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CERTAIN NEW INTEGRAL FORMULAS INVOLVING THE GENERALIZED BESSEL FUNCTIONS

Junesang Choi, Praveen Agarwal, Sudha Mathur, and Sunil Dutt Purohit

Abstract. A remarkably large number of integral formulas involving a variety of special functions have been developed by many authors. Also many integral formulas involving various Bessel functions have been presented. Very recently, Choi and Agarwal derived two generalized integral formulas associated with the Bessel function $J_\nu(z)$ of the first kind, which are expressed in terms of the generalized (Wright) hypergeometric functions. In the present sequel to Choi and Agarwal's work, here, in this paper, we establish two new integral formulas involving the generalized Bessel functions, which are also expressed in terms of the generalized (Wright) hypergeometric functions. Some interesting special cases of our two main results are presented. We also point out that the results presented here, being of general character, are easily reducible to yield many diverse new and known integral formulas involving simpler functions.

1. Introduction and preliminaries

A remarkably large number of integral formulas involving a variety of special functions have been developed by many authors (see, *e.g.*, [5], [7] and [9]; for a very recent work, see also [6]). Many integral formulas involving products of Bessel functions have been developed and play an important role in several physical problems. In fact, Bessel functions are associated with a wide range of problems in diverse areas of mathematical physics, for example, those in acoustics, radio physics, hydrodynamics, and atomic and nuclear physics. These connections of Bessel functions with various other research areas have led many researchers to the field of special functions. Among many properties of Bessel functions, they also have investigated some possible extensions of the Bessel functions. A useful generalization $\mathrm{w}_\nu(z)$ of the Bessel function has been introduced and studied in [1, 2, 3] and [4]. The generalized Bessel function $\mathrm{w}_\nu(z)$ of the first kind is defined for $z \in \mathbb{C} \setminus \{0\}$ and $b, c, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$

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by the following series (see, *e.g.*, [4, p. 10, Eq. (1.15)]; for very recent works, see also [1, 2, 3] and [10, p. 2, Eq. (8)]):

$$\mathrm{w}_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+\frac{1+b}{2})}, \tag{1.1}$$

where $\mathbb{C}$ denotes the set of complex numbers, $\Gamma(z)$ is the familiar Gamma function (see, *e.g.*, [12, Section 1.1]), and $\mathrm{w}_\nu(0)=0$.

Here we aim at presenting two generalized integral formulas, which are expressed in terms of the generalized (Wright) hypergeometric functions, by inserting the generalized Bessel function (1.1) with suitable arguments into the integrand of (1.7). Some interesting special cases of our main results are also considered.

Remark 1. A special case of $\mathrm{w}_\nu(z)$ in (1.1) when $c=1$ and $b=1$ becomes the Bessel function of the first kind $J_\nu(z)$. Another case of $\mathrm{w}_\nu(z)$ in (1.1) when $c=-1$ and $b=1$ reduces to the modified Bessel function of purely imaginary argument $I_\nu(z)$. Similarly, for $c=1$ and $b=2$, $\mathrm{w}_\nu(z)$ reduces to $\frac{2j_\nu}{\sqrt{\pi}}$, while, for $c=-1$ and $b=2$, $\mathrm{w}_\nu(z)$ becomes $\frac{2i_\nu}{\sqrt{\pi}}$. Therefore the results for the generalized Bessel function $\mathrm{w}_\nu(z)$ of the first kind presented here may yield those corresponding ones for the specialized Bessel functions by simply making some suitable parametric replacements.

For our purpose, we recall two functions and a known formula. Fox [8] and Wright [14, 15, 16] introduced and investigated the generalized (Wright) hypergeometric function ${}_p\Psi_q$ defined by (see, *e.g.*, [13, p. 21])

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,\,A_1),\,\ldots,\,(\alpha_p,\,A_p)\,;\\ (\beta_1,\,B_1),\,\ldots,\,(\beta_q,\,B_q)\,;\end{matrix}\ z\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j+A_j\,k\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j+B_j\,k\right)}\,\frac{z^k}{k!}, \tag{1.2}$$

where the coefficients $A_1,\,\ldots,\,A_p$ and $B_1,\,\ldots,\,B_q$ are positive real numbers such that

$$1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j \geqq 0. \tag{1.3}$$

It is noted that the generalized (Wright) hypergeometric function ${}_p\Psi_q$ in (1.2) whose asymptotic expansion was investigated by Fox [8] and Wright [14, 15, 16] is an interesting further generalization of the generalized hypergeometric series ${}_pF_q$ (1.5) as follows:

