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REPRESENTATIONS OF SOLUTIONS TO PERIODIC CONTINUOUS LINEAR SYSTEM AND DISCRETE LINEAR SYSTEM

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Abstract. We give a representation of the component of solutions with characteristic multiplier 1 in a periodic linear inhomogeneous continuous system. It follows from this representation that asymptotic behaviors of the component of solutions to the system and to its associated homogeneous system are quite different, though they are similar in the case where the characteristic multiplier is not 1. Moreover, the representation is applicable to linear discrete systems with constant coefficients.

1. Introduction and preliminaries

We consider periodic linear inhomogeneous differential equations of the form

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t), \quad x(0) = w, \tag{1}$$

where $A(t)$ is a periodic continuous $p \times p$ matrix function with period $\tau > 0$ and $f : \mathbb{R} \to \mathbb{C}^p$ a τ-periodic continuous function. Here $\mathbb{C}$ is the set of all complex numbers and $\mathbb{R}$ the set of all real numbers.

Representations of solutions to the equation (1) have been given in [2, 5, 6]. These reformulate the variation of constants formula into the sum of a τ-periodic function and an exponential-like function. In particular, the representation [6] of the component of solutions with characteristic multiplier $\mu \neq 1$ shows that asymptotic behavior of the component of solutions to the equation (1) is the sum of asymptotic behavior of the component of some solution to the homogeneous equation associated with the equation (1) and the τ-periodic solution decided from the equation (1). Roughly speaking, asymptotic behavior of the component of solutions to the system and to the homogeneous system associated with the equation (1) are similar (refer to Remark 3.4). However, it is predicted that in the case where the characteristic multiplier is 1, they

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are not similar. In order to verify this predict, we must give another representation (Theorem 3.3) of the component of solutions to the equation (1). The purpose of this paper is to give another representation. The proof is based on a property (Theorem 2.2) of factorial functions, which is closely related to the Vandermonde equality. Moreover, we will discuss the same problem for discrete linear systems.

In order to state our results, we will state shortly some notations and basic facts used in linear algebra. We set $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. For a complex $p \times p$ matrix H we denote by $\sigma(H)$ the set of all eigenvalues of H and by $G_H(\eta) = \mathbf{N}((H - \eta E)^{h_H(\eta)})$ the generalized eigenspace corresponding to $\eta \in \sigma(H)$, where E is the unit $p \times p$ matrix and $h_H(\eta)$ the geometric multiplicity of $\eta \in \sigma(H)$. $Q_\eta(H) : \mathbb{C}^p \to G_H(\eta)$ stands for the projection corresponding to the direct sum decomposition

$$\mathbb{C}^p = \bigoplus_{\eta \in \sigma(H)} G_H(\eta).$$

Now, we assume that H is nonsingular, that is, $H = e^{\tau L}$, $\tau > 0$ for a complex $p \times p$ matrix L. By the spectral mapping theorem it is easy to see that $\sigma(H) = e^{\tau \sigma(L)}$ and $\sigma_\mu(L) := \{\lambda \in \sigma(L) \mid \mu = e^{\tau \lambda}\} \neq \emptyset$ for every $\mu \in \sigma(H)$. Moreover, the following relations hold:

$$h_H(\mu) = \max\{h_L(\lambda) \mid \lambda \in \sigma_\mu(L)\}, \quad G_H(\mu) = \bigoplus_{\lambda \in \sigma_\mu(L)} G_L(\lambda),$$

$$HQ_\lambda(L) = Q_\lambda(L)H \quad \text{and} \quad Q_\mu(H) = \sum_{\lambda \in \sigma_\mu(L)} Q_\lambda(L).$$

