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HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD

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Abstract. We study half lightlike submanifold M of an indefinite trans-Sasakian manifold such that its structure vector field is tangent to M. First we study the general theory for such half lightlike submanifolds. Next we prove some characterization theorems for half lightlike submanifolds of an indefinite generalized Sasakian space form.

1. Introduction

The theory of lightlike submanifolds is an important topic of research in modern differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [5] and later studied by many authors (see recent results in two books [7, 11]). The class lightlike submanifolds of codimension 2 is compose of two classes by virtue of the rank of its radical distribution, which are called half lightlike submanifold or coisotropic submanifold [6]. Half lightlike submanifold is a special case of r-lightlike submanifold [5] such that $r = 1$ and its geometry is more general form than that of coisotrophic submanifolds or lightlike hypersurfaces. Much of the theory on half lightlike submanifolds will be immediately generalized in a formal way to r-lightlike submanifolds. For this reason, we study only half lightlike submanifolds in this article.

Recently many authors have studied lightlike submanifolds M of indefinite Sasakian manifolds ([8]$\sim$[13], [19]) or indefinite Kenmotsu manifolds ([15], [16]) or indefinite cosymplectic manifolds ([14], [18]). Oubina [20] introduced the notion of a trans-Sasakian manifold of type (α, β). Indefinite Sasakian manifold is an important kind of indefinite trans-Sasakian manifold with $\alpha = 1$ and $\beta = 0$. Indefinite cosymplectic manifold is another kind of indefinite trans-Sasakian manifold such that $\alpha = \beta = 0$. Indefinite Kenmotsu manifold is also an example with $\alpha = 0$ and $\beta = 1$. Alegre, Blair and Carriazo [2] introduced the notion of indefinite generalized Sasakian space form.

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In this article, we study half lightlike submanifolds of an indefinite trans-Sasakian manifold $\bar{M}$ such that its structure vector field is tangent to M. In Section 3, we obtains some new results which are related to the structure tensor on M induced by the structure tensor J on $\bar{M}$. Furthermore we study screen conformal half lightlike submanifolds M of an indefinite trans-Sasakian manifold $\bar{M}$. In Section 4, we prove some characterization theorems for half lightlike submanifolds M of an indefinite generalized Sasakian space form.

2. Half lightlike submanifold

It is well-known [6] that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of half lightlike submanifold (M, g) of a semi-Rimannian manifold $(\bar{M}, \bar{g})$ of codimension 2 is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank 1. Thus there exist complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called the *screen distribution* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $M = (M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M, by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M and by $(-.-)_i$ the i-th equation of $(-.-)$. We use same notations for any others. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$, certainly $TM^{\perp}$ is a vector subbundle of $S(TM)^{\perp}$. As $S(TM^{\perp})$ is a non-degenerate subbundle of $S(TM)^{\perp}$, the orthogonal complementary distribution $S(TM^{\perp})^{\perp}$ of $S(TM^{\perp})$ in $S(TM)^{\perp}$ is also a non-degenerate vector bundle such that

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp}.$$

Clearly $Rad(TM)$ is a subbundle of $S(TM^{\perp})^{\perp}$. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit spacelike vector field without loss of generality. It is well-known [6] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\bar{g}(\xi, N) = 1,\ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0,\ \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$, respectively. Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{2.2}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.4}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N; \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad \forall\, X,\, Y \in \Gamma(TM), \tag{2.7}$$

where ∇ and ∇^* are induced connections on TM and $S(TM)$, respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are called the *shape operators*, and τ, ρ and ϕ are 1-forms on TM. We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental form tensor* of M.

Since the connection $\bar{\nabla}$ on $\bar{M}$ is torsion-free, the induced connection ∇ on M is also torsion-free, and B and D are symmetric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.8}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{2.9}$$

$$D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \quad \bar{g}(A_L X, N) = \rho(X), \tag{2.10}$$

for any X, $Y \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM).$$

From (2.8), (2.9) and (2.10), we see that B and D satisfy

$$B(X,\,\xi) = 0, \quad D(X,\,\xi) = -\phi(X), \quad \forall\, X \in \Gamma(TM), \tag{2.11}$$

A^*_ξ and A_N are $S(TM)$-valued, and A^*_ξ is self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{2.12}$$

Replacing Y by ξ to (2.3) and using (2.6) and (2.11), we have

$$\bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi - \phi(X)L, \quad \forall\, X \in \Gamma(TM). \tag{2.13}$$

Definition. A half lightlike submanifold M of $\bar{M}$ is said to be

(1) *totally umbilical* [5] if there is a smooth vector field $\mathcal{H}$ on $tr(TM)$ on any coordinate neighborhood $\mathcal{U}$ such that

$$h(X,Y) = \mathcal{H}g(X,Y), \; \forall\, X,\, Y \in \Gamma(TM).$$

In case $\mathcal{H} = 0$, *i.e.*, $h = 0$ on $\mathcal{U}$, we say that M is *totally geodesic.*

(2) *screen totally umbilical* [5] if there exist a smooth function γ on $\mathcal{U}$ such that $A_N X = \gamma PX$, or equivalently,

$$C(X,PY) = \gamma g(X,Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.14}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that M is *screen totally geodesic.*

(3) *screen conformal* [6] if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that $A_{N} = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{2.15}$$

It is easy to see that M is totally umbilical if and only if there exist smooth functions σ and δ on each coordinate neighborhood $\mathcal{U}$ such that

$$B(X, Y) = \sigma g(X, Y),\ D(X, Y) = \delta g(X, Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{2.16}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y) \tag{2.17}$$

for all $X,\, Y,\, Z \in \Gamma(TM)$. But the connection ∇^* on $S(TM)$ is metric.

