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L^p-SOBOLEV REGULARITY FOR INTEGRAL OPERATORS OVER CERTAIN HYPERSURFACES

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Abstract. In this paper we establish sharp L^p-regularity estimates for averaging operators with convolution kernel associated to hypersurfaces in $\mathbb{R}^d (d \geq 2)$ of the form $y \mapsto (y, \gamma(y))$ where $y \in \mathbb{R}^{d-1}$ and $\gamma(y) = \sum_{i=1}^{d-1} \pm|y_i|^{m_i}$ with $2 \leq m_1 \leq \cdots \leq m_{d-1}$.

1. Introduction

In this paper we consider averaging operators along hypersurfaces in $\mathbb{R}^d$ $(d \geq 2)$ of the form $y \mapsto (y, \gamma(y))$ where $y \in \mathbb{R}^{d-1}$ and $\gamma(y) = \sum_{i=1}^{d-1} \pm|y_i|^{m_i}$ with $2 = m_0 \leq m_1 \leq \cdots \leq m_{d-1} < m_d = \infty$. For smooth functions f on $\mathbb{R}^d$, we consider averaging operators $\mathcal{A}$ defined by

$$\mathcal{A}f(x) = \int_{\mathbb{R}^{d-1}} f(x - (y, \gamma(y))\, \chi(y)\, dy, \tag{1.1}$$

where χ is a smooth function with a compact support near the origin with $\chi(0) \neq 0$. For $\alpha \geq 0$ and $1 < p < \infty$ we denote by $L^p_\alpha(\mathbb{R}^d)$ the L^p-Sobolev space with the norm

$$\|f\|_{L^p_\alpha(\mathbb{R}^d)} = \big\|[\,(1 + |\cdot|^2)^{\frac{\alpha}{2}} \widehat{f}\,]^\vee\big\|_{L^p(\mathbb{R}^d)}. \tag{1.2}$$

When $d = 2$ and $m_1 = 2$, the curve $y_1 \mapsto (y_1, \gamma(y_1)) = (y_1, y_1^2)$ has nonvanishing Gaussian curvature, and the operator $\mathcal{A}$ maps L^p into L^p_α, where $\alpha = \alpha(p) = 1/p$ for $2 \leq p < \infty$. It is well-known that the value of $\alpha(p) = 1/p$ is optimal for all p in this range. By duality if $1 < p < 2$, the value for α is $1/p'$, where p' is the Hölder conjugate of p. The case of curves in $\mathbb{R}^2$ with vanishing

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Gaussian curvature, that is, $m_1 > 2$ and $\gamma(y_1) = y_1^{m_1}$, has been considered by M. Christ in [1]. He proved that $\mathcal{A}$ maps L^p into L^p_β if either $p \neq m_1$ and $\beta \leq \min(1/p, 1/m_1)$, or $p = m_1$ and $\beta < \min(1/p, 1/m_1)$. He also proved that the results can not be improved in the sense that when $p = m_1$, strong estimates for $\beta = \min(1/p, 1/m_1)$ is not available. Higher dimensional situations, that is, the cases $d \geq 3$ have been investigated by Nagel, Seeger and Wainger in [4]. They obtained a sharp condition which leads to optimal L^p-Sobolev estimates for maximal operators associated with convex hypersurfaces of finite type on the edges of $1/p$ near 0 and 1. We refer interested readers to results by Iosevich, Sawyer and Seeger in [3] with hypersurfaces in $\mathbb{R}^3$ satisfying 'finite line type conditions'.

The purpose of this paper is to develop tools for drawing complete pictures of the sharp L^p-Sobolev estimates for averaging operator $\mathcal{A}$ for $d \geq 2$. It is worthy of pointing out that the surfaces we consider in this paper are not necessarily convex because of the $\pm$ signs and one can easily see that the arguments are independent of choices of signs. So in what follows we only consider the case where $\gamma(y) = \sum_{i=1}^{d-1} |y_i|^{m_i}$. To state the main theorem we first let $2 \leq p < \infty$ and define ν_k and $\alpha(p)$ by

$$\nu_k = \sum_{j=k}^{d-1} \frac{1}{m_j},\ k = 1, \dots, d-1;\ \nu_d = 0 \tag{1.3}$$

and

$$\alpha(p) := \min_{k=1}^{d} \left(\nu_k + \frac{k-1}{p} \right). \tag{1.4}$$

α is a piecewise linear function of $1/p$ whose linear piece can be written as follows: for each $k = 1, \dots, d$,

$$\alpha(p) := \nu_k + \frac{k-1}{p} \quad \text{if } \frac{1}{m_k} < \frac{1}{p} \leq \frac{1}{m_{k-1}}.$$

Figure 1 illustrates the graph of the function α in $\frac{1}{p}\,\alpha$-plane when $d = 4$. We note that the graph for $1/2 \leq 1/p < 1$ is obtained by reflection about the vertical line $1/p = 1/2$.

In this paper we shall prove the following theorem:

Theorem 1.1. *For $2 \leq p < \infty$ and $2 \leq m_1 \leq \cdots \leq m_{d-1}$, the operator $\mathcal{A}$ maps L^p to L^p_α if and only if either $p = m_i$ and $\alpha < \alpha(p)$, or $p \neq m_i$ and $\alpha \leq \alpha(p)$ where $1 \leq i \leq d-1$ for $d \geq 2$.*

Remark 1.2. (1) The indicated range of parameters p and α can not be improved in the sense of unboundedness of $L^p \to L^q$ estimates under the appropriate affine transformation between the optimal domain of $L^p \to L^p_\alpha$ bounds and that of $L^p \to L^q$ bounds (See Section 3).

(2) As is elucidated in Figure 1, estimates for the cases $1 < p < 2$ can be immediately established by duality arguments as soon as we prove Theorem

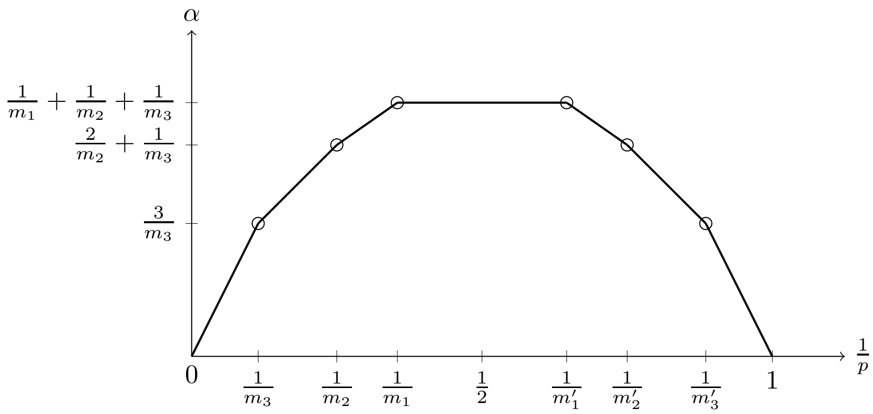

FIGURE 1. Boundedness of $\mathcal{A}$ when $d = 4$

1.1 and this is the reason why we only consider the cases $2 \leq p < \infty$ in the theorem.