Set

$$L_{k,\lambda} = \frac{\tau^k}{k!}(L - \lambda E)^k \quad (\lambda \in \sigma(L)), \quad H_{[k,\mu]} = \frac{1}{k!\mu^k}(H - \mu E)^k \quad (\mu \in \sigma(H))$$

and define matrix functions as

$$Y_\lambda(L) = \sum_{k=0}^{h_L(\lambda)-1} B_k L_{k,\lambda} \quad (\lambda \in i\omega\mathbb{Z} \cap \sigma(L)),$$

where $\omega = 2\pi/\tau$ and $B_0 = 1, B_k, k = 1, 2, \ldots$, are Bernoulli's numbers (refer to [4]). For $\lambda \in \sigma(L)$ and $\mu = 1 \in \sigma(H)$ we set

$$\beta_\lambda(w, b; L) = \tau(L - \lambda E)Q_\lambda(L)w + Y_\lambda(L)Q_\lambda(L)b \quad (\lambda \in i\omega\mathbb{Z})$$

and

$$\delta(w, b; H) = (H - E)Q_1(H)w + Q_1(H)b \quad (\mu = 1),$$

where $w, b \in \mathbb{C}^p$.

2. Factorial functions

To find representations of the component of solutions to periodic continuous linear systems and to discrete linear systems, some results on factorial functions are needed. If $x \in \mathbb{R}$ and $k \in \mathbb{N}_0$, then we define the well-known factorial function $(x)_k$ as

$$(x)_k = \begin{cases} 1, & (k = 0) \\ x(x-1)(x-2)\cdots(x-k+1) & (k \in \mathbb{N}). \end{cases}$$

In particular, if $x = n$ is a positive integer, then

$$\frac{(n)_k}{k!} = \binom{n}{k} := \frac{n!}{k!(n-k)!}, \quad (n)_k = 0 \quad (k > n).$$

Note that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = \begin{cases} 1 & (n = 0) \\ 0 & (n \geq 1). \end{cases} \tag{2}$$

The equality

$$\sum_{k=0}^{n} \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n} \quad (n, r, s \in \mathbb{Z}) \tag{3}$$

is called the Vandermonde equality (refer to [1]).

The following lemma is closely related to the Vandermonde equality (3).

Lemma 2.1 ([1])**.** *Let $x, y \in \mathbb{R}$. Then*

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k} = \sum_{k=0}^{n} \binom{n}{k} (x)_{n-k} (y)_k, \quad n = 0, 1, 2, \ldots.$$

In particular, if $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then

$$(x+m)_n = \sum_{k=0}^{n} \binom{n}{k} (m)_k (x)_{n-k} = \sum_{k=0}^{m} \binom{m}{k} (n)_k (x)_{n-k}.$$

Theorem 2.2. *Let $x \in \mathbb{R}$ and $i \in \mathbb{N}_0$. Then*

$$\sum_{j=0}^{i} (-1)^j \binom{i+1}{j+1} (x+j)_i = (x-1)_i. \tag{4}$$

Proof. Using Lemma 2.1 we have

$$\begin{aligned} \sum_{j=0}^{i} (-1)^j \binom{i+1}{j+1} (x+j)_i &= \sum_{j=0}^{i} (-1)^j \binom{i+1}{j+1} (x-1+j+1)_i \\ &= \sum_{j=0}^{i} (-1)^j \binom{i+1}{j+1} \sum_{k=0}^{i} \binom{i}{k} (j+1)_k (x-1)_{i-k}. \end{aligned}$$

Note that if $k \geq j+2$, $(j+1)_k = 0$ holds. Set $i \vee j = \max\{i, j\}$, $i \wedge j = \min\{i, j\}$. Then

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^j \binom{i+1}{j+1} \sum_{k=0}^{i} \binom{i}{k} (j+1)_k (x-1)_{i-k} \\
&= \sum_{j=0}^{i}(-1)^j \binom{i+1}{j+1} \sum_{k=0}^{i \wedge (j+1)} \binom{i}{k} (j+1)_k (x-1)_{i-k} \\
&= \sum_{k=0}^{i} \sum_{j=(k-1)\vee 0}^{i} (-1)^j \binom{i+1}{j+1} (j+1)_k \binom{i}{k} (x-1)_{i-k} \\
&= \sum_{k=0}^{i} \sum_{j=(k-1)\vee 0}^{i} (-1)^j \binom{i+1}{j+1} \binom{j+1}{k} (i)_k (x-1)_{i-k}.
\end{aligned}$$

