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AUTOCOMMUTATORS AND AUTO-BELL GROUPS

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Abstract. Let x be an element of a group G and α be an automorphism of G. Then for a positive integer n, the autocommutator $[x,_n \alpha]$ is defined inductively by $[x, \alpha] = x^{-1}x^{\alpha} = x^{-1}\alpha(x)$ and $[x,_{n+1} \alpha] = [[x,_n \alpha], \alpha]$. We call the group G to be n-auto-Engel if $[x,_n \alpha] = [\alpha,_n x] = 1$ for all $x \in G$ and every $\alpha \in \mathrm{Aut}(G)$, where $[\alpha, x] = [x, \alpha]^{-1}$. Also, for any integer $n \neq 0, 1$, a group G is called an n-auto-Bell group when $[x^n, \alpha] = [x, \alpha^n]$ for every $x \in G$ and each $\alpha \in \mathrm{Aut}(G)$. In this paper, we investigate the properties of such groups and show that if G is an n-auto-Bell group, then the factor group $G/L_3(G)$ has finite exponent dividing $2n(n-1)$, where $L_3(G)$ is the third term of the upper autocentral series of G. Also, we give some examples and results about n-auto-Bell abelian groups.

1. Introduction

Let G be a group and let $\mathrm{Aut}(G)$ denote the automorphism group of G. For $\alpha \in \mathrm{Aut}(G)$ and $x \in G$, the *autocommutator* of x and α is defined to be $[x, \alpha] = x^{-1}x^{\alpha} = x^{-1}\alpha(x)$. The *absolute centre* and the *autocommutator subgroup* of G are the subgroups $L(G) = \{x \in G : [x, \alpha] = 1 \text{ for all } \alpha \in \mathrm{Aut}(G)\}$ and $K(G) = \langle [x, \alpha] : x \in G, \alpha \in \mathrm{Aut}(G)\rangle$, respectively (see [6]). Clearly, the absolute centre is a characteristic subgroup contained in the centre of G and the autocommutator subgroup is a characteristic subgroup containing the derived subgroup of G. Hegarty [6] uses the notation G^* for $K(G)$ and proves that if $G/L(G)$ is finite, then so is $K(G)$. Autocommutator subgroup and absolute centre are already studied in [3, 11].

Let n be a positive integer. The autocommutator $[x,_n \alpha]$ is defined inductively by $[x,_1 \alpha] = [x, \alpha]$ and $[x,_n \alpha] = [[x,_{n-1} \alpha], \alpha]$ for $n \geq 2$. The group G is said to be *n-auto-Engel* if $[x,_n \alpha] = [\alpha,_n x] = 1$ for all $x \in G$ and every $\alpha \in \mathrm{Aut}(G)$, where $[\alpha, x] = [x, \alpha]^{-1}$. Auto-Engel groups are already studied by Moghaddam et al. (see [9]).

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For any integer $n \neq 0, 1$, a group G is called *n-auto-Bell* if $[x^n, \alpha] = [x, \alpha^n]$ for every $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. In particular, a group G satisfying the previous identity for all inner automorphisms $\alpha \in \mathrm{Inn}(G)$ is an *n-Bell* group. The study of n-Bell groups was the subject of several articles, see for instance Brandl and Kappe [1], Kappe and Morse [8], Delizia et al. [4] and Tortora [13].

A group G is called *n-Kappe* if the factor group $G/R_2(G)$ has finite exponent dividing n, where $R_2(G) = \{g \in G : [g, x, x] = 1 \text{ for all } x \in G\}$ is the set of all right 2-Engel elements of G. It is well known that every n-Bell group is $n(n-1)$-Kappe (see Brandl and Kappe [1]).

In [9], it is proved that the set of all right 2-auto-Engel elements of G, $AR_2(G) = \{g \in G : [g, \alpha, \alpha] = 1 \text{ for all } \alpha \in \mathrm{Aut}(G)\}$ is a characteristic subgroup of G. Here, we call a group G an *n-auto-Kappe group* when the factor group $G/AR_2(G)$ has finite exponent dividing n. In this paper, we study some connections of such groups with n-auto-Bell groups.