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,\,1),\,\ldots,\,(\alpha_p,\,1)\,;\\ (\beta_1,\,1),\,\ldots,\,(\beta_q,\,1)\,;\end{matrix}\ z\right] = \frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j\right)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,;\\ \beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\ z\right], \tag{1.4}$$

where ${}_pF_q$ is the *generalized hypergeometric series* defined by (see, *e.g.*, [12, Section 1.5])

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\
&= {}_pF_q(\alpha_1, \ldots, \alpha_p;\, \beta_1, \ldots, \beta_q;\, z),
\end{aligned}
\tag{1.5}
$$

$(\lambda)_n$ being the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see, *e.g.*, [12, p. 2 and pp. 4–6]):

$$
\begin{aligned}
(\lambda)_n :&= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \end{cases} \\
&= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-)
\end{aligned}
\tag{1.6}
$$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers.

For our present investigation, we also need to recall the following Oberhettinger's integral formula [11]:

$$
\int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} dx = 2\lambda\, a^{-\lambda} \left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\,\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)},
\tag{1.7}
$$

provided $0 < \Re(\mu) < \Re(\lambda)$.

2. Main results

We establish two generalized integral formulas in Theorems 1 and 2 below, which are expressed in terms of the generalized (Wright) hypergeometric functions (1.2), by inserting the generalized Bessel function (1.1) with suitable arguments into the integrand of the integral (1.7).

Theorem 1. *The following integral formula holds true*:

$$
\begin{aligned}
&\int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} w_\nu\left(\frac{y}{x + a + \sqrt{x^2 + 2ax}}\right) dx \\
&= 2^{1-\nu-\mu}\, a^{\mu-\nu-\lambda}\, y^{\nu}\, \Gamma(2\mu) \\
&\quad \cdot {}_2\Psi_3\left[\begin{matrix} (\lambda-\mu+\nu,\, 2)\,,\, (1+\lambda+\nu,\, 2)\,; \\ \left(\nu + \dfrac{1+b}{2},\, 1\right),\, (1+\lambda+\mu+\nu,\, 2)\,,\, (\lambda+\nu,\, 2)\,; \end{matrix}\; -\frac{c\,y^2}{4\,a^2}\right]
\end{aligned}
\tag{2.1}
$$

$$
(x > 0;\; \lambda,\, \mu,\, \nu,\, b,\, c \in \mathbb{C} \;\; \textit{with} \;\; \Re(\nu) > -1 \;\; \textit{and} \;\; 0 < \Re(\mu) < \Re(\lambda+\nu))\,.
$$

Theorem 2. *The following integral formula holds true*:

$$
\begin{aligned}
&\int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda} w_\nu\left(\frac{x\,y}{x + a + \sqrt{x^2 + 2ax}}\right) dx \\
&= 2^{1-2\nu-\mu}\, a^{\mu-\lambda}\, y^{\nu}\, \Gamma(\lambda-\mu)
\end{aligned}
\tag{2.2}
$$

$$\cdot\, {}_2\Psi_3 \left[\begin{array}{c} (2\mu+2\nu,\,4)\,,\,(1+\lambda+\nu,\,2)\,; \\ \left(\nu+\dfrac{1+b}{2},\,1\right),\,(1+\lambda+\mu+2\nu,\,4)\,,\,(\lambda+\nu,\,2)\,; \end{array} -\frac{c\,y^2}{16} \right]$$

$$(x>0;\ \lambda,\,\mu,\,\nu,\,b,\,c\in\mathbb{C}\ \textit{with}\ \Re(\nu)>-1\ \textit{and}\ 0<\Re(\mu)<\Re(\lambda+\nu))\,.$$

Proof. By making use of (1.1) in the integrand of (2.1) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

(2.3)

$$\begin{aligned} &\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} W_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx \\ &= \sum_{k=0}^\infty (-c)^k\,\frac{(y/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+\frac{1+b}{2})}\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda-\nu-2k} dx. \end{aligned}$$

In view of the conditions given in Theorem 1, since

$$\Re(\nu)>-1,\ \ 0<\Re(\mu)<\Re(\lambda+\nu)\le\Re(\lambda+\nu+2k)\quad (k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})\,,$$

we can apply the integral formula (1.7) to the integral in (2.3) and obtain the following expression:

$$\begin{aligned} &\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \mathrm{w}_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx \\ &= 2^{1-\nu-\mu}\,a^{\mu-\nu-\lambda}\,y^\nu\,\Gamma(2\mu) \\ &\quad\cdot\sum_{k=0}^\infty \frac{(-c)^k\,\Gamma(1+\nu+\lambda+2k)\,\Gamma(\nu+\lambda-\mu+2k)}{k!\,\Gamma(\frac{1+b}{2}+\nu+k)\,\Gamma(1+\nu+\lambda+\mu+2k)\,\Gamma(\nu+\lambda+2k)}\left(\frac{y}{2a}\right)^{2k}, \end{aligned}$$

which, upon using (1.2), yields (2.1). This completes the proof of Theorem 1.