3. Indefinite trans-Sasakian manifolds

An odd-dimensional semi-Riemannian manifold $(\bar{M},\, \bar{g})$ is said to be an *indefinite almost contact metric manifold* ([8]$\sim$[19]) if there exists a structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$, where J is a (1, 1)-type tensor field, ζ is a vector field which is called the *structure vector field* of $\bar{M}$ and θ is a 1-form such that

$$J^2X = -X + \theta(X)\zeta,\ \bar{g}(JX,\, JY) = \bar{g}(X,\, Y) - \epsilon\theta(X)\theta(Y),\ \theta(\zeta) = 1, \tag{3.1}$$

for any vector fields X and Y on $\bar{M}$, where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike respectively. In this case, the structure set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$.

In an indefinite almost contact metric manifold, we show that $J\zeta = 0$ and $\theta \circ J = 0$. Such a manifold is said to be an *indefinite contact metric manifold* if $d\theta = \Phi$, where $\Phi(X, Y) = g(X,\, JY)$ is called the *fundamental 2-form* of $\bar{M}$. The indefinite almost contact metric structure of $\bar{M}$ is said to be *normal* if $[J,\, J](X, Y) = -2d\theta(X, Y)\zeta$ for any vector fields X and Y on $\bar{M}$, where $[J,\, J]$ denotes the Nijenhuis (or torsion) tensor field of J given by

$$[J, J](X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY].$$

An indefinite almost contact metric manifold $\bar{M} = (\bar{M}, J, \zeta, \theta, \bar{g})$ is called

(1) *indefinite Sasakian manifold* [8]$\sim$[13] if

$$(\bar{\nabla}_X J)Y = \bar{g}(X, Y)\zeta - \epsilon\,\theta(Y)X,$$

(2) *indefinite Kenmotsu manifold* [15, 16] if

$$(\bar{\nabla}_X J)Y = \bar{g}(JX, Y)\zeta - \epsilon\,\theta(Y)JX,$$

(3) *indefinite cosymplectic* [14, 18] if $\bar{\nabla}_X J = 0$,

for any vector fields X and Y on $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$ with respect to the semi-Riemannian metric $\bar{g}$.

Definition. An indefinite almost contact metric manifold $\bar{M}$ is called *indefinite trans-Sasakian manifold* [2, 17, 20] if, for any vector fields X and Y on $\bar{M}$, there exist smooth functions α and β on $\bar{M}$ such that

$$(\bar{\nabla}_X J)Y = \alpha\{\bar{g}(X, Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \epsilon\theta(Y)JX\}. \tag{3.2}$$

We say that $\{J, \zeta, \theta, \bar{g}\}$ is an *indefinite trans-Sasakian structure of type* (α, β).

Replacing Y by ζ in (3.2), we get

$$\bar{\nabla}_X \zeta = -\epsilon\alpha JX + \epsilon\beta(X - \theta(X)\zeta). \tag{3.3}$$

Remark 3.1. If $\beta = 0$, then $\bar{M}$ is said to be an *indefinite α-Sasakian manifold*. Indefinite Sasakian manifolds appear as examples of indefinite α-Sasakian manifolds, with $\alpha = 1$. Another important kind of indefinite trans-Sasakian manifold is that of indefinite cosymplectic manifolds obtained for $\alpha = \beta = 0$. If $\alpha = 0$, then $\bar{M}$ is said to be an *indefinite β-Kenmotsu manifold*. Indefinite Kenmotsu manifolds are particular examples with $\alpha = 0$ and $\beta = 1$.

From now, let M be a half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. It is known [13, 14] that, for any half lightlike submanifold M of an indefinite almost contact metric manifold $\bar{M}$, $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are subbundles of $S(TM)$, of rank 1. In the entire discussion of this article, we shall assume that ζ to be tangent vector field to M, such an M is called a *tangential half lightlike submanifold* of $\bar{M}$. Călin [3] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which many authors assumed in their works [9, 10, 11, 14, 17, 19]. We also assume this result. Therefore

$$\theta(\xi) = \epsilon g(\zeta, \xi) = 0, \quad \theta(N) = \epsilon g(\zeta, N) = 0, \quad \theta(L) = \epsilon g(\zeta, L) = 0. \tag{3.4}$$

In this case, there exists a non-degenerate almost complex distribution H_o with respect to the structure tensor field J, *i.e.*, $J(H_o) = H_o$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o.$$

Denote by H the almost complex distribution with respect to J such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Therefore the general decomposition $(2.1)_1$ of TM is reduced to

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{3.5}$$

Consider a pair of local null vector fields $\{U, V\}$, a local unit spacelike vector field W on $S(TM)$ and their 1-forms u, v and w defined by

$$U = -JN, \qquad V = -J\xi, \qquad W = -JL, \tag{3.6}$$

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = g(X, W). \tag{3.7}$$

Denote by S the projection morphism of TM on H with respect to (3.5). Then any vector field X on M and its action JX by J are expressed as follows:

$$X = SX + u(X)U + w(X)W, \tag{3.8}$$

$$JX = FX + u(X)N + w(X)L, \tag{3.9}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by

$$FX = JSX, \quad \forall\, X \in \Gamma(TM).$$

Applying $\bar{\nabla}_X$ to $(3.6)_{1,2,3}$ and (3.9) by turns and using (2.3), (2.4), (2.5), (2.8)$\sim$(2.10), (2.13) and (3.6)$\sim$(3.9), for all $X, Y \in \Gamma(TM)$, we have