Also, Delizia et al. [4] proved that for an n-Bell group G, the exponent of $G/Z_3(G)$ divides $2n(n-1)$.

In [10], Moghaddam et al. studied the concept of *lower autocentral series* and its properties. We define the *upper autocentral series* by a similar manner. The n-th absolute centre of G is defined in the following way: $L_1(G) = L(G)$ and $L_n(G) = \{x \in G : [x, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_i \in \mathrm{Aut}(G)\}$. One obtains an ascending chain of characteristic subgroups of G as follows:

$$1 = L_0(G) \leq L_1(G) \leq \cdots \leq L_n(G) \leq \cdots,$$

which we may call the upper autocentral series of G.

In Section 3, we show that if G is an n-auto-Bell group, then the factor group $G/L_3(G)$ has finite exponent dividing $2n(n-1)$.

2. Auto-Bell and auto-Kappe groups

First, we state a result about 2-auto-Engel groups, which is proved in [9].

Lemma 2.1 ([9]). *Let G be a 2-auto-Engel group. Then for every $x, y \in G$, $\alpha \in \mathrm{Aut}(G)$ and $n \in \mathbb{Z}$ the following properties hold*:

(a) $[x, x^\alpha] = 1$;
(b) $[x, \alpha^n] = [x, \alpha]^n = [x^n, \alpha]$;
(c) $[x^\alpha, y] = [x, y^\alpha]$;
(d) $[\alpha, x, y] = [\alpha, y, x]^{-1}$.

By the above lemma, every 2-auto-Engel group is an n-auto-Bell group for any integer $n \neq 0, 1$. Now, suppose that G is a 2-auto-Bell group. Then the identity $[x^2, \alpha] = [x, \alpha^2]$ implies that $([x, \alpha]^x [x, \alpha])^{\alpha^{-1}} = ([x, \alpha][x, \alpha]^\alpha)^{\alpha^{-1}}$. Hence $([x, \alpha]^{\alpha^{-1}})^x = [x, \alpha]$ and so $[x, \alpha, \alpha^{-1}\varphi_x] = 1$, where φ_x is the inner automorphism defined by x. If we replace the automorphism α by $\varphi_x \alpha^{-1}$, then we have $[[x, \alpha^{-1}][x, \varphi_x]^{\alpha^{-1}}, \alpha] = 1$. Hence $[x, \alpha, \alpha] = 1$ and since a right 2-auto-Engel element is also a left one (see [9]), G is a 2-auto-Engel group. Thus for any 2-auto-Bell group G, we have $[G, \alpha] \subseteq C_G(\alpha)$ for every $\alpha \in \mathrm{Aut}(G)$

and hence $[\mathrm{Aut}(G), x, x] = 1$ for every x in G (i.e., x is also a left 2-auto-Engel element). Therefore

$$\mathrm{Aut}(G) = \mathcal{A}(G) = \{\alpha \in \mathrm{Aut}(G) : xx^{\alpha} = x^{\alpha}x \text{ for all } x \in G\},$$

the set of *commuting automorphisms* of the group G (see [2]). It is easy to see that every 2-auto-Bell group satisfies the identity $\alpha(x)\alpha^{-1}(x) = x^2$. In Section 4, we discuss a family of infinitely many non-abelian finite 2-groups which are 2-auto-Bell.

In what follows, we determine the structure of the abelian 2-auto-Bell groups. Let $G = \langle x, y : x^4 = y^2 = 1, xy = yx\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Consider the automorphism α of G given by $\alpha(x) = xy$ and $\alpha(y) = yx^2$. Clearly, $[x, \alpha, \alpha] = x^2$ and hence G is not a 2-auto-Bell group. Now, assume that G is a 2-auto-Bell abelian group, then for the automorphism $\alpha : x \mapsto x^{-1}$, we have $x^4 = [x, \alpha, \alpha] = 1$ for every $x \in G$. Therefore G is a direct sum of cyclic groups of order 2 or 4. On the other hand, $[x, \alpha^4] = [x, \alpha]^4 = 1$ and so $\exp(\mathrm{Aut}(G))$ divides 4. Using the above example and the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ have an automorphism of order 3, it follows that $G \cong 1, \mathbb{Z}_2$ or $\mathbb{Z}_4$. Recall that the structure of non-abelian 2-auto-Bell (2-auto-Engel) 2-groups is studied in [9].