It is easy to see that a similar argument as in the proof of Theorem 1 will establish the integral formula (2.2). □

Remark 2. Setting $b=1=c$ in (2.1) and (2.2) with some suitable parametric replacements, is easily seen to give, respectively, the known integral formulas involving the Bessel functions $J_\nu(z)$, Equations (2.1) and (2.2) in Choi and Agarwal [6].

3. Special cases

Here we consider some cases of (2.1) and (2.2). If we set $\nu=-\frac{b}{2}$ in the generalized Bessel function $\mathrm{w}_\nu(\mathrm{z})$ in (1.1) with c replaced by c^2, we have the following relation between $\mathrm{w}_\nu(z)$ and a cosine function (see [10]):

$$\mathrm{w}_{-b/2,c^2}(z) := \left(\frac{2}{z}\right)^{\frac{b}{2}}\frac{\cos cz}{\sqrt{\pi}}. \tag{3.1}$$

Similarly, setting $\nu = -\frac{b}{2}$ in (1.1) with c replaced by $-c^2$ yields the following relation between $\mathrm{w}_\nu(z)$ and a hyperbolic cosine function:

$$\mathrm{w}_{-b/2,-c^2}(z) := \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cosh cz}{\sqrt{\pi}}. \tag{3.2}$$

Now, setting $\nu = -\frac{b}{2}$ and replacing c by c^2 in (2.1) and (2.2) and make use of the relation (3.1), we obtain the following interesting integral formulas involving cosine functions given, respectively, in Corollaries 1 and 2 below.

Corollary 1. *The following integral formula holds true*:

$$\begin{aligned}(3.3)\quad &\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cdot \cos\left(\frac{yc}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= 2^{1-\mu}\sqrt{\pi}\, a^{\mu+\frac{b}{2}-\lambda}\,\Gamma(2\mu) \\ &\cdot {}_2\Psi_3\left[\begin{matrix} \left(\lambda-\mu-\frac{b}{2},2\right), \left(1+\lambda-\frac{b}{2},2\right); \\ \left(\frac{1}{2},1\right), \left(1+\lambda+\mu-\frac{b}{2},2\right), \left(\lambda-\frac{b}{2},2\right); \end{matrix} -\frac{(cy)^2}{4a^2}\right],\end{aligned}$$

provided λ, μ, b, $c \in \mathbb{C}$ *with* $0 < \Re(\mu) < \Re(\lambda)$ *and* $x > 0$.

Corollary 2. *The following integral formula holds true*:

$$\begin{aligned}(3.4)\quad &\int_0^\infty x^{\mu-\frac{b}{2}-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cdot \cos\left(\frac{x\,y\,c}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= 2^{1+\frac{b}{2}-\mu}\sqrt{\pi}\, a^{\mu-\lambda}\,\Gamma(\lambda-\mu) \\ &\cdot {}_2\Psi_3\left[\begin{matrix} (2\mu-b,4), \left(1+\lambda-\frac{b}{2},2\right); \\ \left(\frac{1}{2},1\right), (1+\lambda+\mu-b,4), \left(\lambda-\frac{b}{2},2\right); \end{matrix} -\frac{(cy)^2}{16}\right],\end{aligned}$$

provided λ, μ, b, $c \in \mathbb{C}$ *with* $0 < \Re(\mu) < \Re(\lambda)$ *and* $x > 0$.

Similarly, setting $\nu = -\frac{b}{2}$ with replaced c by $-c^2$ in (2.1) and (2.2) and taking the relation (3.2) into account yields the following integral formulas involving hyperbolic cosine functions asserted, respectively, in Corollaries 3 and 4 below.