(3) It is unlikely that $\mathcal{A}$ has sharp $L^p \to L^p_{\alpha(p)}$ property at the corner points of the optimal domain, which are circled dots in Figure 1. The best results up to this point are $L^{m_1,2} \to L^{m_1}_{1/m_1}$ estimates obtained by Seeger and Tao in [6] when $d = 2$ and $\gamma(y_1) = y_1^{m_1}$.

We shall need the following notation:

Notation. (1) For two quantities A and B, we shall write $A \lesssim B$ if $A \leq CB$ for some positive constant C, depending on the dimension and possibly other parameters apparent form the context. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

(2) The Lebesgue measure of a set E is denoted by $|E|$.

(3) The set of all integers, nonnegative integers, and positive integers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$, respectively.

(4) For a set A and a positive integer n, $A^n := \{(a_1, \dots, a_n) \,|\, a_i \in A\}$.

(5) For a positive integer n and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ we define $|\boldsymbol{\ell}|_1$ by $|\mathbf{a}|_1 = |a_1| + \cdots + |a_n|$.

The idea of proving Theorem 1.1 starts from taking a look into averaging operators T_i along curves in $\mathbb{R}^2$ of the form $y_i \mapsto (y_i, |y^{m_i}|)$ where $i = 1, \dots, d-1$. According to the results by M. Christ in [1], critical indices $\alpha_i(p)$ of T_i are of the form $\alpha_i(p) = \min(1/p, 1/m_i)$. It is easy to see that the critical index $\alpha(p)$ of our operator $\mathcal{A}$ can be written as

$$\alpha(p) = \sum_{i=1}^{d-1} \alpha_i(p). \tag{1.5}$$

The fact that the function $\gamma(y) = \sum_{i=1}^{d-1} |y_i|^{m_i}$ has no mixed term tempts us to take into account modified operators $\widetilde{T}_i$ averaging along curves $y_i \mapsto y_i\mathbf{u}_i + |y_i|^{m_i}\mathbf{u}_d$ where $\mathbf{u}_j$ is the standard unit vector in $\mathbb{R}^d$ whose j-th component is equal to 1 and other components are all 0's. In crude terms, the operator $\mathcal{A}$ can be realized by composing $\widetilde{T}_i$'s, that is, $\mathcal{A} = \widetilde{T}_1 \circ \cdots \circ \widetilde{T}_{d-1}$ modulo ignorance of the smooth cut-off function χ which is a localized factor of the averaging operator $\mathcal{A}$. It is highly likely that during composing $d-1$ operators $\widetilde{T}_i$, properties of $\widetilde{T}_i$'s, which improve differentiability of input function f, are added up to our aimed critical index $\alpha(p)$. However it would be a little bit rash if one deems that this explains all of the details of Theorem 1.1 because $\widetilde{T}_i$ improves the differentiability along only two directions $\mathbf{u}_i$ and $\mathbf{u}_d$.

Before we proceed to the next section, we make a preliminary remark on the cut-off function χ in (1.1). Without loss of generality we may assume that our original cut-off function χ is a tensor product of $d-1$ cut-off functions of one variable. To see this we first write $\chi(y) = \chi(y)\bar{\chi}(y)$ where $\bar{\chi}$ is a smooth cut-off function whose values are identically equal to 1 on the support of χ and $\bar{\chi}$ is of the form $\bar{\chi}(y) = \bar{\chi}(y_1)\cdots\bar{\chi}(y_{d-1})$. If we express χ as the Fourier series, say, $\chi(y) = \sum_{\mathbf{n}\in\mathbb{Z}^{d-1}} c_{\mathbf{n}} e^{i\mathbf{n}\cdot y}$ where $c_{\mathbf{n}}$ has a fast decay in $|\mathbf{n}|$. We then write

$$\chi(y) = \sum_{\mathbf{n}=(n_1,\ldots,n_{d-1})\in\mathbb{Z}^{d-1}} c_{\mathbf{n}} \prod_{j=1}^{d-1} e^{in_j y_j}\bar{\chi}(y_j),$$

that is, χ is the infinite summation of functions of the type of tensor product of $d-1$ one-variable cut-off functions. Due to the fast decay of $c_{\mathbf{n}}$ in $|\mathbf{n}|$, the results with $e^{in_j y_j}\bar{\chi}(y_j)$ implies those with $\chi(y)$.

2. Proof of Theorem 1.1

We take the Fourier transform $\widehat{\mathcal{A}f}$ of $\mathcal{A}f$ to write

$$\widehat{\mathcal{A}f}(\xi) = \widehat{f}(\xi)\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(y,\gamma(y))}\chi(y)\,dy = m(\xi)\widehat{f}(\xi).$$

As is explained in the previous section we may assume that the cut-off function χ in (1.1) is a tensor product of $d-1$ cut-off functions of one variable and we abuse notation to write $\chi(y) = \chi(y_1)\cdots\chi(y_{d-1})$, then we are able to write m as

$$m(\xi) = \prod_{i=1}^{d-1}\int_{\mathbb{R}} e^{i(\xi_i y_i + \xi_d |y_i|^{m_i})}\chi(y_i)\,dy_i. \tag{2.1}$$