(3.10) $B(X,U) = C(X,V),\ B(X,W) = D(X,V),\ C(X,W) = D(X,U),$

(3.11) $\nabla_X U = F(A_{_N} X) + \tau(X)U + \rho(X)W - \{\alpha\eta(X) + \beta v(X)\}\zeta,$

(3.12) $\nabla_X V = F(A^*_\xi X) - \tau(X)V - \phi(X)W - \beta u(X)\zeta,$

(3.13) $\nabla_X W = F(A_{_L} X) + \phi(X)U - \beta w(X)\zeta,$

(3.14) $(\nabla_X F)(Y) = u(Y)A_{_N} X + w(Y)A_{_L} X - B(X,Y)U - D(X,Y)W + \alpha\{g(X,Y)\zeta - \epsilon\theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \epsilon\theta(Y)FX\}.$

Theorem 3.2. (1) *Any indefinite trans-Sasakian manifold $\bar{M}$ admitting a totally umbilical tangential half lightlike submanifold M is an indefinite β-Kenmotsu manifold, i.e., $\alpha = 0$. In this case M is totally geodesic.*

(2) *Any indefinite trans-Sasakian manifold $\bar{M}$ admitting either a screen conformal or a screen totally umbilical tangential half lightlike submanifold is an indefinite cosymplectic manifold, i.e., $\alpha = \beta = 0$. In case M is screen totally umbilical, it is screen totally geodesic.*

Proof. Applying $\bar{\nabla}_X$ to $(3.4)_{1,2,3}$ and using (3.1) and (3.3), we have

(3.15)
$$\begin{aligned} &B(X,\zeta) = -\epsilon\alpha u(X), \qquad D(X,\zeta) = -\epsilon\alpha w(X), \\ &C(X,\zeta) = \epsilon\beta\eta(X) - \epsilon\alpha v(X), \qquad \forall\, X \in \Gamma(TM). \end{aligned}$$

(1) In case M is totally umbilical: From (2.16) and $(3.15)_{1,2}$, we have

$$\sigma\theta(X) = -\alpha u(X), \quad \delta\theta(X) = -\alpha w(X), \quad \forall\, X \in \Gamma(TM).$$

Taking $X = \zeta$ and $X = U$ or W, we get $\sigma = \delta = 0$ and $\alpha = 0$ respectively. Thus $\bar{M}$ is an indefinite β-Kenmotsu manifold and M is totally geodesic.

(2) In case M is screen conformal: From (2.15) and $(3.15)_{1,3}$, we have

$$\alpha\varphi u(X) = \alpha v(X) - \beta\eta(X), \quad \forall\, X \in \Gamma(TM).$$

Taking $X = V$ and $X = \xi$ to this equation by turns, we have $\alpha = 0$ and $\beta = 0$ respectively. Thus $\bar{M}$ is an indefinite cosymplectic manifold.

In case M is screen totally umbilical: From (2.14) and $(3.15)_3$, we have

$$\gamma\theta(X) = \beta\eta(X) - \alpha v(X), \quad \forall\, X \in \Gamma(TM).$$

Taking $X = \zeta$, $X = V$ and $X = \xi$ to this equation by turns, we have $\gamma = 0$, $\alpha = 0$ and $\beta = 0$ respectively. Thus $\bar{M}$ is an indefinite cosymplectic manifold and M is screen totally geodesic. □

Theorem 3.3. *Let M be a tangential half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If either U or V is parallel with respect to ∇, then $\bar{M}$ is an indefinite cosymplectic manifold, i.e., $\alpha = \beta = 0$, and $\tau = 0$. In case U is parallel, we get $\rho = 0$. In case V is parallel, we get $\phi = 0$.*

Proof. In case U is parallel: From (3.9) and (3.11) we have

$$\begin{aligned}J(A_N X) - u(A_N X)N - w(A_N X)L + \tau(X)U + \rho(X)W \\ -\{\alpha\eta(X) + \beta v(X)\}\zeta = 0, \quad \forall X \in \Gamma(TM).\end{aligned} \tag{3.16}$$

Taking the scalar product with ζ to (3.16) and using (3.1) and (3.4), we get $\alpha\eta(X) + \beta v(X) = 0$ for all $X \in \Gamma(TM)$. Taking $X = \xi$ and $X = V$ to this equation by turns, we have $\alpha = 0$ and $\beta = 0$ respectively. Thus $\bar{M}$ is an indefinite cosymplectic manifold. Taking the scalar product with V and W to (3.16) by turns and using (3.1) and the facts $\bar{g}(J(A_N X), V) = 0$ and $\bar{g}(J(A_N X), W) = 0$, we have $\tau = 0$ and $\rho = 0$ respectively.