Now, we discuss the relations between auto-Bell and auto-Kappe groups after some preliminary results.

Lemma 2.2. *Let G be an n-auto-Bell group, $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. Then*
(i) $[x^n, \alpha, x^{1-n}] = 1$;
(ii) $x^{n(1-n)} \in Z(x^{\mathrm{Aut}(G)})$, *where* $x^{\mathrm{Aut}(G)} = \langle x^{\alpha} : \alpha \in \mathrm{Aut}(G)\rangle$.

Proof. (i) Since G is n-auto-Bell,

$$[x^n, \alpha]^{-x^{-n}} = [x^{-n}, \alpha] = [x^{-1}, \alpha^n] = [x, \alpha^n]^{-x^{-1}}.$$

Conjugating with x and taking the inverse yields $[x^n, \alpha][x^n, \alpha, x^{1-n}] = [x, \alpha^n]$. Hence $[x^n, \alpha, x^{1-n}] = 1$.

(ii) Using the Jacobi identity, one obtains $[x, \alpha, y^x][y, x, \alpha^{\varphi_y}][\alpha, y, x^{\alpha}] = 1$ for every x and y in G and $\alpha \in \mathrm{Aut}(G)$, where φ_y is the inner automorphism of G defined by y. From this identity and (i) it follows that

$$1 = [\alpha, x^{1-n}, x^{n\alpha}] = [x^{(n-1)\alpha}x^{1-n}, x^{n\alpha}] = [x^{n-n^2}, x^{\alpha}].$$

Hence $x^{n(1-n)} \in Z(x^{\mathrm{Aut}(G)})$. □

Proposition 2.3. *Every n-auto-Bell group is also $(1-n)$-auto-Bell and hence $n(1-n)$-auto-Bell.*

Proof. Since G is an n-auto-Bell group, $[x^n, \alpha^{-1}]^{\alpha} = [x, \alpha^{-n}]^{\alpha}$. Therefore $x^{-n\alpha}x^n = x^{-\alpha}x^{\alpha^{1-n}}$ and hence

$$x^{n-1}x^{(1-n)\alpha}x^n = x^{n-1}x^{\alpha^{1-n}}.$$

So $[x^{1-n}, \alpha]^{x^n} = [x, \alpha^{1-n}]$. Finally, $[x^{1-n}, \alpha][x^{1-n}, \alpha, x^n] = [x, \alpha^{1-n}]$ and by Lemma 2.2(i), G is a $(1-n)$-auto-Bell group. Clearly, it follows that G is also $n(1-n)$-auto-Bell. □

Observe that an n-Bell group need not be a $(-n)$-Bell group, in general. Clearly, by Proposition 2.3 an n-auto-Bell group need not be $(-n)$-auto-Bell. In the following theorem, we show that every n-auto-Bell group is also $n(n-1)$-auto-Bell.

Theorem 2.4. *Every n-auto-Bell group is also $n(n-1)$-auto-Kappe and hence $n(n-1)$-auto-Bell.*

Proof. Let G be an n-auto-Bell group, $x \in G$, $\alpha \in \mathrm{Aut}(G)$. Using Lemma 2.2(ii) and Proposition 2.3, we get $x^{n(1-n)} \in Z(x^{\mathrm{Aut}(G)})$ and hence

$$1 = [x^{n(1-n)}, x^{\alpha}] = [x, x^{n(1-n)\alpha}] = [x, [x^{n(1-n)}, \alpha]].$$