Ψ_1 be a smooth radial function supported in $\{\xi \in \mathbb{R}^d : |\xi| \le 1\}$, which satisfies $\Psi_1(\xi) = 1$ when $|\xi| \le \frac{1}{2}$. One can easily see that there are homogeneous functions Ψ_2 and Ψ_3 satisfying the conditions that Ψ_2 is supported in $\{\xi = (\xi_1,\ldots,\xi_d) : |\xi| \ge \frac{1}{2}$ and $|\xi_d| \le \frac{|\xi|}{M}\}$, Ψ_3 is supported in $\{\xi = (\xi_1,\ldots,\xi_d) : |\xi| \ge \frac{1}{2}$ and $|\xi_d| \ge \frac{|\xi|}{2M}\}$, and $\Psi_1+\Psi_2+\Psi_3 \equiv 1$, where M is chosen to be so large that the following arguments hold. We first decompose $\mathcal{A}$ as $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$, where $\widehat{\mathcal{A}_i f}(\xi) = \Psi_i(\xi)\widehat{\mathcal{A}_i f}(\xi)$. We can see that in view of the compactness of the support Ψ_1 the operator $\mathcal{A}_1$ has an enough L^p-Sobolev estimates for our purpose. Due to the support condition of Ψ_2, there exists at least one $j \in \{1, \dots, d-1\}$ such that $|\xi| \approx |\xi_j| >> |\xi_d|$. In this case we perform integration by parts in the j-th factor of the right-hand side of (2.1) as many time as we obtain enough decay of $|\xi|$ for proving desired L^p-Sobolev estimates for $\mathcal{A}_2$. Hence it suffice to only consider $\mathcal{A}_3$. To avoid the complexity of indices we abuse the notation to set $\mathcal{A} := \mathcal{A}_3$ with the assumption that the multiplier of the operator $\mathcal{A}$ is supported in $\{\xi = (\xi_1, \dots, \xi_d) : |\xi| \geq \frac{1}{2} \text{ and } |\xi_d| \geq \frac{|\xi|}{2M}\}$. Throughout this section we fix index sets $\mathfrak{I} = \{1, \dots, \mu\}$ and $\mathfrak{I}' = \{\mu + 1, \dots, d-1\}$. Since the proof will be gone through via decomposing the operator $\mathcal{A}$ into dyadic pieces, we shall need various types of cut-off functions.

Definition. (1) Let ψ be a smooth radial function in $\mathbb{R}^d$ whose Fourier transform $\widehat{\psi}$ is supported in $\{\xi : 1/2 < |\xi| \leq 2\}$.

(2) Let η_0 be a function on $\mathbb{R}$ such that $\widehat{\eta_0} \in C_0^\infty(\mathbb{R})$, $\widehat{\eta_0}(s) = 1$ for $|s| \leq 1/2$, and $\widehat{\eta_0}(s) = 0$ for $|s| > 1$ and let η be a function defined by $\widehat{\eta}(s) = \widehat{\eta_0}(s) - \widehat{\eta_0}(2s)$.

(3) Let φ_0 be a function on $\mathbb{R}$, which has the same properties as $\widehat{\eta_0}$ above and let φ be a function defined by $\varphi(t) = \varphi_0(t) - \varphi_0(2t)$.

(4) $\mathbf{N} = \{(n_1, \dots, n_{d-1}) \in \mathbb{Z}^{d-1} : n_j \geq 0 \text{ if } j \in \mathfrak{I}, \text{ and } n_j \leq \frac{k}{m_j} \text{ if } j \in \mathfrak{I}'\}$.

(5) $\boldsymbol{L} = \{(\ell_{\mu+1}, \dots, \ell_{d-1}) \in \mathbb{Z}^{d-\mu-1} : j \in \mathfrak{I} \text{ and } \ell_j \geq -\frac{k}{m_j}\}$.

(6) For a complex number z, the real part of z is denoted by $\Re(z)$.

(7) $\nu(\mathfrak{I}) = \sum_{i=\mu+1}^{d-1} \frac{1}{m_i}$.

If ϕ is either φ or $\widehat{\eta}$ in Definition (2), then we clearly have

$$1 = \phi_0(t) + \sum_{n=1}^{\infty} \phi(2^{-n}t) := \phi_0(t) + \sum_{n=1}^{\infty} \phi_n(t) \quad \text{for all } t \tag{2.2}$$

and

$$1 = \sum_{n=1}^{\infty} \phi(2^n t) = \sum_{n=1}^{\infty} \phi_{-n}(t) \quad \text{for all } 0 < |t| < 1/4. \tag{2.3}$$

For $k \in \mathbb{N}$, we define operators $\mathcal{P}^k$ by

$$\widehat{\mathcal{P}^k(f)}(\xi) = \widehat{\psi}(2^{-k}\xi)\widehat{f}(\xi). \tag{2.4}$$

For $k \in \mathbb{N}$, $\mathbf{n} = (n_1, \dots, n_{d-1}) \in \mathbf{N}$, $\boldsymbol{\ell} = (\ell_{\mu+1}, \dots, \ell_{d-1}) \in \mathbf{L}$, $y = (y_1, \dots, y_{d-1}) \in \mathbb{R}^{d-1}$, and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, we define $\psi_{\mathfrak{I}'}^k$ and $\varphi_{\mathbf{n}}^k$ by

$$\widehat{\psi_{\boldsymbol{\ell}}^k}(\xi) = \widehat{\psi}(2^{-k}\xi) \prod_{j \in \mathfrak{I}'} \widehat{\eta_{\ell_j}}(2^{-\frac{k}{m_j}}\xi_j)$$

and

$$\varphi_{\mathbf{n}}^k(y) = \prod_{i \in \mathfrak{I}} \varphi_{-n_i}(y_i)\chi_i(y_i) \times \prod_{j \in \mathfrak{I}'} \varphi_{n_j}(2^{\frac{k}{m_j}} y_j)\chi_j(y_j).$$

Now we define operators $\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}$ by

$$\widehat{\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f}(\xi) = \widehat{f}(\xi)\widehat{\psi^k_{\boldsymbol{\ell}}}(\xi)\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(y,\gamma(y))}\varphi^k_{\mathbf{n}}(y)\,dy \tag{2.5}$$

and the multiplier $J^k_{\mathbf{n},\boldsymbol{\ell}}$ of $\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}$ by

$$J^k_{\mathbf{n},\boldsymbol{\ell}}(\xi) = \widehat{\psi^k_{\boldsymbol{\ell}}}(\xi)\int_{\mathbb{R}^{d-1}} e^{i\xi\cdot(y,\gamma(y))}\varphi^k_{\mathbf{n}}(y)\,dy.$$

Lemma 2.1 (Van der Corput)**.** *Suppose that ϕ is real-valued and smooth in (a,b), and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$ with the additional conditions $k \geq 2$, or $k = 1$ and $\phi'(x)$ is monotonic. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}\psi(x)dx\right| \leq C_k\lambda^{-1/k}\left[|\psi(b) + \int_a^b |\psi'(x)|dx\right],$$

where C_k is independent of ϕ, ψ, and λ.