In case V is parallel: From (3.9) and (3.12), for all $X \in \Gamma(TM)$, we have

$$J(A^*_\xi X) - u(A^*_\xi X)N - w(A^*_\xi X)L - \tau(X)V - \phi(X)W - \beta u(X)\zeta = 0. \tag{3.17}$$

Taking the scalar product with ζ to (3.17) and using (3.1) and (3.4), we get $\beta u(X) = 0$. Taking $X = U$ to this equation, we have $\beta = 0$. Taking the scalar product with U and W to (3.17) by turns and using (3.1) and the facts $\bar{g}(J(A^*_\xi X), U) = \bar{g}(J(A^*_\xi X), W) = 0$, we have $\tau = \phi = 0$. Applying J to (3.17) and using (3.1) and $\tau = \phi = \beta = 0$, we have

$$A^*_\xi X = \theta(A^*_\xi X)\zeta + u(A^*_\xi X)U + w(A^*_\xi X)W, \quad \forall X \in \Gamma(TM). \tag{3.18}$$

Taking the scalar product with U to this equation, we get

$$B(X, U) = g(A^*_\xi X, U) = v(A^*_\xi X) = 0. \tag{3.19}$$

Replacing X by U in $(3.15)_1$ and using (3.19), we get

$$-\epsilon\alpha = -\epsilon\alpha u(U) = B(U, \zeta) = 0.$$

Thus $\alpha = \beta = 0$ and $\bar{M}$ is an indefinite cosymplectic manifold. □

Theorem 3.4. *Let M be a tangential half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If W is parallel with respect to ∇, then $\bar{M}$ is an indefinite α-Sasakian manifold, i.e., $\beta = 0$, and $\phi = \rho = 0$.*

Proof. If W is parallel, then, for all $X \in \Gamma(TM)$, from (3.9) and (3.13) we get

$$J(A_L X) - u(A_L X)N - w(A_L X)L + \phi(X)U - \beta w(X)\zeta = 0. \tag{3.20}$$

Taking the scalar product with ζ to (3.20) and using (3.1) and (3.4), we have $\beta w(X) = 0$. Taking $X = W$ to this, we have $\beta = 0$. Thus $\bar{M}$ is an indefinite α-Sasakian manifold. Taking the scalar product with V to (3.20) and using (3.1) and the fact $\bar{g}(J(A_L X), V) = 0$, we have $\phi = 0$. Applying J to (3.20) and using (3.1) and the fact $\tau = \phi = \beta = 0$, we have

$$A_L X = \theta(A_L X)\zeta + u(A_L X)U + w(A_L X)W, \quad \forall X \in \Gamma(TM).$$

Taking the scalar product with N to this and using $(2.10)_2$, we obtain $\rho = 0$. □

Theorem 3.5. *Let M be a tangential half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If all of $\{V, U, W\}$ are parallel with respect to ∇, then M is screen totally geodesic.*

Proof. As U is parallel, we get $\alpha = \beta = 0$. By $(3.15)_3$, we have $C(X, \zeta) = \theta(A_N X) = 0$. Applying J to (3.16) and using (3.1) and (3.6), we have

$$A_N X = u(A_N X)U + w(A_N X)W.$$

From the fact $\alpha = 0$ and $(3.15)_1$, we have $B(X, \zeta) = \theta(A^*_\xi X) = 0$. As V is parallel, from (3.18) we have

$$A^*_\xi X = u(A^*_\xi X)U + w(A^*_\xi X)W, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with U to this, we obtain $u(A_N X) = v(A^*_\xi X) = 0$. Thus we show that $A_N = w(A_N X)W$. As W is parallel and $\alpha = 0$, by $(3.15)_2$ we obtain $D(X, \zeta) = \theta(A_L X) = 0$. Thus we have

$$A_L X = u(A_L X)U + w(A_L X)W, \quad \forall\, X \in \Gamma(TM).$$

Taking the scalar product with U to this equation, we get $v(A_L X) = w(A_N X) = 0$. Thus $A_N = 0$. Consequently M is screen totally geodesic. □

Let m =rank$(S(TM))$. Denote by H' the distribution on $S(TM)$ such that

$$H' = J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)).$$

Then the decomposition (3.5) of TM is reduced to $TM = H \oplus H'$.

Theorem 3.6. *Let M be a tangential half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If F is parallel with respect to ∇, then $\alpha = \beta = 0$ and $\bar{M}$ is an indefinite cosymplectic manifold. Furthermore H and H' are parallel distributions on M and M is locally a product manifold $M^2 \times M^{m-2}$, where M^2 and M^{m-2} are leaves of H' and H respectively.*

Proof. If F is parallel with respect to ∇, then, taking the scalar product with U to (3.14) and using the facts $g(\zeta, U) = 0$ and $g(FX, U) = -\eta(X)$, we get

$$u(Y)v(A_N X) + w(Y)v(A_L X) - \epsilon\theta(Y)\{\alpha v(X) - \beta\eta(X)\} = 0$$

for all X, $Y \in \Gamma(TM)$. Taking $Y = U$, $Y = W$ and $Y = \zeta$ by turns, we get

$$v(A_N X) = 0, \quad v(A_L X) = 0, \quad \alpha v(X) - \beta\eta(X) = 0. \tag{3.21}$$

Taking $X = V$ and $X = \xi$ to $\alpha v(X) - \beta\eta(X) = 0$ by turns, we have $\alpha = \beta = 0$. Thus $\bar{M}$ is an indefinite cosymplectic manifold. From (3.14) we have

$$u(Y)A_N X + w(Y)A_L X = B(X, Y)U + D(X, Y)W \tag{3.22}$$

for all X, $Y \in \Gamma(TM)$. Replacing Y by ξ to (3.22) and using (2.11), we have

$$D(X, \xi) = -\phi(X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{3.23}$$

Taking $Y \in \Gamma(D)$ to (3.22), we have $B(X, Y)U + D(X, Y)W = 0$. Therefore

$$B(X, Y) = 0, \quad D(X, Y) = 0, \quad \forall\, X \in \Gamma(TM),\ Y \in \Gamma(D). \tag{3.24}$$