Since

$$\binom{i+1}{j+1}\binom{j+1}{k} = \binom{i+1}{k}\binom{i+1-k}{j+1-k},$$

it yields that

$$\begin{aligned}
&\sum_{k=0}^{i} \sum_{j=(k-1)\vee 0}^{i} (-1)^j \binom{i+1}{j+1} \binom{j+1}{k} (i)_k (x-1)_{i-k} \\
&= \sum_{k=0}^{i} \binom{i+1}{k} \sum_{j=(k-1)\vee 0}^{i} (-1)^j \binom{i+1-k}{j+1-k} (i)_k (x-1)_{i-k}.
\end{aligned}$$

If $k = 0$ in the above, then

$$\begin{aligned}
&\sum_{j=0}^{i}(-1)^j \binom{i+1-k}{j+1-k} (i)_k (x-1)_{i-k} \\
&= \sum_{j=0}^{i}(-1)^j \binom{i+1}{j+1} (i)_0 (x-1)_i = \sum_{m=1}^{i+1} (-1)^{m-1} \binom{i+1}{m} (x-1)_i \\
&= -\left(\sum_{m=0}^{i+1} (-1)^m \binom{i+1}{m} - 1 \right) (x-1)_i = (x-1)_i,
\end{aligned}$$

because of (2). If $k \geq 1$, then

$$\sum_{j=k-1}^{i} (-1)^j \binom{i+1-k}{j+1-k} = \sum_{m=0}^{i+1-k} (-1)^{m-1+k} \binom{i+1-k}{m} = 0.$$

Therefore, the equality (4) holds. $\square$

From (4) we have

$$\sum_{j=0}^{i}(-1)^j\binom{i+1}{j+1}x\,(x+j)_i=(x)_{i+1}. \tag{5}$$

Corollary 2.3. *Let $x\in\mathbb{R}$ and $i\in\mathbb{N}_0$. Then*

$$\sum_{j=0}^{i}(-1)^j\binom{x}{i-j}\binom{x+j}{j+1}=\binom{x}{i+1}. \tag{6}$$

Proof. If $i=0$, then it is obvious. Let $i\geq 1$. Then it follows from (5) that

$$(x)_{i+1}=\sum_{j=0}^{i}(-1)^j\binom{i+1}{j+1}x(x+j)_i$$

holds. Dividing by $(i+1)!$ the both sides of the above relation, we have

$$\binom{x}{i+1}=\frac{1}{(i+1)!}\sum_{j=0}^{i}(-1)^j\binom{i+1}{j+1}x(x+j)_i=\sum_{j=0}^{i}(-1)^j\frac{(x+j)_i x}{(j+1)!(i-j)!}.$$

If $j=i$, then $(x+i)_i x=(x+i)_{i+1}$; if $j<i$, then $(x+j)_i x=(x+j)_{j+1}(x)_{i-j}$. Therefore, the equality (6) holds. □

3. The case of periodic continuous linear systems

First, we shall give another representation of the component of solutions of the equation (1). The solution operators $U(t,s)$ of the homogeneous equation

$$\frac{d}{dt}x(t)=A(t)x(t) \tag{7}$$

is defined as $U(t,s)w=u(t;s,w),\ w\in\mathbb{C}^p$, where $u(t;s,w)$ stands for the unique solution of the equation (7) with the initial condition $u(s)=w\in\mathbb{C}^p$. Since $U(\tau,0)$ is a nonsingular matrix, we can take a matrix A such that $U(\tau,0)=e^{\tau A}$. Thus the representation by Floquet is expressed as follows: $U(t,0)=P(t)e^{tA}$, where $P(t)$ is a τ-periodic function. Define the well-known periodic map $V(t),t\in\mathbb{R}$ by $V(t)=U(t+\tau,t)$. Set $Q_\mu(t)=Q_\mu(V(t)),t\in\mathbb{R}$ and

$$R_1(t)=P(t)\sum_{\lambda\in\sigma_1(A)}e^{t\lambda}Q_\lambda(A),$$

where $\sigma_1(A)=\{\lambda\in\sigma(A)\mid 1=e^{\tau\lambda}\}$. Then $R_1(t)$ is τ-periodic and $R_1(n\tau)=R_1(0)=Q_1(0),n\in\mathbb{Z}$.

The two lemmas below are concerned with representations of solutions to the equations (7) and (1).