Lemma 2.2. *We define $\mathfrak{I}''$ by $\mathfrak{I}'' = \{\, j \in \mathfrak{I}' : (m_j - 1)n_j \lesssim \ell_j \,\}$. Then for $\mathbf{n} = (n_1, \ldots, n_{d-1}) \in \mathbf{N}$ and $\boldsymbol{\ell} = (\ell_{\mu+1}, \ldots, \ell_{d-1}) \in \mathbf{L}$, and for any $N > 0$ we have*

$$|J^k_{\mathbf{n},\boldsymbol{\ell}}(\xi)| \lesssim 2^{-k(\frac{\mu}{2}+\nu(\mathfrak{I}))}2^{\sum_{i\in\mathfrak{I}}\frac{(m_i-2)n_i}{2}}2^{-\sum_{j\in\mathfrak{I}'}\frac{(m_j-2)n_j}{2}}2^{-\sum_{j\in\mathfrak{I}''}|\ell_j|}. \tag{2.6}$$

Proof. To prove the lemma we consider two cases, $j \in \mathfrak{I}\cup(\mathfrak{I}'\setminus\mathfrak{I}'')$ and $j \in \mathfrak{I}''$. When $j \in \mathfrak{I}\cup(\mathfrak{I}'\setminus\mathfrak{I}'')$ and $j \in \mathfrak{I}''$, we apply Lemma 2.1 and integration by parts with respect to y_j, respectively. The reason why we are able to use integration by parts when $j \in \mathfrak{I}''$ is because the phase $\xi_j y_j + \xi_d|y_j|^{m_j}$ is dominated by the linear term in this case. If we define $J^k_{\mathbf{n},\boldsymbol{\ell},j}(\xi)$ as

$$J^k_{\mathbf{n},\boldsymbol{\ell},j}(\xi) = \int_{\mathbb{R}} e^{i(\xi_j y_j+\xi_d|y_j|^{m_j})}\varphi_{-n_j}(y_j)\chi_j(y_j)\,dy_j$$

for $j \in \mathfrak{I}$ and

$$J^k_{\mathbf{n},\boldsymbol{\ell},j}(\xi) = \widehat{\eta_{\ell_j}}(2^{-\frac{k}{m_j}}\xi_j)\int_{\mathbb{R}} e^{i(\xi_j y_j+\xi_d|y_j|^{m_j})}\varphi_{n_j}(2^{\frac{k}{m_j}}y_j)\chi_j(y_j)\,dy_j$$

for $j \in \mathfrak{I}'$, then we can write $J^k_{\mathbf{n},\boldsymbol{\ell}}(\xi)$ as

$$J^k_{\mathbf{n},\boldsymbol{\ell}}(\xi) \quad = \quad \widehat{\psi}(2^{-k}\xi)\prod_{j=1}^{d-1}J^k_{\mathbf{n},\boldsymbol{\ell},j}(\xi).$$

When $j \in \mathfrak{I}\cup(\mathfrak{I}'\setminus\mathfrak{I}'')$ we apply Lemma 2.1 above and use the fact that $|\xi_d| \sim |\xi| \sim 2^k$ in the support of $\widehat{\psi}(2^{-k}\xi)$ to obtain

$$\left|J^k_{\mathbf{n},\boldsymbol{\ell},j}(\xi)\right| \lesssim 2^{-\frac{k}{2}}2^{\frac{(m_j-2)n_j}{2}}$$

for $j \in \mathfrak{I}$ and

$$\left|J^k_{\mathbf{n},\boldsymbol{\ell},j}(\xi)\right| \lesssim 2^{-\frac{k}{2}}(2^{-\frac{k}{m_j}+n_j})^{-\frac{m_j-2}{2}} = 2^{-\frac{k}{m_j}}2^{-\frac{(m_j-2)n_j}{2}}$$

for $j \in \mathfrak{I}' \setminus \mathfrak{I}''$.

When $j \in \mathfrak{I}''$, we first observe that the definition $\ell_j \gtrsim (m_j - 1)n_j$ of $\mathfrak{I}''$ and the support condition $t \sim 2^{-\frac{k}{m_j}+n_j}$ of $\varphi_{n_j}(2^{\frac{k}{m_j}} y_j)$ imply

$$|\partial_{y_j}(\xi_j y_j + \xi_d |y_j|^{m_j})| \gtrsim |\xi_j|,$$

and employ integration by parts with respect to y_j twice to obtain

$$|J^k_{\mathbf{n},\boldsymbol{\ell},j}(\xi)| \lesssim (2^{\frac{k}{m_j}-n_j})^{2-1}|\xi_j|^{-2} \lesssim 2^{\frac{k}{m_j}-n_j}(2^{\frac{k}{m_j}+\ell_j})^{-2} \lesssim 2^{-\frac{k}{m_j}}2^{-\frac{(m_j-2)n_j}{2}}2^{-\ell_j}.$$

Now it is easy to see that by taking the product of all factors we finally obtain the desired estimates. $\square$

Lemma 2.3. *For $1/m_{\mu+1} < 1/p < 1/m_\mu$ there exists an $\epsilon(p) > 0$ such that*

$$\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^p \to L^p} \lesssim 2^{-\epsilon(p)(|\mathbf{n}|_1 + |\boldsymbol{\ell}|_1)} 2^{-k\alpha(p)}.$$

Proof. In view of Lemma 2.2 and the support conditions of y_j's we obtain

$$\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^2 \to L^2} \lesssim 2^{-k(\frac{\mu}{2}+\nu(\mathfrak{I}))} 2^{\sum_{j\in\mathfrak{I}} \frac{(m_j-2)n_j}{2}} 2^{-\sum_{j\in\mathfrak{I}'} \frac{(m_j-2)n_j}{2}} 2^{-\sum_{j\in\mathfrak{I}''}|\ell_j|}$$

and

$$\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^\infty \to L^\infty} \lesssim 2^{-k\nu(\mathfrak{I})} 2^{-\sum_{j\in\mathfrak{I}} n_j} 2^{\sum_{j\in\mathfrak{I}'} n_j},$$

respectively. We apply the interpolation to obtain

$$\begin{aligned}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^p \to L^p} &\lesssim \|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|^{\frac{2}{p}}_{L^2 \to L^2} \|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|^{1-\frac{2}{p}}_{L^\infty \to L^\infty} \\ &\lesssim 2^{-k\alpha(p)} 2^{-\sum_{j\in\mathfrak{I}}(1-\frac{m_j}{p})n_j} 2^{-\sum_{j\in\mathfrak{I}'}(\frac{m_j}{p}-1)n_j} 2^{-(1-\frac{2}{p})\sum_{j\in\mathfrak{I}''}|\ell_j|},\end{aligned}$$

which completes the proof. $\square$

2.1. Endpoint estimates for $\frac{1}{m_{\mu+1}} < \frac{1}{p} < \frac{1}{m_\mu}$

By Littlewood-Payley theory it suffices to prove the vector-valued inequality

$$\Big\|\Big(\sum_{k>0} |2^{k\alpha(p)}\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}} f_k|^2\Big)^{1/2}\Big\|_p \lesssim 2^{-\epsilon(p)(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)} \Big\|\Big(\sum_{k>0}|f_k|^2\Big)^{1/2}\Big\|_p \tag{2.7}$$

for $\frac{1}{m_{\mu+1}} < \frac{1}{p} < \frac{1}{m_\mu}$.