Taking the scalar product with $Z \in \Gamma(D_o)$ to (3.22), we get $u(Y)C(X, Z) + w(Y)D(X, Z) = 0$ for all X, $Y \in \Gamma(TM)$. Taking $Y = U$ to this, we have

$$C(X, Y) = 0, \quad \forall\, X \in \Gamma(TM),\ Y \in \Gamma(D_o). \tag{3.25}$$

Taking the scalar product with N to (3.22) and then, taking $Y = W$, we have

$$\rho(X) = 0, \quad \forall\, X \in \Gamma(TM). \tag{3.26}$$

By using (2.3), (3.1), (3.9), (3.12) and (3.24), we derive

$$\begin{aligned}
&g(\nabla_X\xi, V) = -g(\xi, \bar{\nabla}_X V) = -B(X, V) = 0, \quad g(\nabla_X V, V) = 0,\\
&g(\nabla_X Y, V) = -g(Y, \nabla_X V) = g(A^*_\xi X, JY) = B(X, FY) = 0,\\
&g(\nabla_X\xi, W) = -D(X, V) = 0, \quad g(\nabla_X V, W) = -\phi(X) = 0,\\
&g(\nabla_X Y, W) = -g(Y, \nabla_X W) = D(X, FY) + u(Y)\rho(X) = 0,
\end{aligned}$$

for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H_o)$, or equivalently, we get

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \;\; \forall\, Y \in \Gamma(H).$$

This result implies that H is a parallel distribution on M.

For all $X \in \Gamma(TM)$ and $Y \in \Gamma(H_o)$, using (3.11) and (3.25), we derive

$$\begin{aligned}
&g(\nabla_X U, N) = v(A_N X) = 0, \quad g(\nabla_X U, U) = -g(A_N X, N) = 0,\\
&g(\nabla_X U, Y) = g(F(A_N X), Y) = -g(A_N X, JY) = -C(X, FY) = 0,\\
&g(\nabla_X W, N) = v(A_L X) = 0, \quad g(\nabla_X W, U) = -\rho(X) = 0,\\
&g(\nabla_X W, Y) = -g(A_L X, JY) = D(X, FY) - u(Y)\rho(X) = 0,
\end{aligned}$$

that is, $\nabla_X Z \in \Gamma(H')$ for all $X \in \Gamma(TM)$ and $Z \in \Gamma(H')$. Thus $J(H')$ is also a parallel distribution of M.

As $TM = H \oplus H'$, and H and H' are parallel distributions, by the decomposition theorem of de Rham [4], M is locally a product manifold $M^2 \times M^{m-2}$, where M^2 and M^{m-2} are leaves of H' and H respectively. □

Corollary 3.7. *Let M be a tangential half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If F and V are parallel with respect to ∇, then M is screen totally geodesic.*

Proof. As F is parallel with respect to ∇, from (3.10) and $(3.21)_2$ we have $D(X, U) = C(X, W) = 0$. Taking $Y = U$ to (3.22) and using (3.10), we have

$$A_N X = u(A_N X)U + w(A_N X)W = u(A_N X)U.$$

As V is also parallel with respect to ∇, from (3.10) and (3.19), we have $u(A_N X) = 0$. Thus $A_N = 0$ and M is screen totally geodesic. □

Theorem 3.8. *Let M be a screen conformal tangential half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If M is totally umbilical, then $\bar{M}$ is an indefinite cosymplectic manifold and M is locally a product manifold $M^2 \times M^{m-2}$, where M^2 and M^{m-2} are leaves of H' and H respectively. Moreover, M is totally geodesic and screen totally geodesic.*

Proof. By straightforward calculations from (2.15) and (3.10), we have

$$B(X, U - \varphi V) = 0, \qquad D(X, U - \varphi V) = 0 \tag{3.27}$$

for any $X \in \Gamma(TM)$. Assume that M is totally umbilical. Then we have

$$\sigma g(X, U - \varphi V) = 0, \qquad \delta g(X, U - \varphi V) = 0.$$

Replacing X by V to these, we have $\rho = B = 0$ or $\delta = C = 0$. As $C = \varphi B$, we have $C = 0$. Thus M is totally geodesic and screen totally geodesic. As M is screen totally geodesic, by Theorem 3.2, we shown that $\alpha = \beta = 0$. Thus $\bar{M}$ is an indefinite cosymplectic manifold. As $B = D = C = 0$, F is parallel with respect to ∇ by (3.14). Thus, from Theorem 3.6, we have our assertion. □

Definition. A half lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is said to be *irrotational* [7] if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

From (2.3) we see that a necessary and sufficient condition for M to be irrotational is $D(X, \xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

Define a non-null vector field ω on $S(TM)$ and a vector bundle $H^{\natural}$ by

$$\omega = U + \varphi V, \qquad H^{\natural} = H_o \oplus_{orth} J(S(TM^{\perp})) \oplus Rad(TM).$$

In this case, we show that $\omega \in \Gamma(J(Rad(TM)) \oplus J(tr(TM)))$ and

$$TM = \{J(Rad(TM)) \oplus J(tr(TM))\} \oplus_{orth} H^{\natural}.$$

Theorem 3.9. *Let M be a screen conformal irrotational tangential half lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$. If ω is parallel with respect to ∇, then $\tau = 0$ and φ is a constant and M is locally a product manifold $\mathcal{C}_u \times \mathcal{C}_v \times M^{\natural}$, where $\mathcal{C}_u$ and $\mathcal{C}_u$ are null curves tangent to $J(tr(TM))$ and $J(Rad(TM))$ respectively and $M^{\natural}$ is a leaf of $H^{\natural}$.*

