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FINITE GROUPS WITH SOME SEMI-p-COVER-AVOIDING OR ss-QUASINORMAL SUBGROUPS

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Abstract. Suppose that G is a finite group and H is a subgroup of G. H is said to be an ss-quasinormal subgroup of G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B; H is said to be semi-p-cover-avoiding in G if there is a chief series $1 = G_0 < G_1 < \cdots < G_t = G$ of G such that, for every $i = 1, 2, \ldots, t$, if G_i/G_{i-1} is a p-chief factor, then H either covers or avoids G_i/G_{i-1}. We give the structure of a finite group G in which some subgroups of G with prime-power order are either semi-p-cover-avoiding or ss-quasinormal in G. Some known results are generalized.

1. Introduction

All groups considered in this paper are finite. G always means a group, $|G|$ denotes the order of G and $\pi(G)$ denotes the set of all primes dividing $|G|$.

Let $\mathscr{F}$ be a class of groups. We call $\mathscr{F}$ a formation, provided that (1) if $G \in \mathscr{F}$ and $H \trianglelefteq G$, then $G/H \in \mathscr{F}$, and (2) if G/M and G/N are in $\mathscr{F}$, then $G/(M \cap N)$ is in $\mathscr{F}$ for any normal subgroups M, N of G. A formation $\mathscr{F}$ is said to be saturated if $G/\Phi(G) \in \mathscr{F}$ implies that $G \in \mathscr{F}$. In this paper, $\mathscr{U}$ will denote the class of all supersolvable groups. Clearly, $\mathscr{U}$ is a saturated formation.

A famous topic in group theory is to study the influence of some subgroups with prime-power order on the structure of G. In [6], Li, Shen and Liu generalized s-quasinormal subgroups to ss-quasinormal subgroups. A subgroup H of G is said to be an ss-quasinormal subgroup of G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. Recently, Fan, Guo and Shum [1] introduced the semi-p-cover-avoiding property. A subgroup H of G is said to be semi-p-cover-avoiding in G if there is a chief

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series $1 = G_0 < G_1 < \cdots < G_t = G$ of G such that H either covers or avoids G_i/G_{i-1} whenever G_i/G_{i-1} is a p-chief factor.

Some interesting results have been obtained about the structure of a group G under assumption that some subgroups of G are ss-quasinormal or semi-p-cover-avoiding in G (see: [1, 3, 5, 6]).

There are examples to show ss-quasinormal and semi-p-cover-avoiding are two different properties of subgroups.

Example 1.1. Let $G = A_5$, the alternative group of degree 5. Then A_4 is an ss-quasinormal subgroup of G but not semi-p-cover-avoiding in G.

Example 1.2 ([2, Example 2.4])**.** Let A_4 be the alternative group of degree 4 and $C_2 = \langle c \rangle$ a cyclic group of order 2, generated by an element c. Let $G = C_2 \times A_4$. Then $A_4 = K_4 \cdot \langle t \rangle$, where $K_4 = \langle a, b \rangle$ is the Klein four-group with generators a and b of order 2 and $\langle t \rangle$ is a cyclic group of order 3. Take $H = \langle ac \rangle$ to be the subgroup of G generated by ac. It is clear that the following series

$$1 < K_4 < A_4 < C_2 \times A_4 = G$$

is a chief series of G such that H covers $G = A_4$ and avoids the rest. This is to say that H has the semi-cover-avoiding property in G. Of course, H is semi-p-cover-avoiding in G. However, H is not ss-quasinormal in G.

The aim of this article is to unify and improve some earlier results using ss-quasinormal and semi-p-cover-avoiding subgroups. Our main theorems are as follows:

Theorem 3.1. *Let G be a group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Let P be a Sylow p-subgroup of G. If all maximal subgroups of P are either semi-p-cover-avoiding or ss-quasinormal subgroups in G, then G is p-nilpotent.*

Theorem 3.6. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are either semi-p-cover-avoiding or ss-quasinormal in G, then $G \in \mathscr{F}$.*

Theorem 3.7. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are either semi-p-cover-avoiding or ss-quasinormal in G, then $G \in \mathscr{F}$.*

2. Basic definitions and preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 ([6])**.** *Let H be an ss-quasinormal in a group G, $K \leq G$ and N a normal subgroup of G.*

(i) *If $H \leq K$, then H is ss-quasinormal in K;*

(ii) *HN/N is ss-quasinormal in G/N;*

(iii) *If* $N \leq K$ *and* K/N *is ss-quasinormal in* G/N, *then* K *is ss-quasinormal in* G;

(iv) *If* K *is quasinormal in* G, *then* HK *is ss-quasinormal in* G.