Let us consider the anisotropic dilations

$$x \to t^P x = \exp(P \log t)x,$$

where P is a real $n \times n$-matrix with the real parts of the eigenvalues being contained in (a_0, a^0), $a_0 > 0$. Define the P homogeneous distance function; this means $\rho(t^P x) = t\rho(x)$, $x \in \mathbb{R}^d$, $t > 0$, and $\rho(x) > 0$, $x \neq 0$. Let $\mathcal{W}$ be the collection of all ρ-balls

$$Q = \{x : \rho(x - x_0) \leq 2^k\}, \quad x_0 \in \mathbb{R}^d,\ k \in \mathbb{Z}.$$

The Hardy-Littlewood maximal operator with respect to $\mathcal{W}$ is defined for the functions with values in a Banach-space B by

$$\mathcal{M}f(x) := \sup_{x\in Q\in\mathcal{W}} \frac{1}{|Q|}\int_Q |f(y)|_B \, dy.$$

By $f^\sharp$ we denote the Fefferman-Stein sharp maximal function, defined by

$$f^\sharp(x) = \sup_{x\in Q\in\mathcal{W}} \frac{1}{|Q|}\int_Q |f(y) - f_Q|_B dy,$$

where $f_Q = |Q|^{-1}\int_Q f(y)dy$. The following proposition is taken from [5].

Proposition 2.4. *Assume that $1 < p < \infty$, $1 \le p_0 \le p$ and $f \in L^{p_0}(\mathbb{R}^d, B)$. If $f^\sharp \in L^p(\mathbb{R}^d)$, then $\mathcal{M}f \in L^p(\mathbb{R}^d)$ and $\|\mathcal{M}f\|_p \le c\|f^\sharp\|_p$.*

We consider a $d\times d$ diagonal matrix P of the form

$$P = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & a_{d-1} & 0 \\ 0 & \cdots & 0 & 0 & a_d \end{pmatrix},$$

where $a_i = 1$ for $1 \le i \le \mu$ or $i = d$ and $a_i = \frac{1}{m_i}$ for $\mu + 1 \le i \le d-1$. Then we have

$$\rho(x) := \max_{i=1}^{d}\left(|x_i|^{1/a_i}\right), \quad \text{and } B = \ell^2(\mathbb{N}).$$

We let $\beta(z) = \frac{\mu z}{2} + \nu(\mathfrak{I})$ and define a complex family of operators $S^z_{\mathbf{n},\boldsymbol{\ell}}$ on $L^p(\ell^2)$ by

$$S^z_{\mathbf{n},\boldsymbol{\ell}}F(x) = \left\{2^{k\beta(z)}\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k(x)\right\}_{k=1}^{\infty} \quad \text{where } F = \{f_k\} \in L^p(\ell^2).$$

We note that $\beta(2/p) = \alpha(p)$. The remaining of this section is devoted to prove the following lemma:

Lemma 2.5. *If $\frac{1}{m_{\mu+1}} < \frac{1}{p} < \frac{1}{m_\mu}$ and $z = \frac{2}{p}$, then*

$$\left\|(S^z_{\mathbf{n},\boldsymbol{\ell}}F)^\sharp\right\|_{L^p} \lesssim 2^{-\epsilon(p)(|\boldsymbol{\ell}|_1+|\mathbf{n}|_1)}\|F\|_{L^p(\ell^2)}.$$

If we prove Lemma 2.5, then

$$\left\|S^z_{\mathbf{n},\boldsymbol{\ell}}F\right\|_{L^p(\ell^2)} \le \left\|\mathcal{M}\left(S^z_{\mathbf{n},\boldsymbol{\ell}}F\right)\right\|_{L^p} \lesssim \left\|(S^z_{\mathbf{n},\boldsymbol{\ell}}F)^\sharp\right\|_{L^p} \lesssim 2^{-\epsilon(p)(|\boldsymbol{\ell}|_1+|\mathbf{n}|_1)}\|F\|_{L^p(\ell^2)},$$

which gives (2.7).

Proof of Lemma 2.5

In order to apply interpolation arguments as in [5], we use linearized operators $T^z_{\mathbf{n},\boldsymbol{\ell}}$ of the operators $F \to (S^z_{\mathbf{n},\boldsymbol{\ell}}F)^\sharp$, which is defined as follows:
We first define operators $T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}$ of the form

$$T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}f(x) = 2^{k\beta(z)}\frac{1}{|Q_x|}\int_{Q_x}\left[\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f(y) - [\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f]_{Q_x}\right]g_k(x,y)dy,$$

where Q_x is a ball in $\mathcal{W}$ containing $x \in \mathbb{R}^d$ with radius δ_x, $g_k(x,y)$'s are measurable functions with

$$\left(\sum_k |g_k(x,y)|^2\right)^{1/2} \leq 1$$

for $y \in Q_x$, and $[\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f]_{Q_x} \equiv \frac{1}{|Q_x|}\int_{Q_x}\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f(u)du$. We now define $T^z_{\mathbf{n},\boldsymbol{\ell}}$ as

$$T^z_{\mathbf{n},\boldsymbol{\ell}}F(x) = \sum_{k>0} T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}f_k(x).$$

The ball $Q_x \in \mathcal{W}$ and measurable functions $g_k(x,y)$ can be suitably chosen so that the following inequality holds:

$$(S^z_{\mathbf{n},\boldsymbol{\ell}}F)^\sharp(x) \leq 2|T^z_{\mathbf{n},\boldsymbol{\ell}}F(x)|.$$

Hence the proof of Lemma 2.5 can be completed if one is able to show that for $z = \frac{2}{p}$,

$$\|T^z_{\mathbf{n},\boldsymbol{\ell}}F\|_{L^p} \leq C2^{-\epsilon(p)(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\|F\|_{L^p(\ell^2)},$$

with a constant C independent of the choice of Q_x and g_k. We define index sets $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ for k as

$$\mathcal{I}_1 = \{k \in \mathbb{Z}_+ \,:\, 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)} \leq 2^k\delta_x \leq 2^{N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\};$$
$$\mathcal{I}_2 = \{k \in \mathbb{Z}_+ \,:\, 2^k\delta_x \geq 2^{N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\};$$
$$\mathcal{I}_3 = \{k \in \mathbb{Z}_+ \,:\, 2^k\delta_x \leq 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\},$$

where $N >$ is chosen so that the following arguments hold. We split $T^z_{\mathbf{n},\boldsymbol{\ell}}F$ as $T^z_{\mathbf{n},\boldsymbol{\ell}}F = I^z_{\mathbf{n},\boldsymbol{\ell}}F + II^z_{\mathbf{n},\boldsymbol{\ell}}F + III^z_{\mathbf{n},\boldsymbol{\ell}}F$, where

$$I_{\mathbf{n},\boldsymbol{\ell}}F(x) = \sum_{k\in\mathcal{I}_1} T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}f_k(x);$$
$$II_{\mathbf{n},\boldsymbol{\ell}}F(x) = \sum_{k\in\mathcal{I}_2} T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}f_k(x);$$
$$III_{\mathbf{n},\boldsymbol{\ell}}F(x) = \sum_{k\in\mathcal{I}_3} T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}f_k(x).$$