Proof. From (3.11), (3.12) and the fact $A_{N} = \varphi A^*_{\xi}$, we have

$$\nabla_X \omega \;=\; 2F(A_{N} X) + \tau(X)U + \{X[\varphi] - \varphi\tau(X)\}V + \rho(X)W$$

for all $X \in \Gamma(TM)$. From this and the facts $g(F(A_{N} X), V) = g(F(A_{N} X), U) = g(F(A_{N} X), W) = 0$, we show that ω is parallel with respect to ∇ if and only if $\tau = \rho = 0$, φ is a constant and $F(A_{N} X) = F(A^*_{\xi} X) = 0$. Thus if ω is parallel with respect to ∇, then U and V are also parallel by (3.11) and (3.12), *i.e.*, $J(tr(TM))$ and $J(Rad(TM))$ are parallel distributions of M. As V is parallel with respect to ∇, from Theorem 3.5, we have $B(X, U) = 0$ by (3.19). Using (2.15) and $(3.10)_1$, we get $B(X, V) = \varphi^{-1}C(X, V) = \varphi^{-1}B(X, U) = 0$.

As U and V are parallel with respect to ∇, we get

$$\begin{aligned}
&g(\nabla_X \xi, U) = -B(X, U) = 0, \quad g(\nabla_X \xi, V) = -B(X, V) = 0,\\
&g(\nabla_X W, U) = -\rho(X) = 0, \qquad g(\nabla_X W, V) = \phi(X) = 0,\\
&g(\nabla_X Y, U) = -g(Y, \nabla_X U) = 0, \quad g(\nabla_X Y, V) = -g(Y, \nabla_X V) = 0,
\end{aligned}$$

for any $X \in \Gamma(TM)$ and $Y \in \Gamma(H_o)$. These equations imply

$$\nabla_X Y \in \Gamma(H^{\natural}), \quad \forall\, X \in \Gamma(TM), \;\; Y \in \Gamma(H^{\natural}).$$

Thus $H^{\natural}$ is also a parallel distribution. We have our assertion. □

4. Indefinite generalized Sasakian space form

An indefinite almost contact metric manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ is called an *indefinite generalized Sasakian space form* [2] and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \bar{R}(X,Y)Z = &\ f_1\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ &+ f_2\{\bar{g}(X,JZ)JY - \bar{g}(Y,JZ)JX + 2\bar{g}(X,JY)JZ\} \\ &+ f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ &+ \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\}, \end{aligned} \tag{4.1}$$

for any vector fields X, Y and Z on $\bar{M}$, where $\bar{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$. This kind of a manifold appears as a natural generalization of the well-known indefinite Sasakian space form, which can be obtained as a particular case of indefinite generalized Sasakian space forms by taking $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ and c is a constant. Moreover, we can also find some other examples (see [1]).

Example 4.1. An *indefinite Kenmotsu space form*, *i.e.*, an indefinite Kenmotsu manifold with constant J-sectional curvature c, such that ζ is spacelike, is an indefinite generalized Sasakian space form with $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$.

Example 4.2. An *indefinite cosymplectic space form*, *i.e.*, an indefinite cosymplectic manifold with constant J-sectional curvature c, such that ζ is spacelike, is an indefinite generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

Example 4.3. An indefinite almost contact metric manifold is said to be an *indefinite almost $C(\alpha)$-manifold* if its semi-Riemannian curvature satisfies

$$\begin{aligned} \bar{R}(X,Y,Z,W) = &\ \bar{R}(X,Y,JZ,JW) \\ &+ \alpha\{\bar{g}(X,W)\bar{g}(Y,Z) - \bar{g}(X,Z)\bar{g}(Y,W) \\ &+ \bar{g}(X,JY)\bar{g}(Y,JW) - \bar{g}(X,JW)\bar{g}(Y,JZ)\}, \end{aligned}$$

where α is a real number. Moreover, if such a manifold has constant J-sectional curvature equal to c, then its curvature tensor is given by

$$\begin{aligned} \bar{R}(X,\, Y)Z = &\ \frac{c+3\alpha^2}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ &+ \frac{c-\alpha^2}{4}\{\bar{g}(X,JZ)JY - \bar{g}(Y,JZ)JX + 2\bar{g}(X,JY)JZ\} \\ &+ \frac{c-\alpha^2}{4}\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X \\ &+ \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\}. \end{aligned}$$

It is an indefinite generalized Sasakian space form such that ζ is spacelike and

$$f_1 = \tfrac{c+3\alpha^2}{4}, \quad f_2 = f_3 = \tfrac{c-\alpha^2}{4}.$$

Denote by R and R^* the curvature tensors of ∇ and ∇^* respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, for any $X, Y, Z, W \in \Gamma(TM)$, we have the Gauss-Codazzi equations for M and $S(TM)$ such that

$$(4.2)\quad \begin{aligned}\bar{g}(\bar{R}(X,Y)Z, PW) = g(R(X,Y)Z, PW)& \\ + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW)& \\ + D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW)&,\end{aligned}$$

$$(4.3)\quad \begin{aligned}\bar{g}(\bar{R}(X,Y)Z, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)& \\ + B(Y,Z)\tau(X) - B(X,Z)\tau(Y)& \\ + D(Y,Z)\phi(X) - D(X,Z)\phi(Y)&,\end{aligned}$$

$$(4.4)\quad \begin{aligned}\bar{g}(\bar{R}(X,Y)Z, N) = \bar{g}(R(X,Y)Z, N)& \\ + D(X,Z)\rho(Y) - D(Y,Z)\rho(X)&,\end{aligned}$$

$$(4.5)\quad \begin{aligned}\bar{g}(\bar{R}(X,Y)Z, L) = (\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z)& \\ + B(Y,Z)\rho(X) - B(X,Z)\rho(Y)&,\end{aligned}$$

$$(4.6)\quad \begin{aligned}\bar{g}(R(X,Y)PZ, PW) = g(R^*(X,Y)PZ, PW)& \\ + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW)&,\end{aligned}$$

$$(4.7)\quad \begin{aligned}g(R(X,Y)PZ, N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)& \\ + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X)&.\end{aligned}$$