Corollary 3. *The following integral formula holds true*:

$$\begin{aligned}(3.5)\quad &\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cdot \cosh\left(\frac{yc}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= 2^{1-\mu}\sqrt{\pi}\, a^{\mu+\frac{b}{2}-\lambda}\,\Gamma(2\mu) \\ &\cdot {}_2\Psi_3\left[\begin{matrix} \left(\lambda-\mu-\frac{b}{2},2\right), \left(1+\lambda-\frac{b}{2},2\right); \\ \left(\frac{1}{2},1\right), \left(1+\lambda+\mu-\frac{b}{2},2\right), \left(\lambda-\frac{b}{2},2\right); \end{matrix} \frac{(cy)^2}{4a^2}\right],\end{aligned}$$

provided λ, μ, b, $c \in \mathbb{C}$ *with* $0 < \Re(\mu) < \Re(\lambda)$ *and* $x > 0$.

Corollary 4. *The following integral formula holds true*:

$$\begin{aligned}(3.6)\quad & \int_0^\infty x^{\mu-\frac{b}{2}-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cdot \cosh\left(\frac{x\,y\,c}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= 2^{1+\frac{b}{2}-\mu}\sqrt{\pi}\,a^{\mu-\lambda}\,\Gamma(\lambda-\mu) \\ &\cdot {}_2\Psi_3\left[\begin{matrix} (2\mu-b,\,4)\,,\ \left(1+\lambda-\dfrac{b}{2},\,2\right); \\ \left(\dfrac{1}{2},\,1\right),\ (1+\lambda+\mu-b,\,4)\,,\ \left(\lambda-\dfrac{b}{2},\,2\right); \end{matrix}\ \frac{(cy)^2}{16}\right],\end{aligned}$$

provided λ, μ, b, $c \in \mathbb{C}$ *with* $0 < \Re(\mu) < \Re(\lambda)$ *and* $x > 0$.

We also recall the following well known formulas (see, *e.g.*, [10]):

$$\mathrm{w}_{1-b/2,c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sin cz}{\sqrt{\pi}} \tag{3.7}$$

and

$$\mathrm{w}_{1-b/2,-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sinh cz}{\sqrt{\pi}}. \tag{3.8}$$

Considering (3.7) and (3.8) and making use of (2.1) and (2.2), we get the following interesting integral formulas involving sine and hyperbolic sine functions asserted by Corollary 5 below.

Corollary 5. *Each of the following integral formulas holds true*:

$$\begin{aligned}(3.9)\quad & \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cdot \sin\left(\frac{yc}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= 2^{-\mu}\sqrt{\pi}\,a^{\mu-\lambda+\frac{b}{2}-1}\,\Gamma(2\mu)\,y \\ &\cdot {}_2\Psi_3\left[\begin{matrix} \left(\lambda-\mu+1-\dfrac{b}{2},\,2\right),\ \left(2+\lambda-\dfrac{b}{2},\,2\right); \\ \left(\dfrac{3}{2},\,1\right),\ \left(2+\lambda+\mu-\dfrac{b}{2},\,2\right),\ \left(1+\lambda-\dfrac{b}{2},\,2\right); \end{matrix}\ -\frac{(cy)^2}{4\,a^2}\right];\end{aligned}$$

$$\begin{aligned}(3.10)\quad & \int_0^\infty x^{\mu-\frac{b}{2}-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}} \cdot \sin\left(\frac{x\,y\,c}{x+a+\sqrt{x^2+2ax}}\right) dx \\ &= 2^{\frac{b}{2}-\mu-1}\sqrt{\pi}\,a^{\mu-\lambda}\,\Gamma(\lambda-\mu)\,y \\ &\cdot {}_2\Psi_3\left[\begin{matrix} (2+2\mu-b,\,4)\,,\ (2+\lambda-b,\,2)\,; \\ \left(\dfrac{3}{2},\,1\right),\ (3+\lambda+\mu-b,\,4)\,,\ \left(1+\lambda-\dfrac{b}{2},\,2\right); \end{matrix}\ -\frac{(cy)^2}{16}\right];\end{aligned}$$

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}}\cdot\sinh\left(\frac{yc}{x+a+\sqrt{x^2+2ax}}\right)dx \tag{3.11}$$
$$= 2^{1+\frac{b}{2}-\mu}\sqrt{\pi}\,a^{\mu+\frac{b}{2}-\lambda}\,\Gamma(2\mu)$$
$$\cdot\,{}_2\Psi_3\left[\begin{array}{c} \left(\lambda-\mu+1-\frac{b}{2},\,2\right),\,\left(2+\lambda-\frac{b}{2},\,2\right);\\ \left(\frac{3}{2},\,1\right),\,\left(2+\lambda+\mu-\frac{b}{2},\,2\right),\,\left(1+\lambda-\frac{b}{2},\,2\right);\end{array} -\frac{(cy)^2}{4\,a^2}\right];$$