Therefore

$$[\alpha^{n(1-n)}, x, x] = 1, \tag{1}$$

and hence, $\alpha^{n(n-1)} \in \mathcal{A}(G)$. So, in every n-auto-Bell group, we have the following identity,

$$[x^{n(n-1)\alpha}, x] = 1 = [x^{\alpha}, x^{n(n-1)}]. \tag{2}$$

Now, put $m = n(n-1)$. For the n-auto-Bell group G, it is easy to see that

$$x^{(n-1)\alpha} x^{\alpha^{1-n}} = x^{n}. \tag{3}$$

Replacing x by x^n yields

$$x^{n\alpha^{1-n}} = x^{-m\alpha} x^{n^2}. \tag{4}$$

In the equation (4), if we replace α by α^{-1} and conjugate with α, we get $x^{n\alpha^{n}} = x^{-m} x^{n^2\alpha}$. Now, conjugating the equation (3) with α^n and using the latter equality yields

$$x^{(n-1)\alpha^{n+1}} = x^{n\alpha^{n}} x^{-\alpha} = x^{-m} x^{(n^2-1)\alpha}. \tag{5}$$

By the equation (2), clearly $[x^m, \alpha, x] = 1$ and so $[x^m, \alpha^{-1}, x]^{\alpha} = 1$. It follows that

$$[x^m, \alpha, x^{\alpha}] = 1. \tag{6}$$

Therefore by Proposition 2.3, equations (4), (5) and (6)

$$\begin{aligned}
[x^m, \alpha, \alpha] &= [x^{-m}, \alpha][x^{m\alpha}, \alpha] \\
&= [x^n, \alpha^{1-n}][x^{(n-1)\alpha}, \alpha^n] \\
&= x^{-n} x^{n\alpha^{1-n}} x^{(1-n)\alpha} x^{(n-1)\alpha^{n+1}} \\
&= x^{-n} x^{-m\alpha} x^{n^2} x^{(1-n)\alpha} x^{n\alpha^{n}} x^{-\alpha} \\
&= x^{m} x^{-m\alpha} x^{(1-n)\alpha} x^{-m} x^{(n^2-1)\alpha} \\
&= x^{(-n(n-1)+(1-n)+n^2-1)\alpha} \\
&= 1.
\end{aligned}$$

Thus G is $n(n-1)$-auto-Kappe. It also follows that $[x^{-1}, \alpha^{-m}]^{\alpha} = [x^{-1}, \alpha^{-m}]$. Therefore $[x, \alpha^m]^{\alpha^{-1}} = [x, \alpha^m]$ and hence by Proposition 2.3 and (1) we get:

$$\begin{aligned} [x^m, \alpha] &= [(x^{-m})^{-1}, \alpha] = [x, \alpha^{-m}]^{-x^{-m}} \\ &= [x, \alpha^{-m}]^{-1} = [x, \alpha^m]^{\alpha^{-m}} \\ &= [x, \alpha^m]. \end{aligned}$$

So G is an $n(n-1)$-auto-Bell group. □

Remark 2.5. Some connections are held between Kappe and Bell groups, which may not be true for auto-Kappe and auto-Bell groups. For example in [4, Theorem 2.1], it is pointed out that every n-Kappe group is an n^2-Bell group. If G is the elementary abelian 2-group of order 4, then G is a 2-auto-Kappe, but as G has an automorphism of order 3, it cannot be a 4-auto-Bell group.

We end this section by pointing a result, which gives some relations about auto-Bell groups.

Proposition 2.6. *Let G be a group and $n \neq 0, 1$ be an integer.*

(i) *If G is an $(n-1)$-auto-Kappe and n-auto-Bell group, then G is also $(n-1)$-auto-Bell.*

(ii) *If G is an n-auto-Kappe and n-auto-Bell group, then G is also an $(n+1)$-auto-Bell group.*

Proof. (i) Let $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. Since G is an n-auto-Bell group (and hence $(1-n)$-auto-Bell) and also an $(n-1)$-auto-Kappe, we get

$$[x^{1-n}, \alpha] = [x, \alpha^{1-n}] = [x, \alpha^{n-1}]^{-\alpha^{1-n}} = [x, \alpha^{n-1}]^{-1}.$$

On the other hand, since x^{n-1} is a right 2-auto-Engel element, it is also a left one and so $[\alpha, x^{n-1}, x^{n-1}] = 1$. Therefore $[x^{1-n}, \alpha] = [x^{n-1}, \alpha]^{-1}$. This implies that $[x^{n-1}, \alpha] = [x, \alpha^{n-1}]$ and hence G is an $(n-1)$-auto-Bell group.