Theorem 4.1. *Let M be a tangential half lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. If α is non-zero constant, then $\alpha^2 = \epsilon f_1 - f_3$; $\beta = 0$ and $\bar{M}(f_1, f_2, f_3)$ is an indefinite α-Sasakian space form.*

Proof. First we prove $\beta = 0$: Substituting (3.9) into (3.3), we have

$$(4.8)\qquad \nabla_X \zeta = -\epsilon\alpha FX + \epsilon\beta(X - \theta(X)\zeta), \quad \forall\, X \in \Gamma(TM).$$

Applying $\bar{\nabla}_X$ to $u(Y) = g(Y, V)$ and using (3.9) and (3.12), we get

$$(4.9)\qquad (\nabla_X u)(Y) = -u(Y)\tau(X) - w(Y)\phi(X) - \epsilon\beta\theta(Y)u(X) - B(X, FY)$$

for all X, $Y \in \Gamma(TM)$. Substituting (4.1) into (4.3), we have

$$\begin{aligned} &f_2\{u(Y)\bar{g}(X, JZ) - u(X)\bar{g}(Y, JZ) + 2u(Z)\bar{g}(X, JY)\} \\ =\ &(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\ &+ D(Y,Z)\phi(X) - D(X,Z)\phi(Y).\end{aligned}$$

Replacing Z by ζ to this equation and using (3.15), we have

$$(4.10)\qquad \begin{aligned}&(\nabla_X B)(Y,\zeta) - (\nabla_Y B)(X,\zeta) + \epsilon\alpha\{u(X)\tau(Y) - u(Y)\tau(X)\} \\ &+ \epsilon\alpha\{w(X)\phi(Y) - w(Y)\phi(X)\} = 0, \quad \forall\, X, Y \in \Gamma(TM).\end{aligned}$$

Applying ∇_X to $B(Y, \zeta) = -\epsilon\alpha u(Y)$ and using (4.8) and (4.9), we have

$$\begin{aligned}(\nabla_X B)(Y,\zeta) = -\epsilon\beta B(X,Y) + \alpha\beta\{\theta(Y)u(X) - \theta(X)u(Y)\}& \\ + \epsilon\alpha\{u(Y)\tau(X) + w(Y)\phi(X) + B(X,FY) + B(Y,FX)\}&\end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Substituting this equation into (4.10), we have

$$\alpha\beta\{\theta(X)u(Y) - \theta(Y)u(X)\} = 0, \quad \forall\, X, Y \in \Gamma(TM).$$

Taking $X = \zeta$ and $Y = U$ and using $\alpha \neq 0$, we get $\beta = 0$.

Next we prove $\alpha^2 = \epsilon f_1 - f_3$: Applying $\bar{\nabla}_X$ to $v(Y) = g(Y, U)$ and using (3.7), (3.9), $(3.10)_1$, (3.11) and the fact $\beta = 0$, we get

$$(\nabla_X v)(Y) = v(Y)\tau(X) + w(Y)\rho(X) - g(A_{N}X, FY) - \epsilon\alpha\theta(Y)\eta(X) \tag{4.11}$$

for all $X, Y \in \Gamma(TM)$. Substituting (4.1) and (4.7) into (4.4), we have

$$\begin{aligned} &f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &+ f_2\{v(Y)\bar{g}(X, JPZ) - v(X)\bar{g}(Y, JPZ) + 2v(PZ)\bar{g}(X, JY)\} \\ &+ f_3\{\theta(X)\theta(PZ)\eta(Y) - \theta(Y)\theta(PZ)\eta(X)\} \\ = &\ (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ &+ C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X) \\ &+ D(X, PZ)\rho(Y) - D(Y, PZ)\rho(X). \end{aligned} \tag{4.12}$$

Replacing Z by ζ to the last equation and using $(3.15)_1$, we have

$$\begin{aligned} &(\epsilon f_1 - f_3)\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\} \\ = &\ (\nabla_X C)(Y, \zeta) - (\nabla_Y C)(X, \zeta) + \epsilon\alpha\{v(Y)\tau(X) - v(X)\tau(Y)\} \\ &+ \epsilon\alpha\{w(Y)\rho(X) - w(X)\rho(Y)\}. \end{aligned} \tag{4.13}$$

Applying ∇_X to $C(Y, \zeta) = -\epsilon\alpha v(Y)$ and using (4.8) and (4.11), we have

$$\begin{aligned} (\nabla_X C)(Y, \zeta) = -\epsilon\alpha\{&v(Y)\tau(X) + w(Y)\rho(X) - g(A_{N}X, FY) \\ &- g(A_{N}Y, FX)\} + \alpha^2\theta(Y)\eta(X), \end{aligned}$$

due to $\beta = 0$. Substituting this equation into (4.13), we have

$$(\epsilon f_1 - f_3 - \alpha^2)\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}, \quad \forall\, X, Y \in \Gamma(TM).$$

