpha} - 1| \ge 1). \end{cases}$$

By Lemma 7

$$\begin{aligned} \limsup_{r\to 1, |s|<1} \frac{N_\phi(re^{i\alpha})}{(\log r)^2} &= -\frac{1}{2} \liminf_{r\to 1, |s|<1} \sum_{j=1}^{2} \frac{\log|b_j|^2}{(\log r)^2} \\ &\ge -\frac{1}{2} \limsup_{r\to 1, |s|<1} \sum_{j=1}^{2} \frac{\log|b_j|^2}{(\log|s|)^2} \frac{(\log|s|)^2}{(\log r)^2} \\ &\ge -\frac{1}{2} \sum_{j=1}^{2} \limsup_{r\to 1, |s|<1} \frac{\log|b_j|^2}{(\log|s|)^2} \frac{(\log|s|)^2}{(\log r)^2} = \infty. \end{aligned}$$

$\square$

Corollary 6. *If $\phi = (1+q)/2$ and $q = \frac{z-a}{1-az} \cdot \frac{z-b}{1-bz}$ with $-1 < a < 1$ and $-1 < b < 1$, then C_ϕ is not bounded.*

Proof. Put $s = 2re^{i\delta} - 1$ when $|s| < 1$ and $\Phi(z) = s - \frac{z-a}{1-az} \frac{z-b}{1-bz}$. Let α and β such that $\Phi(\alpha) = \Phi(\beta) = 0$, $|\alpha| < 1$ and $|\beta| < 1$. Then

$$s = \frac{\alpha - a}{1 - a\alpha} \cdot \frac{\alpha - b}{1 - b\alpha} = \frac{\beta - a}{1 - a\beta} \cdot \frac{\beta - b}{1 - b\beta}.$$

Put $s_1 = (\alpha - a)/(1 - a\alpha)$, $s_2 = (\alpha - b)/(1 - b\alpha)$, $s_3 = (\beta - a)/(1 - a\beta)$ and $s_4 = (\beta - b)/(1 - b\beta)$. Then

$$\frac{b-a}{1-ab} = \frac{s_1 - s_2}{1 - s_1 s_2} = \frac{s_3 - s_4}{1 - s_3 s_4}.$$

As in Corollary 5, for nearly all $re^{i\alpha}$

$$N_\phi(re^{i\delta}) = \begin{cases} \log\frac{1}{|\alpha|} + \log\frac{1}{|\beta|} & (|2re^{i\delta} - 1| < 1) \\ 0 & (|2re^{i\delta} - 1| \ge 1). \end{cases}$$

Hence

$$\begin{aligned}\frac{N_\phi(re^{i\delta})}{(\log r)^2} &= \frac{-\log\left|\frac{s_1+a}{1+as_1}\right|\left|\frac{s_3+a}{1+as_3}\right|}{(\log r)^2}\\ &= \frac{1}{(\log r)^2}\cdot\frac{1}{2}\left(-\log\left|\frac{s_1+a}{1+as_1}\right|^2 - \log\left|\frac{s_3+a}{1+as_3}\right|^2\right)\\ &\geq \frac{1}{(\log r)^2}\left\{\left(\log\left|\frac{s_1+a}{1+as_1}\right|^2\right)\left(\log\left|\frac{s_3+a}{1+as_3}\right|^2\right)\right\}^{1/2}\\ &= \frac{(\log|s_1|)(\log|s_3|)}{(\log r)^2}\left\{\frac{-\log\left|\frac{s_1+a}{1+as_1}\right|^2}{(\log|s_1|)^2}\right\}^{1/2}\left\{\frac{-\log\left|\frac{s_3+a}{1+as_3}\right|^2}{(\log|s_3|)^2}\right\}^{1/2}.\end{aligned}$$

By Lemma 7

$$\lim_{\substack{|s_1|\to 1\\ |s_2|\to 1}}\left\{\frac{-\log\left|\frac{s_1+a}{1+as_1}\right|^2}{(\log|s_1|)^2}\right\}^{1/2}\left\{\frac{-\log\left|\frac{s_3+a}{1+as_3}\right|^2}{(\log|s_3|)^2}\right\}^{1/2} = \infty.$$

Hence by Lemmas 7 and 8

$$\frac{\log|s_1|}{\log|s|} = \frac{\log|s_1|}{\log|s_1|+\log|s_2|} = \frac{\log|s_1|}{\log|s_1|+\log\left|\frac{x-s_1}{1-s_1x}\right|}$$

and so

$$\frac{(\log|s_1|)(\log|s_3|)}{(\log r)^2} = \frac{(\log|s|)^2}{(\log r)^2}\cdot\frac{\log|s_1|}{\log|s|}\cdot\frac{\log|s_3|}{\log|s|} \to \infty \text{ as } |s|\to 1.$$

Therefore

$$\limsup_{r\to 1, |s|<1}\frac{N_\phi(re^{i\alpha})}{(\log r)^2} = \infty.$$

□

5. Bounded composition operator from L_a^2 onto H^2

We would like to prove that there does not exist any bounded composition operator from L_a^2 onto H^2.