Lemma 2.2 ([1, 3])**.** *Let* H *be a semi-p-cover-avoiding subgroup of a group* G *and* N *a normal subgroup of* G. *Then*

(i) H *is semi-p-cover-avoiding in* K *for every subgroup* K *of* G *with* $H \leq K$;

(ii) HN/N *is semi-p-cover-avoiding in* G *if one of the following holds*:

(1) $N \subseteq H$;

(2) $gcd(|H|, |N|) = 1$, *where* $\gcd(\cdot, \cdot)$ *denotes the greatest common divisor.*

Lemma 2.3 ([7])**.** *Let* G *be a group and* p *a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *Let* P *be a Sylow* p*-subgroup of* G. *If all maximal subgroups of* P *are semi-p-cover-avoiding or* s*-quasinormally embedded subgroups in* G, *then* G *is* p*-nilpotent.*

Lemma 2.4 ([8])**.** *Let* G *be a group and* p *a prime dividing* $|G|$ *with* $(|G|, p-1) = 1$.

(i) *If* N *is normal in* G *of order* p, *then* $N \leq Z(G)$;

(ii) *If* G *has a cyclic Sylow* p*-subgroup, then* G *is* p*-nilpotent*;

(iii) *If* $M \leq G$ *and* $[G : M] = p$, *then* $M \trianglelefteq G$.

Lemma 2.5 ([6])**.** *Let* H *be a nilpotent subgroup of* G. *Then the following statements are equivalent.*

(i) H *is* s*-quasinormal in* G;

(ii) $H \leq F(G)$ *and* H *is ss-quasinormal in* G;

(iii) $H \leq F(G)$ *and* H *is* s*-quasinormally embedded in* G.

Lemma 2.6 ([7])**.** *Let* $\mathscr{F}$ *be a saturated formation containing* $\mathscr{U}$, G *is a group with a normal subgroup* H *such that* $G/H \in \mathscr{F}$. *Then* $G \in \mathscr{F}$ *if one of the following holds*:

(i) *all maximal subgroups of all non-cyclic Sylow subgroups of* H *are either semi-p-cover-avoiding or* s*-quasinormally embedded in* G;

(ii) *all maximal subgroups of all non-cyclic Sylow subgroups of* $F^*(H)$ *are either semi-p-cover-avoiding or* s*-quasinormally embedded in* G.

Lemma 2.7 ([4, X.13])**.** *Let* G *be a group and* M *a subgroup of* G.

(i) *If* M *is normal in* G, *then* $F^*(M) \leq F^*(G)$;

(ii) $F^*(G) \neq 1$ *if* $G \neq 1$; *in fact,* $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$;

(iii) $F^*(F^*(G)) = F^*(G) \geq F(G)$; *if* $F^*(G)$ *is solvable, then* $F^*(G) = F(G)$.

3. Main results

Theorem 3.1. *Let* G *be a group and* p *a prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *Let* P *be a Sylow* p*-subgroup of* G. *If all maximal subgroups of* P *are either semi-p-cover-avoiding or ss-quasinormal subgroups in* G, *then* G *is* p*-nilpotent.*

Proof. If all maximal subgroups of P are semi-p-cover-avoiding in G, then G is p-nilpotent by Lemma 2.3. Hence there exists a maximal subgroup P_1 of P such that P_1 is ss-quasinormal in G. Firstly, fix an H which is a maximal subgroup of P such that H is ss-quasinormal in G.

Now we prove that there exists a Hall p'-subgroup K of G such that HK is a subgroup of index p in G.

By conditions, there is a subgroup $B \leq G$ such that $G = HB$ and $HX = XH$ for all $X \in \mathrm{Syl}(B)$, and $H \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $H \cap B$. Thus $S \nsubseteq H$ and $S \cap H = B \cap H$ for all $S \in \mathrm{Syl}_p(B)$. So $B \cap H = \bigcap_{b \in B} (S^b \cap H) \leq \bigcap_{b \in B} S^b = O_p(B)$.

We claim that B has a Hall p'-subgroup. Because $|O_p(B) : B \cap H| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. As $(|G|, p-1)$=1, then $B/O_p(B)$ is p-nilpotent by Lemma 2.4, and hence B is p-solvable. So B has a Hall p'-subgroup. Thus the claim holds. Now, let K be a Hall p'-subgroup of B. $\pi(K) = \{p_2, \ldots, p_s\}$ and $P_i \in \mathrm{Syl}_{p_i}(K)$. By the conditions, H is ss-quasinormal in G, so H permute with subgroup $\langle P_2, \ldots, P_s \rangle = K$ and $HK \leq G$. Moreover, $[G : HK] = p$ as desired.

Now, for every H_i which is a maximal subgroup of P (H_i is ss-quasinormal in G), there exists a Hall p'-subgroup K_i of G such that $M_i = H_iK_i$, which is a subgroup of index p in G. As $(|G|, p-1) = 1$, by Lemma 2.4, $M_i \trianglelefteq G$. Obviously, H_i is s-quasinormally embedded in G. Thus every maximal subgroup of G is either semi-p-cover-avoiding or s-quasinormally embedded in G. By Lemma 2.3, G is p-nilpotent. □

Corollary 3.2. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If all maximal subgroups of P are semi-p-cover-avoiding or ss-quasinormal subgroups in G, then G is p-nilpotent.*

Corollary 3.3. *Suppose that G is a group. If all maximal subgroups of all Sylow subgroups of G are either semi-p-cover-avoiding or ss-quasinormal in G, then G has Sylow tower of supersolvable type.*

Proof. It is clear that $(|G|, p-1) = 1$, if p is the smallest prime dividing $|G|$. By the hypothesis, all maximal subgroups of all Sylow subgroups of G are either semi-p-cover-avoiding or ss-quasinormal in G, so G satisfies the condition of Theorem 3.1, and hence G is p-nilpotent. Let U be the normal p-complement of G, then U satisfies the condition by induction, hence G possesses Sylow tower property of supersolvable type. □

Theorem 3.4. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If for every prime p dividing $|H|$ and $P \in Syl_p(H)$, If all maximal subgroups of P are either semi-p-cover-avoiding or ss-quasinormal in G, then $G \in \mathscr{F}$.*

Proof. Assume that the theorem is not true and let G be a minimal counter-example.