By Hölder's inequality we obtain the pointwise estimates for the main term $I^z_{\mathbf{n},\boldsymbol{\ell}}F(x)$ of the form

$$|I^{\frac{2}{p}}_{\mathbf{n},\boldsymbol{\ell}}F(x)| \lesssim \frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{k\in\mathcal{I}_1} 2^{2k\beta(2/p)}\left|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k(y)-[\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k]_{Q_x}\right|^2\Big)^{1/2}dy$$
$$\lesssim (1+|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)^{1/2-1/p}\bigg(\sum_{k>0}\Big[2^{k\beta(2/p)}\mathcal{M}(\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k)\Big]^p\bigg)^{1/p}(x).$$

Now we apply Lemma 2.3 to obtain

$$\left\|I^{\frac{2}{p}}_{\mathbf{n},\boldsymbol{\ell}}F\right\|_p \lesssim (1+|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)^{1/2-1/p}\sup_k\Big(2^{k\beta(2/p)}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^p\to L^p}\Big)\|F\|_{L^p(\ell^p)}$$
$$\lesssim (1+|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)^{1/2-1/p}2^{-\epsilon(p)(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\|F\|_{L^p(\ell^2)}.$$

For the operators $II_{\mathbf{n},\boldsymbol{\ell}}$ and $III_{\mathbf{n},\boldsymbol{\ell}}$ we prove that if $\Re(z)=1$, then

$$\|II^z_{\mathbf{n},\boldsymbol{\ell}}F\|_2+\|III^z_{\mathbf{n},\boldsymbol{\ell}}F\|_2 \lesssim \sup_k\Big(2^{k\beta(1)}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^2\to L^2}\Big)\|F\|_{L^2(\ell^2)}, \tag{2.8}$$

and if $\Re(z)=0$, then

$$\begin{aligned}&\|II^z_{\mathbf{n},\boldsymbol{\ell}}F\|_\infty+\|III^z_{\mathbf{n},\boldsymbol{\ell}}F\|_\infty\\ &\lesssim\Big[\sup_k\Big(2^{k\beta(0)}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^2\to L^2}\Big)+2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\Big]\|F\|_{L^\infty(\ell^2)}.\end{aligned} \tag{2.9}$$

Then by interpolating (2.8) and (2.9) we obtain that when $z=\frac{2}{p}$,

$$\begin{aligned}&\sum_{\mathbf{n},\boldsymbol{\ell}}\Big(\left\|II^z_{\mathbf{n},\boldsymbol{\ell}}F\right\|_p+\left\|III^z_{\mathbf{n},\boldsymbol{\ell}}F\right\|_p\Big)\\ &\lesssim \sum_{\mathbf{n},\boldsymbol{\ell}}\Big[\sup_k\big(2^{k\beta(1)}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^2\to L^2}\big)\Big]^{\frac{2}{p}}\\ &\quad\times\Big[\sup_k\big(2^{k\beta(0)}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^2\to L^2}\big)+2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\Big]^{1-\frac{2}{p}}\|F\|_{L^p(\ell^2)}\\ &\lesssim \|F\|_{L^p(\ell^2)}.\end{aligned}$$

For (2.8), we first obtain the pointwise estimates of the form

$$|II_{\mathbf{n},\boldsymbol{\ell}}F(x)| \le \frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{k\in\mathcal{I}_2}2^{k\beta(1)}\Big|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k(y)-[\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k]_{Q_x}\Big||g_k(x,y)|\Big)\,dy$$
$$\le \Big(\sum_{k>0}2^{2k\beta(1)}\Big[\mathcal{M}(\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k)\Big]^2\Big)^{1/2}(x).$$

We therefore have

$$\|II^z_{\mathbf{n},\boldsymbol{\ell}}F\|_2 \le \Big(\sum_{k>0}2^{2k\beta(1)}\|\mathcal{M}(\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k)\|_2^2\Big)^{1/2}$$

$$\leq \Big(\sum_{k>0} 2^{2k\beta(1)}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}} f_k\|_2^2\Big)^{1/2}$$

$$\leq \sup_k \Big(2^{k\beta(1)}\|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}\|_{L^2\to L^2}\Big)\|F\|_{L^2(\ell^2)}$$

when $\Re(z) = 1$. The argument for $III^z_{\mathbf{n},\boldsymbol{\ell}}$ is exactly analogous.
For (2.9), we note that for $\Re(z) = 0$

$$II^z_{\mathbf{n},\boldsymbol{\ell}}F(x) \leq \frac{2}{|Q_x|}\int_{Q_x}\left(\sum_{k\in\mathcal{I}_2} 2^{2k\beta(0)}\left|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}} f_k(y)\right|^2\right)^{1/2} dy.$$

For each $x \in \mathbb{R}^d$, we put

$$\mathcal{U}_{\mathbf{n}}(x) = \bigcup_{k\in\mathcal{I}_2}\Big\{y : \rho(x-y+\gamma(s)) \lesssim \delta_x \text{ for some } s \in \operatorname{supp}(\varphi^k_{\mathbf{n}})\Big\}.$$

Lemma 2.6. *If $k \in \mathcal{I}_2$, then*

$$|\mathcal{U}_{\mathbf{n}}(x)| \lesssim \delta_x^{\beta(0)+1}.$$

Proof. By the definition of $\mathcal{I}_2$, $\delta_x \geq 2^{-k}2^{N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}$. We note that if $s = (s_1,\dots,s_{d-1}) \in \operatorname{supp}(\varphi^k_{\mathbf{n}})$, then $|s_i| \sim 2^{-n_i}$ for $i = 1,\dots,\mu$ and $|s_i| \sim 2^{-\frac{k}{m_i}+n_i}$ for $i = \mu+1,\dots,d-1$. We define $\mathcal{U}^1_{\mathbf{n}} \subset \mathbb{R}^{d-\mu-1}$ and $\mathcal{U}^2_{\mathbf{n}} \subset \mathbb{R}^{\mu+1}$ as

$$\mathcal{U}^1_{\mathbf{n}} = \{(z_{\mu+1},\dots,z_{d-1}) \,:\, |x_i - z_i + s_i| \lesssim \delta_x^{\frac{1}{m_i}}\}$$

and

$$\mathcal{U}^2_{\mathbf{n}} = \Big\{(z_1,\dots,z_\mu,z_d) \,: |x_i - z_i + s_i| \lesssim \delta_x (i = 1,\dots,\mu) \text{ and } \left|x_d - y_d + \sum_{i=1}^{d-1}|s_i|^{m_i}\right| \lesssim \delta_x\Big\}.$$