Proposition 1. *If C_ϕ is bounded and onto, then ϕ is one-to-one on D and $\phi(D) \subsetneq D$.*

Proof. If $C_\phi L_a^2 = H^2$, then there exists f in L_a^2 such that $f\circ\phi(z) = z$ and ϕ is one-to-one on D. If $\phi(D) = D$, then $\phi(z) = \alpha\frac{z-a}{1-\bar{a}z}$ when $|\alpha| = 1$ and $a \in D$, and $f(z) = \frac{\bar{\alpha}z+a}{1+\bar{\alpha}\bar{a}z}$. Then $\phi\circ f(z) = z$. Suppose $F \in L_a^2$ but $F \notin H^2$. Since $F\circ\phi \in H^2$, $F\circ\phi\circ f$ belongs to H^2. It contradicts that $F = F\circ\phi\circ f$. Thus $C_\phi L_a^2 \neq H^2$. □

Lemma 9. *If $\phi(z) = \alpha z$ and $|\alpha| < 1$, then $C_\phi L_a^2 \subsetneq H^2$.*

Proof. It is easy to see $C_\phi L_a^2 \subseteq H^2$. If $C_\phi L_a^2 = H^2$, then $\|f \circ \phi\|_{H^2} \geq \delta \|f\|_{L_a^2}$ $(f \in L_a^2)$. Hence

$$\sum_{n=0}^{\infty} |\alpha|^{2n} |a_n|^2 \geq \delta^2 \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)}$$

when $f = \sum_{n=0}^{\infty} a_n z^n$ and $f \in L_a^2$. If $f_\varepsilon = \sum_{n=0}^{\infty} (n+1)^{-\varepsilon} z^n$, then $\|f_\varepsilon \circ \phi\|_{H^2}^2 = \sum_{n=0}^{\infty} |\alpha|^{2n} (n+1)^{-2\varepsilon}$ and $\|f_\varepsilon\|_{L_a^2}^2 = \sum_{n=0}^{\infty} (n+1)^{-(1+2\varepsilon)}$. Hence $\|f_\varepsilon\|_{L_a^2}^2 \to \infty$ and $\|f_\varepsilon\|_{H^2}^2 = \sum_{n=0}^{\infty} |\alpha|^{2n}$ as $\varepsilon \to 0$. This contradiction shows $C_\phi L_a^2 \subsetneq H^2$. □

Proposition 2. *If $\overline{\phi(D)} \subsetneq D$, then C_ϕ is bounded but not onto.*

Proof. If $\overline{\phi(D)} \subsetneq D$, then $\alpha = \|\phi\|_\infty < 1$. Put $\psi(z) = \phi(z)/\alpha$ and $\phi_\alpha(z) = \alpha z$. Then C_ψ is bounded from H^2 to H^2 and C_{ϕ_α} is not onto from L_a^2 to H^2 by Lemma 9. Suppose C_ϕ is bounded from L_a^2 onto H^2. Since C_ϕ is bounded from L_a^2 to H^2, there exist $0 < \varepsilon, \gamma < \infty$ such that

$$\begin{aligned} \varepsilon \|f\|_{L_a^2}^2 &\leq \|C_\phi f\|_{H^2} = \|C_\psi C_{\phi_\alpha} f\|_{H^2} \\ &\leq \gamma \|C_{\phi_\alpha} f\|_{H^2} \ (f \in L_a^2). \end{aligned}$$

The inequality above contradicts Lemma 9. □

If ϕ is inner it is known that C_ϕ is not bounded and onto. In fact, by (2) of Theorem 2 C_ϕ is not bounded. But we can give a direct simple proof. If ϕ is inner and C_ϕ is bounded and onto by Proposition 1 ϕ is one to one and so ϕ is a single Blaschke product. Hence $\phi(D) = D$ and this contradicts Proposition 1.

When $\phi = (1+q)/2$ and q is inner, we could not show whether C_ϕ is bounded or not in general (see Section 4). But C_ϕ is not onto in general. For, by Proposition 1, q is a single Blaschke product and by Corollary 4, C_ϕ is not bounded and so not onto.

References

[1] C. Bishop, *Orthogonal functions in H^∞*, Pacific J. Math. **220** (2005), no. 1, 1–31.

[2] J. Moorhouse and C. Toews, *Differences of composition opertors*, Contemporary Mathematics **321** (2003), 207–213.

[3] T. Nakazi, *The Nevanlinna counting functions for Rudin's orthogonal functions*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1267–1271.

[4] ———, *Isometric composition operators between two weighted Hardy spaces*, Nihonkai Math. J. **17** (2006), no. 2, 111–124.

[5] T. Nakazi and T. Watanabe, *Properties of a Rudin's orthogonal function which is a linear combination of two inner functions*, Sci. Math. Jpn. **57** (2003), no. 2, 413–418.

[6] W. Smith, *Composition operators between Bergman and Hardy spaces*, Trans. Amer. Math. Soc. **348** (1996), no. 6, 2331–2348.

[7] C. Sundberg, *Measures induced by analytic functions and a problem of Walter Rudin*, J. Amer. Math. Soc. **16** (2003), no. 1, 69–90.

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