Lemma 3.1 ([6])**.** *Let $\mu = 1 \in \sigma(V(0))$. Then*

$$U(t,0)Q_1(0) = R_1(t) \sum_{k=0}^{h_{V(0)}(1)-1} \left(\frac{t}{\tau}\right)_k V(0)_{[k,1]} Q_1(0).$$

Set

$$b_f = \int_0^\tau U(\tau, s) f(s) ds \quad \text{and} \quad \delta(w, b_f) := \delta(w, b_f; V(0)).$$

Lemma 3.2 ([6])**.** *Let $\mu = 1 \in \sigma(V(0))$. Then the component $Q_1(t)x(t)$ of solutions $x(t)$ of the equation* (1) *satisfying the initial condition $x(0) = w$ is expressed by*

$$(8) \qquad Q_1(t)x(t) = R_1(t) \sum_{k=0}^{h_{V(0)}(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{(k+1)!} (V(0) - E)^k \delta(w, b_f) + R_1(t)Q_1(0)w + h_1(t,f) \quad (t \in \mathbb{R}),$$

where $h_1(t,f)$ is a τ-periodic continuous function.

We are now in a position to state a main theorem in this paper.

Theorem 3.3. *Let $\mu = 1 \in \sigma(V(0))$. Then the component $Q_1(t)x(t)$ of solutions $x(t)$ of the equation* (1) *satisfying the initial condition $x(0) = w$ is expressed by*

$$(9) \quad Q_1(t)x(t) = U(t,0) \left(\sum_{k=0}^{h_{V(0)}(1)-1} \left(-\frac{t}{\tau}\right)_{k+1} \frac{(-1)}{(k+1)!} (V(0) - E)^k \delta(w, b_f) \right) + R_1(t)Q_1(0)w + h_1(t,f) \quad (t \in \mathbb{R}).$$

Proof. In view of (8), it suffices to prove the following equality

$$(10) \qquad U(t,0) \sum_{k=0}^{h_{V(0)}(1)-1} \left(-\frac{t}{\tau}\right)_{k+1} \frac{(-1)}{(k+1)!} (V(0) - E)^k Q_1(0) = R_1(t) \sum_{k=0}^{h_{V(0)}(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{(k+1)!} (V(0) - E)^k Q_1(0).$$

Put $h(1) = h_{V(0)}(1)$. Since Lemma 3.1 implies that

$$U(t,0)Q_1(0) = R_1(t) \sum_{k=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_k \frac{1}{k!} (V(0) - E)^k Q_1(0),$$

the left side of (10) may be rewritten as

$$U(t,0) \sum_{k=0}^{h(1)-1} \left(-\frac{t}{\tau}\right)_{k+1} \frac{(-1)}{(k+1)!} (V(0) - E)^k Q_1(0)$$

$$= \left(R_1(t) \sum_{k=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_k \frac{1}{k!}(V(0)-E)^k \right) \times \left(\sum_{j=0}^{h(1)-1} \left(-\frac{t}{\tau}\right)_{j+1} \frac{(-1)}{(j+1)!}(V(0)-E)^j Q_1(0) \right).$$

Therefore we have

$$\begin{aligned}
&\sum_{k=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_k \frac{1}{k!}(V(0)-E)^k \sum_{j=0}^{h(1)-1} \left(-\frac{t}{\tau}\right)_{j+1} \frac{1}{(j+1)!}(V(0)-E)^j Q_1(0) \\
&= \sum_{k=0}^{h(1)-1} \sum_{j=0}^{h(1)-1} \frac{1}{k!(j+1)!} \left(\frac{t}{\tau}\right)_k \left(-\frac{t}{\tau}\right)_{j+1} (V(0)-E)^{k+j} Q_1(0) \\
&= \sum_{j=0}^{h(1)-1} \sum_{k=0}^{h(1)-1} \frac{1}{k!(j+1)!} \left(\frac{t}{\tau}\right)_k \left(-\frac{t}{\tau}\right)_{j+1} (V(0)-E)^{k+j} Q_1(0) \\
&= \sum_{j=0}^{h(1)-1} \sum_{i=j}^{j+h(1)-1} \frac{1}{(i-j)!(j+1)!} \left(\frac{t}{\tau}\right)_{i-j} \left(-\frac{t}{\tau}\right)_{j+1} (V(0)-E)^{i} Q_1(0) \\
&= \sum_{j=0}^{h(1)-1} \sum_{i=j}^{h(1)-1} \frac{1}{(i-j)!(j+1)!} \left(\frac{t}{\tau}\right)_{i-j} \left(-\frac{t}{\tau}\right)_{j+1} (V(0)-E)^{i} Q_1(0) \\
&= \sum_{i=0}^{h(1)-1} \sum_{j=0}^{i} \frac{1}{(i-j)!(j+1)!} \left(\frac{t}{\tau}\right)_{i-j} \left(-\frac{t}{\tau}\right)_{j+1} (V(0)-E)^{i} Q_1(0).
\end{aligned}$$