$$\int_0^\infty x^{\mu-\frac{b}{2}-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda+\frac{b}{2}}\cdot\sinh\left(\frac{x\,y\,c}{x+a+\sqrt{x^2+2ax}}\right)dx \tag{3.12}$$
$$= 2^{1+\frac{b}{2}-\mu}\sqrt{\pi}\,a^{\mu-\lambda}\,\Gamma(\lambda-\mu)$$
$$\cdot\,{}_2\Psi_3\left[\begin{array}{c} (2+2\mu-b,\,4)\,,\,(2+\lambda-b,\,2)\,;\\ \left(\frac{3}{2},\,1\right),\,(3+\lambda+\mu-b,\,4)\,,\,\left(1+\lambda-\frac{b}{2},\,2\right);\end{array} \frac{(cy)^2}{16}\right],$$

provided $\lambda,\ \mu,\ b,\ c\in\mathbb{C}$ *with* $0<\Re(\mu)<\Re(\lambda)$ *and* $x>0$.

Remark 3. Setting $b=1=c$ in (3.3), (3.4), (3.9) and (3.10) with some suitable parametric replacements in the resulting identities is easily seen to yield the corresponding known integral formulas in Choi and Agarwal [6].

4. Concluding remarks

In this section we briefly consider another variation of the results derived in the preceding sections. Bessel functions are important special functions that appear widely in science and engineering. Bessel functions of the first kind $J_\nu(z)$ are oscillatory and may be regarded as generalizations of trigonometric functions. Indeed, for large argument z with $z\geq 1$, the function $\sqrt{\frac{\pi z}{2}}J_\nu(z)$ is well approximated by the trigonometric function $\cos(z-\frac{\pi\nu}{2}-\frac{\pi}{4})$. Similarly, modified Bessel functions of the first kind $I_\nu(z)$, which are Bessel functions of imaginary argument, may be regarded as generalizations of exponentials. Further, it can be easily seen that $J_\nu(z)$, $I_\nu(z)$, $\frac{2j_\nu}{\sqrt{\pi}}$ and $\frac{2i_\nu}{\sqrt{\pi}}$ are special cases of the generalized Bessel function of first kind $w_\nu(\mathrm{z})$ in (1.1). Therefore, the results presented in this paper are easily converted in terms of various Bessel functions after some suitable parametric replacements.

Indeed, it is interesting to observe that, if we use Gauss's multiplication theorem for the Gamma function Γ:

$$\Gamma(mz)=(2\pi)^{\frac{1}{2}(1-m)}\,m^{mz-\frac{1}{2}}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right) \tag{4.1}$$
$$\left(z\neq 0,\,-\frac{1}{m},\,-\frac{2}{m},\,\ldots;\ m\in\mathbb{N}\right),$$

which is equivalently written in terms of the Pochhammer symbol (1.6) as follows (see, *e.g.*, [12, p. 6]):

$$(\lambda)_{mn} = m^{mn} \prod_{j=1}^{m} \left(\frac{\lambda + j - 1}{m} \right)_n \qquad (m \in \mathbb{N};\ n \in \mathbb{N}_0), \tag{4.2}$$

each of the integral formulas presented here can be expressed in terms of the generalized hypergeometric function ${}_pF_q$.

The generalized Bessel function defined by (1.1) possesses the advantage that various Bessel functions, trigonometric functions and hyperbolic functions happen to be particular cases of this function. Therefore, we see that the results deduced above may be significant and can lead to yield numerous other interesting integrals involving various Bessel functions and trigonometric functions by suitable specializations of arbitrary parameters in the theorems. Furthermore, they are expected to find some applications to the solutions of fractional differential and integral equations (see, *e.g.*, [10]). The results thus derived in this paper are general in character and likely to find certain applications in the theory of special functions.

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Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 780-714, Korea
E-mail address: junesang@mail.dongguk.ac.kr

Praveen Agarwal
Department of Mathematics
Anand International College of Engineering
Jaipur-303012, India
E-mail address: goyal.praveen2011@gmail.com

Sudha Mathur
Department of Basic Sciences
College of Technology and Engineering
M. P. University of Agriculture and Technology
Udaipur-313001, India
E-mail address: mathursudhayk@yahoo.co.in

Sunil Dutt Purohit
Department of Basic Sciences
College of Technology and Engineering
M. P. University of Agriculture and Technology
Udaipur-313001, India
E-mail address: sunil_a_purohit@yahoo.com