If $y = (y_1,\dots,y_d) \in \mathcal{U}_{\mathbf{n}}(x)$, then it is clear that there exists $s \in \operatorname{supp}(\varphi^k_{\mathbf{n}})$ such that

$$(y_{\mu+1},\dots,y_{d-1}) \in \mathcal{U}^1_{\mathbf{n}}$$

and

$$(y_1,\dots,y_\mu,y_d) \in \mathcal{U}^2_{\mathbf{n}}.$$

By using this one can easily see that

$$|\mathcal{U}_{\mathbf{n}}(x)| \leq |\mathcal{U}^1_{\mathbf{n}}| \times |\mathcal{U}^2_{\mathbf{n}}|.$$

Since for $(y_{\mu+1},\dots,y_{d-1}) \in \mathcal{U}^1_{\mathbf{n}}$

$$|x_i - y_i| \lesssim \delta_x^{\frac{1}{m_i}} + |s| \lesssim \delta_x^{\frac{1}{m_i}} + 2^{-\frac{k}{m_i}+n_i} \lesssim \delta_x^{\frac{1}{m_i}},$$

we obtain the size estimates for $\mathcal{U}^1_{\mathbf{n}}$

$$|\mathcal{U}^1_{\mathbf{n}}| \lesssim \delta_x^{\beta(0)}.$$

For $\mathcal{U}_{\mathbf{n}}^2$, we make use of a simple size estimates $|\mathcal{U}_{\mathbf{n}}^2| \lesssim \delta_x$ to obtain the desired inequality

$$|\mathcal{U}_{\mathbf{n}}(x)| \lesssim \delta_x^{\beta(0)+1}. \qquad \square$$

Now we turn to the proof of Lemma 2.5. We observe

$$II_{\mathbf{n},\boldsymbol{\ell}}^z F(x)| \leq 2\left[II_{\mathbf{n},\boldsymbol{\ell},1}F(x) + II_{\mathbf{n},\boldsymbol{\ell},2}F(x)\right],$$

where

$$II_{\mathbf{n},\boldsymbol{\ell},1}F(x) = \frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{k\in\mathcal{I}_2} 2^{2k\beta(0)}\big|\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k[\chi_{\mathcal{U}_{\mathbf{n}}(x)}f_k](y)\big|^2\Big)^{1/2}dy,$$
$$II_{\mathbf{n},\boldsymbol{\ell},2}F(x) = \frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{k\in\mathcal{I}_2} 2^{2k\beta(0)}\big|\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k[\chi_{\mathbb{R}^d\setminus\mathcal{U}_{\mathbf{n}}(x)}f_k](y)\big|^2\Big)^{1/2}dy.$$

For $II_{\mathbf{n},\boldsymbol{\ell},1}F(x)$ we use Lemma 2.6 to obtain

$$\begin{aligned}
II_{\mathbf{n},\boldsymbol{\ell},1}^z F(x) &\leq \Big(\frac{1}{|Q_x|}\int_{Q_x}\sum_{k\in\mathcal{I}_2} 2^{2k\beta(0)}\big|\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k[\chi_{\mathcal{U}_{\mathbf{n}}(x)}f_k](y)\big|^2 dy\Big)^{1/2} \\
&\lesssim \sup_{k\in\mathcal{I}_2}\Big(2^{k\beta(0)}\|\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k\|_{L^2\to L^2}\Big)\Big(\frac{1}{|Q_x|}\sum_k \|\chi_{\mathcal{U}_{\mathbf{n}}(x)}f_k\|_2^2\Big)^{1/2} \\
&\lesssim \sup_{k\in\mathcal{I}_2}\Big(2^{k\beta(0)}\|\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k\|_{L^2\to L^2}\Big)\left(\frac{|\mathcal{U}_{\mathbf{n}}(x)|}{|Q_x|}\right)^{\frac{1}{2}}\|F\|_{L^\infty(\ell^2)} \\
&\lesssim \sup_{k\in\mathcal{I}_2}\Big(2^{k\beta(0)}\|\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k\|_{L^2\to L^2}\Big)\delta_x^{-\frac{\mu}{2}}\|F\|_{L^\infty(\ell^2)} \\
&\lesssim 2^{-(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\|F\|_{L^\infty(\ell^2)}.
\end{aligned}$$

We now crudely estimate the terms $II_{\mathbf{n},\boldsymbol{\ell},2}^z F(x)$ and $III_{\mathbf{n},\boldsymbol{\ell}}^z F(x)$ when $\Re(z) = 0$. Let $\widetilde{\psi}$ be a smooth function whose Fourier transform is identically 1 on $|s| < 2$. Let

$$\widetilde{\psi}_{\boldsymbol{\ell}}^k(x) := 2^k\widetilde{\psi}(2^k x_d)\prod_{j=\mu+1}^{d-1}[2^{\frac{k}{m_j}+\ell_j}\eta(2^{\frac{k}{m_j}+\ell_j}x_j)] \tag{2.10}$$

then

$$\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k f(y) = \widetilde{\psi}_{\boldsymbol{\ell}}^k * d\sigma_{\mathbf{n}}^k * (\psi^k * f)(y) = \iint \widetilde{\psi}_{\boldsymbol{\ell}}^k(y-w-\gamma(s))(\psi^k * f)(w)\varphi_{\mathbf{n}}^k(s)dsdw.$$

For $y \in Q_x$ and $w \notin \mathcal{U}_{\mathbf{n}}(x)$, i.e., $\rho(x-\omega+\gamma(s)) \gtrsim \delta_x$, we have

$$\rho(y-w+\gamma(s)) \geq \rho(x-\omega+\gamma(s)) - \rho(x-y) \gtrsim \delta_x$$

for all $s \in \mathrm{supp}(\varphi_{\mathbf{n}}^k)$ and

$$\begin{aligned}
&|\mathcal{A}_{\mathbf{n},\boldsymbol{\ell}}^k[\chi_{\mathbb{R}^d\setminus\mathcal{U}_{\mathbf{n}}(x)}f_k](y)| \\
&\lesssim \iint_{\rho(y-w+\gamma(s))\gtrsim\delta_x} |\widetilde{\psi}_{\boldsymbol{\ell}}^k(y-w-\gamma(s))||\psi^k * f_k(w)||\varphi_{\mathbf{n}}^k(s)|\, dsdw
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{y,s}\Big(\int_{\rho(y-w+\gamma(s))\gtrsim\delta_x}|\widetilde{\psi}^k_{\boldsymbol{\ell}}(y-w-\gamma(s))|\,dw\Big)\|\varphi^k_{\mathbf{n}}\|_{L^1}\|f_k\|_{L^\infty}\\
&\lesssim (1+2^k\delta_x)^{-N}\|\varphi^k_{\mathbf{n}}\|_{L^1}\|f_k\|_{L^\infty}\\
&\lesssim (1+2^k\delta_x)^{-N}2^{-\beta(0)+\sum_{j=\mu+1}^{d-1}n_j}\|f_k\|_{L^\infty}.
\end{aligned}
$$