Using the equality

$$-\left(\frac{t}{\tau}\right)_{i-j} \left(-\frac{t}{\tau}\right)_{j+1} = (-1)^j \frac{t}{\tau} \left(\frac{t}{\tau}+j\right)_i$$

and the equality (5), we obtain

$$\begin{aligned}
&- \sum_{i=0}^{h(1)-1} \sum_{j=0}^{i} \frac{1}{(i-j)!(j+1)!} \left(\frac{t}{\tau}\right)_{i-j} \left(-\frac{t}{\tau}\right)_{j+1} (V(0)-E)^{i} Q_1(0) \\
&= \sum_{i=0}^{h(1)-1} \sum_{j=0}^{i} (-1)^j \frac{1}{(i-j)!(j+1)!} \frac{t}{\tau} \left(\frac{t}{\tau}+j\right)_i (V(0)-E)^{i} Q_1(0) \\
&= \sum_{i=0}^{h(1)-1} \left(\frac{t}{\tau} \sum_{j=0}^{i} (-1)^j \binom{i+1}{j+1} \left(\frac{t}{\tau}+j\right)_i \right) \frac{1}{(i+1)!} (V(0)-E)^{i} Q_1(0) \\
&= \sum_{i=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_{i+1} \frac{1}{(i+1)!} (V(0)-E)^{i} Q_1(0),
\end{aligned}$$

which proves the theorem. □

Remark 3.4. Note that a representation of the component solution $Q_\mu(t)x(t)$ for $\mu(\neq 1) \in \sigma(V(0))$ to the equation (1) is given by

$$Q_\mu(t)x(t) = U(t,0)\left(Q_\mu(0)w - \sum_{k=0}^{h_{V(0)}(\mu)-1} \frac{1}{(1-\mu)^{k+1}}(V(0)-\mu E)^k Q_\mu(0)b_f\right) + h_\mu(t,b_f),$$

where $h_\mu(t,b_f)$ is a τ-periodic solution of the equation (1) in $G_{V(t)}(\mu)$ (refer to [6]). This shows that asymptotic behaviors of the component of solutions to the equation (1) and to its associated homogeneous equation is similar. However, the representation (9) of the component of solutions with characteristic multiplier 1 shows that they are quite different.

Corollary 3.5. *If* $h_{V(0)}(1) = 1$ *in Theorem* 3.3*, then* (9) *becomes*

$$Q_1(t)x(t) = U(t,0)\left(\frac{t}{\tau}\delta(w,b_f) + Q_1(0)w\right) + h_1(t,f) \quad (t \in \mathbb{R}).$$

Next, we consider the special case where $A(t) = A$, a constant matrix, in the equation (1), that is,

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \quad x(0) = w. \tag{11}$$

Then the solution operator is given by $U(t,s) = e^{(t-s)A}$ for all $t, s \in \mathbb{R}$. Put

$$\beta_\lambda(w,b_f) := \beta_\lambda(w,b_f;A) \quad (\lambda \in i\omega\mathbb{Z}).$$

Lemma 3.6 ([6])**.** *Let* $\lambda \in \sigma_1(A)$*. Then*

$$\sum_{k=0}^{h_A(\lambda)-1} t^{k+1}\frac{1}{k+1}A_{k,\lambda}\beta_\lambda(w,b_f) = \sum_{k=0}^{h_{V(0)}(1)-1} (t)_{k+1}\frac{1}{k+1}V(0)_{[k,1]}P_\lambda\delta(w,b_f)$$

for all $t \in \mathbb{R}$.