(ii) Since G is an n-auto-Kappe, one may show that $[x^n, \varphi_x\alpha^{-1}, \alpha] = 1$, where φ_x is the inner automorphism defined by the element x. Replacing α by $\alpha^{-1}\varphi_x$ yields $[x^n, \alpha, \alpha^{-1}\varphi_x] = 1$. Thus $[x^n, \alpha]^{\alpha^{-1}} x = x[x^n, \alpha]$ and hence $[x^n, \alpha]x^{\alpha} = x^{\alpha}[x^n, \alpha]^{\alpha}$. Therefore $x^{-1}[x^n, \alpha]xx^{-1}x^{\alpha} = x^{-1}x^{\alpha}[x^n, \alpha]^{\alpha}$ and from the fact that G is an n-auto-Bell group, it follows that $[x^n, \alpha]^x[x, \alpha] = [x, \alpha][x, \alpha^n]^{\alpha}$. This shows that $[x^{n+1}, \alpha] = [x, \alpha^{n+1}]$. Thus G is an $(n+1)$-auto-Bell. □

3. Upper autocentral series in auto-Bell groups

Given a group G, the n-th *autocommutator subgroup* of G is

$$K_n(G) = \langle [x, \alpha_1, \alpha_2, \ldots, \alpha_n] : x \in G, \alpha_1, \ldots, \alpha_n \in \mathrm{Aut}(G) \rangle.$$

It can be easily seen that for every $n \in \mathbb{N}$, the n-th autocommutator subgroup is a characteristic subgroup of G containing $\gamma_{n+1}(G)$. Now, we obtain the following series of subgroups

$$G = K_0(G) \geq K(G) = K_1(G) \geq K_2(G) \geq \cdots \geq K_n(G) \geq \cdots ,$$

which is called the lower autocentral series of G. In [10], it is proved that for any finite abelian group G and every natural number n, there exists a finite abelian group H such that $G \cong K_n(H)$.

Now, the n-th absolute centre of G is defined inductively by $L_1(G) = L(G)$ and $L_n(G) = \{x \in G : [x, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_i \in \mathrm{Aut}(G)\}$. Clearly, the n-th absolute centre of G is contained in the n-th centre of G, $Z_n(G)$. One obtains an ascending chain of characteristic subgroups of G as follows:

$$1 = L_0(G) \leq L_1(G) \leq \cdots \leq L_n(G) \leq \cdots,$$

which we may call the upper autocentral series of G. In the following theorem, we prove that if G is an n-auto-Bell group, then $[G^{2n(n-1)}, \alpha, \beta, \gamma] = 1$ for every $\alpha, \beta, \gamma \in \mathrm{Aut}(G)$.

Theorem 3.1. *Let G be an n-auto-Bell group. Then the factor group $G/L_3(G)$ has finite exponent dividing $2n(n-1)$.*

Proof. First, we show that for any right 2-auto-Engel element x of G, the subgroup $x^{\mathrm{Aut}(G)} = \langle x^\alpha : \alpha \in \mathrm{Aut}(G)\rangle$ is abelian. Let α and β be automorphisms of G. Then

$$[x^\alpha, x^\beta] = [x^{\alpha\beta^{-1}}, x]^\beta = [x[x, \alpha\beta^{-1}], x]^\beta = [[x, \alpha\beta^{-1}], x]^\beta.$$

On the other hand, every right 2-auto-Engel element is also a left 2-auto-Engel element. Hence $[\alpha\beta^{-1}, x, x] = 1$ and this implies that $[x^\alpha, x^\beta] = 1$ and hence $x^{\mathrm{Aut}(G)}$ is abelian.