Taking $X = \zeta$ and $Y = \xi$ to this equation, we get $\alpha^2 = \epsilon f_1 - f_3$. □

Theorem 4.2. *Let M be a screen conformal tangential half lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is a semi-Euclidean space.*

Proof. Substituting (2.15) into (4.12) and using (4.4) and (4.10), we have

$$\begin{aligned} &f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &+ f_2\{[v(Y) - \varphi u(Y)]\bar{g}(X, JPZ) - [v(X) - \varphi u(X)]\bar{g}(Y, JPZ) \\ &\qquad + 2[v(PZ) - \varphi u(PZ)]\bar{g}(X, JY)\} \\ &+ f_3\{\theta(X)\theta(PZ)\eta(Y) - \theta(Y)\theta(PZ)\eta(X)\} \\ = &\ \{X[\varphi] - 2\varphi\tau(X)\}B(Y, PZ) - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X, PZ) \\ &+ D(X, PZ)\{\rho(Y) + \varphi\phi(Y)\} - D(Y, PZ)\{\rho(X) + \varphi\phi(X)\} \end{aligned}$$

for all $X,\, Y,\, Z \in \Gamma(TM)$. Replacing Y by ξ to the last equation, we obtain

$$\begin{aligned} &\{\xi[\varphi] - 2\varphi\tau(\xi)\}B(X, PZ) - \{\rho(\xi) + \varphi\phi(\xi)\}D(X, PZ) \\ &= f_1 g(X, PZ) + f_2\{v(X) - \varphi u(X)\}u(PZ) \\ &\quad + 2f_2\{v(PZ) - \varphi u(PZ)\}u(X) - f_3\theta(X)\theta(PZ). \end{aligned}$$

Taking $X = PZ = \zeta$ to this equation and using $(3.15)_{1,\,2}$, we obtain $\epsilon f_1 = f_3$. Also taking $X = V,\ PZ = U$ and $X = U,\ PZ = V$ by turns, we have

$$\begin{aligned} &\{\xi[\varphi] - 2\varphi\tau(\xi)\}B(V, U) - \{\rho(\xi) + \varphi\phi(\xi)\}D(V, U) = f_1 + f_2, \\ &\{\xi[\varphi] - 2\varphi\tau(\xi)\}B(U, V) - \{\rho(\xi) + \varphi\phi(\xi)\}D(U, V) = f_1 + 2f_2, \end{aligned}$$

respectively. From these two equations we show that $f_2 = 0$.

As M is screen conformal, $\bar{M}$ is an indefinite cosymplectic manifold by Theorem 3.2 and $f_1 = f_2 = f_3 = \frac{c}{4}$ by Example 4.2. Thus we have $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is a semi-Euclidean space. □

Let $R^{(0,\,2)}$ denote the induced Ricci type tensor of M given by

$$R^{(0,\,2)}(X, Y) = trace\{Z \to R(Z, X)Y\}$$

for any $X,\, Y \in \Gamma(TM)$. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$. Put $m = \mathrm{rank}(S(TM))$. Using this quasi-orthonormal frame field, we obtain

$$R^{(0,\,2)}(X, Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a, X)Y,\, W_a) + \bar{g}(R(\xi, X)Y,\, N)$$

for any $X,\, Y \in \Gamma(TM)$, where $\epsilon_a = g(W_a, W_a)$ is the causal character of W_a.

In general, $R^{(0,\,2)}$ is not symmetric [5, 7]. A tensor field $R^{(0,\,2)}$ of lightlike submanifolds M is called its *induced Ricci tensor* if it is symmetric. A symmetric $R^{(0,\,2)}$ tensor will be denoted by Ric. A lightlike manifold M equipped with an induced Ricci tensor is called *Ricci flat* if its Ricci tensor vanishes. M is called an *Einstein manifold* if the Ricci tensor of M satisfies $Ric = \kappa g$.

Theorem 4.3. *Any screen conformal irrotational Einstein tangential half lightlike submanifold of the space form $\bar{M}(f_1, f_2, f_3)$ is Ricci flat.*

Proof. If M is a screen conformal half lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$, then, using (4.2), (4.4) and the fact $\bar{R} = f_i = 0$ for all i, we have

$$\begin{aligned} R^{(0,\,2)}(X, Y) = \varphi\{&B(X, Y)trA^*_\xi - g(A^*_\xi X, A^*_\xi Y)\} \\ &+ D(X, Y)trA_{_L} - g(A_{_L}X, A_{_L}Y) + \rho(X)\phi(Y). \end{aligned} \tag{4.14}$$

This implies that if M is irrotational, then $R^{(0,\,2)}$ is symmetric.

Let $\mu = U - \varphi V$. It follows from (3.27) that

$$B(X, \mu) = 0, \quad D(X, \mu) = 0, \quad \forall\, X \in \Gamma(TM). \tag{4.15}$$

From (2.8), (2.10) and the last equations, we show that

$$A^*_\xi \mu = 0, \qquad A_L \mu = \rho(\mu)\xi. \tag{4.16}$$

As M is Einstein, from (4.14) and the fact $R^{(0,2)} = \kappa g$

$$\begin{aligned}\kappa g(X,Y) = \varphi\{B(X,Y)trA^*_\xi - g(A^*_\xi X, A^*_\xi Y)\}\\ + D(X,Y)trA_L - g(A_L X, A_L Y).\end{aligned}$$

Taking $X = Y = \mu$ to this equation and using (4.15) and (4.16), we get $\kappa = 0$. Therefore, M is Ricci flat. □

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