Combining Theorem 3.3 with Lemma 3.6, we have the following result.

Theorem 3.7. *Let* $\lambda \in i\omega\mathbb{Z}$*. Then the component* $P_\lambda x(t)$ *of the solution* $x(t)$ *of the equation* (11) *is expressed by*

$$P_\lambda x(t) = e^{tA}\sum_{k=0}^{h_A(\lambda)-1} \frac{(-1)^k t^{k+1}}{\tau(k+1)!}(A-\lambda E)^k\beta_\lambda(w,b_f) + e^{\lambda t}P_\lambda w + r_\lambda(t,f),$$

where $r_\lambda(t,f)$ *is a* τ*-periodic continuous function.*

4. The case of discrete linear systems

Next, we consider linear difference equations of the form

$$x_{n+1} = Bx_n + b, \quad x_0 = w, \quad n \in \mathbb{N}_0, \tag{12}$$

where B is a complex $p \times p$ matrix and $b \in \mathbb{C}^p$. Set $Q_\mu = Q_\mu(B)$. If $B = e^{\tau A}$, then we define the projection P_λ by $P_\lambda = Q_\lambda(A)$. The following lemma is well-known.

Lemma 4.1. *Let B be nonsingular and $\mu \in \sigma(B)$. Then*

$$B^n Q_\mu = \mu^n \sum_{k=0}^{h_B(\mu)-1} \binom{n}{k} \mu^{-k}(B - \mu E)^k Q_\mu, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{13}$$

Lemma 4.2 ([6])**.** *Let $\mu = 1 \in \sigma(B)$ and $n \in \mathbb{N}_0$. Then the component $Q_1 x_n(w,b)$ of the solution $x_n(w,b)$ of the equation* (12) *is expressed by*

$$Q_1 x_n(w,b) = \sum_{k=0}^{h_B(1)-1} \binom{n}{k+1} (B - E)^k \delta(w,b) + Q_1 w. \tag{14}$$

The following theorems are discrete versions of Theorem 3.3 and Theorem 3.7.

Theorem 4.3. *Let B be nonsingular. The component $Q_1 x_n(w,b)$ of the solution $x_n(w,b)$ of the difference equation* (12) *is given as*

$$Q_1 x_n(w,b) = B^n \sum_{k=0}^{h_B(1)-1} (-1)^k \binom{n+k}{k+1} (B - E)^k \delta(w,b) + Q_1 w \quad (n \in \mathbb{Z}). \tag{15}$$

Proof. It suffices to prove the following equality

$$B^n \sum_{k=0}^{h_B(1)-1} (-1)^k \binom{n+k}{k+1} (B - E)^k Q_1 = \sum_{k=0}^{h_B(1)-1} \binom{n}{k+1} (B - E)^k Q_1. \tag{16}$$

Note that there is a τ-periodic continuous linear system $x'(t) = A(t)x(t) + f(t)$ with periodic operators $V(t)$ such that $V(0) = B$ and $b_f = b$. We denote by $x(t)$ the solution of the above equation with the initial condition $x(0) = w$. Then (16) is easily derived from (10) with $t = n\tau$ and $Q_1(0) = Q_1$, because $Q_1(n\tau) = Q_1(0)$ and $R_1(n\tau) = Q_1(0)$. Therefore, the proof of the theorem is complete. □

Corollary 4.4. *If $h_B(1) = 1$ in Theorem* 4.3*, then* (15) *becomes*

$$Q_1 x_n(w,b) = B^n n \delta(w,b) + Q_1 w \quad (n \in \mathbb{Z}).$$

The following result is derived from Theorem 3.7.

Theorem 4.5. *Let $B = e^{\tau A}$ and $\lambda \in i\omega\mathbb{Z}$. Then the component $P_\lambda x_n(w,b)$ of the solution $x_n(w,b)$ of the difference equation* (12) *is given by*

$$P_\lambda x_n(w,b) = e^{n\tau A} \sum_{k=0}^{h_A(\lambda)-1} \frac{(-1)^k \tau^k}{(k+1)!} n^{k+1} (A - \lambda E)^k \beta_\lambda(w,b) + P_\lambda w \quad (n \in \mathbb{Z}).$$

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