Now, by Theorem 2.4, $g := x^{n(n-1)}$ is a right 2-auto-Engel element. So, for each $\alpha \in \mathrm{Aut}(G)$, we have $[g, \alpha^{-1}] = [g, \alpha]^{-1}$. On the other hand, since $g^{\mathrm{Aut}(G)}$ is abelian, we get $[g, \alpha\beta] = [g, \alpha][g, \beta][g, \alpha, \beta]$ (observe that $[g, \alpha] \in g^{\mathrm{Aut}(G)}$) for every $\alpha, \beta \in \mathrm{Aut}(G)$. Hence $[g, \alpha\beta]^{-1} = [g, \beta^{-1}\alpha^{-1}]$ and the above equality shows that

$$[g, \alpha, \beta] = [g, \beta, \alpha]^{-1}.$$

Now, suppose that α, β and γ are arbitrary automorphisms of G. One may check that the equality $[g, \alpha, \beta\gamma] = [g, \beta\gamma, \alpha]^{-1}$ implies that $[g, \alpha, \beta, \gamma]^2 = 1$ and since g is a 2-auto-Engel element, we obtain $[g^2, \alpha, \beta, \gamma] = 1$. Therefore $[x^{2n(n-1)}, \alpha, \beta, \gamma] = 1$ and this completes the proof. □

4. Abelian n-auto-Bell groups

Clearly, every abelian group is an n-Bell group, but this statement is not true for n-auto-Bell groups. In what follows, we give some examples of auto-Bell groups and also discuss some results about n-auto-Bell abelian groups. Observe that by Proposition 2.3, in this section we may suppose that $n \geq 2$.

Example 4.1. (i) Let G be a non-periodic abelian group, and consider the inverting automorphism $\alpha \in \mathrm{Aut}(G)$ and a torsion-free element $x \in G$. Then one can easily see that $[x^n, \alpha] = x^{-2n}$ and $[x, \alpha^n] = x^{(-1)^n - 1}$. If G is an n-auto-Bell group, then we must have $-2n = (-1)^n - 1$, and this implies that $n \in \{0, 1\}$. So, G cannot be an n-auto-Bell group for every integer $n \neq 0, 1$.

(ii) In [7], Jamali constructed the following family of groups. For $m \geq 3$, let G_m be a 2-group with the following presentation

$$G_m = \langle a_1, \dots, a_m, b : a_1^2 = a_2^4 = \cdots = a_m^4 = 1, a_{m-1}^2 = b^2, [a_1, b] = 1,$$
$$[a_m, b] = a_1, [a_{i-1}, b] = a_i^2, [a_j, a_k] = 1, 3 \leq i \leq m, 1 \leq j < k \leq m\rangle.$$

The group G_m is of order 2^{2m} with exponent 4 whose automorphism group is isomorphic to $\mathbb{Z}_2^{m^2}$ and also $Z(G_m) \cong \mathbb{Z}_2^m$. Clearly, for every $m \geq 3$, G_m is a non-abelian 2-auto-Bell (and hence an n-auto-Bell, for every $n \geq 3$) group. By using GAP [5], one can check that $G_3 \cong (\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$.

(iii) Let $G = \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \cong \langle x\rangle \times \langle y\rangle \times \langle z\rangle$. Consider the automorphism α defined by $\alpha(x) = xy$, $\alpha(y) = x^2yz$ and $\alpha(z) = x^4y^2z$. One can easily check that $1 = [x^4, \alpha] \neq [x, \alpha^4]$ and so G is not a 4-auto-Bell group.

Recall that there are only two non-trivial abelian 2-auto-Bell groups, namely $\mathbb{Z}_2$ and $\mathbb{Z}_4$.

Observe that if $G \cong H \times K$ is an n-auto-Bell group, then so are H and K. Now, let G be an abelian n-auto-Bell group ($n \geq 3$) and α be the inverting automorphism. Clearly, the identity $[x^n, \alpha] = [x, \alpha^n]$ implies that $\exp(G)$ divides $2n$ or $2(n-1)$ when n is an even or an odd integer, respectively. By Proposition 2.3, G is also a $(1-n)^2$-auto-Bell group. Hence, the exponent of $\mathrm{Aut}(G)$ divides $n(n-2)$ or $(n-1)^2$ when n is an even or an odd integer, respectively.