Therefore

$$
II^z_{\mathbf{n},\boldsymbol{\ell},2}F(x)\lesssim 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\|F\|_{\boldsymbol{\ell}^\infty(L^\infty)}\lesssim 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\|F\|_{L^\infty(\ell^2)}.
$$

Note that

(2.11)

$$
\begin{aligned}
&|T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}f_k(x)|\\
&\le 2^{\frac{k}{m}}\int_{Q_x}\big|\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k(y)-[\mathcal{A}^k_{\mathbf{n},\boldsymbol{\ell}}f_k]_{Q_x}\big|\frac{dy}{|Q_x|}\\
&\le 2^{\frac{k}{m}}\int_{Q_x}\int_{Q_x}\iint\Big|\widetilde{\psi^k}_{\boldsymbol{\ell}}(y-w-\gamma(s))-\widetilde{\psi}^k_{\boldsymbol{\ell}}(z-w-\gamma(s))\Big|\\
&\qquad\qquad |\psi^k*f_k(w)||\varphi^k_{\mathbf{n}}(s)|dsdw\frac{dz}{|Q_x|}\frac{dy}{|Q_x|}\\
&\le 2^{\frac{k}{m}}\sup_{y,z\in Q_x}\Big(\int\Big|\widetilde{\psi}^k_{\boldsymbol{\ell}}(y-w-\gamma(s))-\widetilde{\psi}^k_{\boldsymbol{\ell}}(z-w-\gamma(s))\Big|\,dw\Big)\|\varphi^k_n\|_{L^1}\|f_k\|_{L^\infty}.
\end{aligned}
$$

Lemma 2.7.

$$
\sup_{y,z\in Q_x}\Big(\int\Big|\widetilde{\psi}^k_{\boldsymbol{\ell}}(y-w-\gamma(s))-\widetilde{\psi}^k_{\boldsymbol{\ell}}(z-w-\gamma(s))\Big|\,dw\Big)\lesssim 2^{\boldsymbol{\ell}}\max_{j=1}^{2}(2^k\delta_x)^{a_j}.
$$

Proof. The proof follows by (2.10) and Mean Value Theorem. We omit the proof. □

By (2.11) and Lemma 2.7, we have

$$
\begin{aligned}
|III^z_{\mathbf{n},\boldsymbol{\ell}}F(x)| &\le \sum_{k>0:\ 2^k\delta_x\le 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}}|T^{k,z}_{\mathbf{n},\boldsymbol{\ell}}f_k(x)|\\
&\lesssim \sum_{k>0:\ 2^k\delta_x\le 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}}2^{\frac{k}{m}}(2^{\boldsymbol{\ell}}\max_{j=1}^{2}(2^k\delta_x)^{a_j})(2^{-\frac{k}{m}+n})\|f_k\|_{L^\infty}\\
&\lesssim 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\|f_k\|_{\boldsymbol{\ell}^\infty(L^\infty)}\lesssim 2^{-N(|\mathbf{n}|_1+|\boldsymbol{\ell}|_1)}\|F\|_{L^\infty(\ell^2)}.
\end{aligned}
$$

3. Necessary conditions

Let $2\le m_1\le\cdots\le m_{d-1}$. For $k=1,\dots,d-1$, let

$$
B_k:=\left(\frac{(1+\alpha(m_i))m_i-1}{(1+\alpha(m_i))m_i},\frac{m_i-1}{(1+\alpha(m_i))m_i}\right).
$$

Let $\Sigma(m_1,\dots,m_{d-1})$ be the convex polygonal region with vertices $(0,0)$, $(1,1)$, B_1, …, B_{d-1} and their dual points $B'_1,\dots,B'_{d-1}$. Let

$$
E_\sigma=\left\{(1/p,1/q):\|\mathcal{A}\|_{L^p\to L^q<\infty,1\le p,q\le\infty}\right\}.
$$

Then it is well-known that $E_\sigma \subset \Sigma(m_1, \ldots, m_{d-1})$ (see [2]).

Corollary 3.1. *$\mathcal{A}$ maps L^p into L^q if $(1/p, 1/q)$ belongs to the interior of $\Sigma(m_1, \ldots, m_{d-1})$.*

Proof. By Theorem 1.1, for each $i = 1, \ldots, d-1$, we have

$$\|\mathcal{P}^k \mathcal{A} f\|_{m_i} \lesssim 2^{-\alpha(m_i)k} \|f\|_{m_i}.$$

And the results follow by interpolating these estimates with the following the trivial estimates

$$\|\mathcal{P}^k \mathcal{A} f\|_\infty \lesssim \|\psi^k * d\sigma\|_\infty \|f\|_1 \lesssim 2^k \|f\|_1. \qquad \square$$

The necessary conditions of Theorem 1.1 is clear from the proof of Corollary 3.1 since $E_\sigma \subset \Sigma(m_1, \ldots, m_{d-1})$.

References

[1] M. Christ, *Failure of an endpoint estimate for integrals along curves*, Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992), 163–168, Stud. Adv. Math. CRC, Boca Raton, FL, 1995.

[2] E. Ferreyra, T. Godoy, and M. Urciuolo, *Endpoint bounds for convolution operators with singular measures*, Colloq. Math. **76** (1998), no. 1, 35–47.

[3] A. Iosevich, E. Sawyer, and A. Seeger, *On averaging operators associated with convex hypersurfaces of finite type*, J. Anal. Math. **79** (1999), 159–187.

[4] A. Nagel, A. Seeger, and S. Wainger, *Averages over convex hypersurfaces*, Amer. J. Math. **115** (1993), no. 4, 903–927.

[5] A. Seeger, *Some inequalities for singular convolution operators in L^p-spaces*, Trans. Amer. Math. Soc. **308** (1988), no. 1, 259–272.

[6] A. Seeger and T. Tao, *Sharp Lorentz space estimates for rough operators*, Math. Ann. **320** (2001), no. 2, 381–415.

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