(1) H has minimal normal subgroup H_1, $H_1 \leq Q \leq H$, $Q \in \mathrm{Syl}_q(H)$ and q is the largest prime in $\pi(H)$.

Obviously, H satisfies the condition of Corollary 3.3, so H possesses Sylow tower property of supersolvable type. Let q is the largest prime dividing $|H|$ and Q is a Sylow q-subgroup of H, then $Q \trianglelefteq H$, so H has minimal normal subgroup H_1, $H_1 \leq Q$ and H_1 is an elementary abelian q-group, as desired.

(2) $G/H_1 \in \mathscr{F}$, $H_1 \nleq \Phi(G)$, $H_1 = Q \in \mathrm{Syl}_q(H)$.

Obviously, $G/H_1 \in \mathscr{F}$. Since $\mathscr{F}$ is a saturated formation, so H_1 is the unique minimal normal subgroup of G containing in H, $H_1 \nleq \Phi(G)$. Moreover, $H_1 = F(H)$. Since H is solvable, so $C_H(H_1) \leq F(H)$ and $C_H(H_1) = H_1 = F(H)$. Since $Q \trianglelefteq H$, $Q \leq F(H)$, thus $H_1 = Q \in \mathrm{Syl}_q(H)$.

(3) The final contradiction.

For any maximal subgroup Q_1 of Q, Q_1 is either semi-p-cover-avoiding or ss-quasinormal in G by (2) and the hypothesis. Thus Q_1 is either semi-p-cover-avoiding or s-quasinormally embedded in G by Lemma 2.5. Hence $G \in \mathscr{F}$ by Lemma 2.6(i). We get the final contradiction. □

Corollary 3.5. *Let G be a group, H a normal subgroup of G such that G/H is supersolvable. If all maximal subgroups of all Sylow subgroups of H are either semi-p-cover-avoiding or ss-quasinormal in G, then G is supersolvable.*

Theorem 3.6. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are either semi-p-cover-avoiding or ss-quasinormal in G, then $G \in \mathscr{F}$.*

Proof. As H is solvable, by Lemma 2.7, $F(H) = F^*(H)$. For any Sylow subgroup P of $F(H)$ and for any maximal subgroup P_1 of P, if P_1 is ss-quasinormal in G, then P_1 is s-quasinormally embedded in G by Lemma 2.5. Applying Lemma 2.6(ii), we can get $G \in \mathscr{F}$. □

Theorem 3.7. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. If all maximal subgroups of all Sylow subgroups of $F^*(H)$ are either semi-p-cover-avoiding or ss-quasinormal in G, then $G \in \mathscr{F}$.*

Proof. Suppose that the theorem is false and let G be a minimal counter-example.

Case 1. $\mathscr{F} = \mathscr{U}$.

Let G be a minimal counter-example.

(1) Every proper normal subgroup N of G containing $F^*(H)$ is supersolvable.

Since $N/N \cap H \cong NH/H$ is supersolvble, we get $F^*(H) = F^*(F^*(H)) \leq F^*(N \cap H) \leq F^*(H)$ by Lemma 2.7. So $F^*(H) = F^*(N \cap H)$ and N, $N \cap H$ satisfy the hypothesis of the theorem. Hence N is supersolvable by the minimal choice of G.

(2) $H = G$ and $1 \neq F^*(G) = F(G) < G$.

If $H < G$, then H is supersolvable as H contains $F^*(H)$ and $F^*(H) = F(H)$, it follows that G is supersolvable by Theorem 3.6, a contradiction.

If $F^*(G) = G$, then G is supersolvable by applying Corollary 3.5, a contradiction. Thus $F^*(G) < G$, it is supersolvable by (1), so $F^*(G) = F(G) \neq 1$ by Lemma 2.7.

(3) The final contradiction.

For any Sylow subgroup P of $F^*(H)$ and for any maximal subgroup P_1 of P, P_1 is either semi-p-cover-avoiding or ss-quasinormal in G by the hypothesis. As $P_1 \leq F(G)$, so P_1 is either semi-p-cover-avoiding or s-quasinormally embedded in G by Lemma 2.5. Applying Lemma 2.6(ii), we can get G is supersolvable, the final contradiction.

Case 2. $\mathscr{F} \neq \mathscr{U}$.

By Case 1, H is supersolvable, so H is solvable and $F^*(H) = F(H)$ by Lemma 2.7. Then $G \in \mathscr{F}$ by Theorem 3.6. $\square$

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