By the above statement, it is easy to see that the 3-auto-Bell abelian groups are actually 2-auto-Bell. Assume that $n = 4$. Therefore G is a direct sum of cyclic groups of order $2, 4$ or 8. Hence, Example 4.1(iii) and the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_8 \times \mathbb{Z}_8$ have an automorphism of order 3, show that G is isomorphic to one of the groups $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_8$.

Finally, let G be a 5-auto-Bell abelian group. It is easy to see that $\exp(G)$ and $\exp(\mathrm{Aut}(G))$ divide 8 and 16, respectively. One may check that the abelian 5-auto-Bell groups are actually 4-auto-Bell.

Remark 4.2. Let p be an odd prime, $n \geq 6$ and G be an abelian n-auto-Bell p-group (if any). By the above statement, it is easy to see that $\exp(\mathrm{Aut}(G))$ divides n or $(n-1)$ when n is an even or an odd integer, respectively.

The following theorem may be considered as a criterion for recognition of abelian p-groups which are not n-auto-Bell.

Theorem 4.3. *Let G be a finite abelian n-auto-Bell group with $|G| = \prod_{i=1}^m {p_i}^{r_i}$. Then for every $1 \leq j \leq m$, the numbers $p_j(p_j-1)$ and $\prod_{i=1}^m p_i$ divide n or $(n-1)$ when n is an even or an odd integer, respectively.*

Proof. Suppose that an arbitrary prime p divides the order of G. Clearly, the Sylow p-subgroup P of G is also an n-auto-Bell group. If $p = 2$ or 3 the result is true. Suppose that n is an even integer and $p \geq 5$. By considering the inverting automorphism, we get $p|n$. Let $\alpha : x \mapsto x^\lambda$ be an automorphism of P, where $1 < \lambda < p$ and $(\lambda, p-1) = 1$. Then the identity $[x^n, \alpha] = [x, \alpha^n]$ implies that $p|(\lambda^n - n\lambda + n - 1)$. Therefore $\lambda^n \equiv 1 \pmod{p}$.

On the other hand, Euler's theorem implies that $\lambda^{p-1} \equiv 1 \pmod{p}$ and since $(\lambda, p-1) = 1$, we get $(p-1)|n$. Therefore $p(p-1)|n$. Similarly, it may be shown that $p(p-1)|(n-1)$, if n is an odd integer. Therefore the proof is complete. □

The above theorem immediately yields the following corollary.

Corollary 4.4. *There is no abelian n-auto-Bell p-group for $n < p(p-1)$.*

Proposition 4.5. *If G is an abelian $p(p-1)n$-auto-Bell p-group (p odd and $1 \leq n \leq p-1$), then $G \cong \mathbb{Z}_p$.*

Proof. Clearly, $\mathbb{Z}_p$ is a $p(p-1)m$-auto-Bell group for every $m \in \mathbb{N}$. It is enough to show that $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^k}$ $(k \geq 2)$ are not $p(p-1)n$-auto-Bell. Since $\mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p) \cong GL(2,p)$ and $\exp(GL(2,p)) = p(p^2-1)$, we obtain $1 = [x^{p(p-1)n}, \alpha] \neq [x, \alpha^{p(p-1)n}]$ for some $x \in \mathbb{Z}_p \times \mathbb{Z}_p$ and $\alpha \in GL(2,p)$.

Also, since p^k does not divide $p(p-1)n$, we get $[x^{p(p-1)n}, \alpha] \neq [x, \alpha^{p(p-1)n}] = 1$ and hence the cyclic group of order p^k $(k \geq 2)$ cannot be a $p(p-1)n$-auto-Bell p-group. □

Remark 4.6. In the previous proposition, it is not difficult to show that if $n = p$, then $G \cong \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$. If $n = p+1$, then $G \cong \mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ and finally if $n = p+2$, then $G \cong \mathbb{Z}_p$. Observe that if $n > p+2$, then the structure of G may depend on the odd prime p.

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