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Default Bayesian testing for the equality of the scale parameters of several inverted exponential distributions

Sang Gil Kang[1] · Dal Ho Kim[2] · Woo Dong Lee[3]

[1]Department of Computer and Data Information, Sangji University
[2]Department of Statistics, Kyungpook National University
[3]Department of Asset Management, Daegu Haany University

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Abstract

This article deals with the problem of testing the equality of the scale parameters of several inverted exponential distributions. We propose Bayesian hypothesis testing procedures for the equality of the scale parameters under the noninformative prior. The noninformative prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we propose the default Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors under the reference priors. Simulation study and an example are provided.

Keywords: Fractional Bayes factor, intrinsic Bayes factor, reference prior, scale parameter.

1. Introduction

The exponential distribution is the most exploited distribution for life time data analysis. However, its suitability is restricted to a constant hazard rate, which is difficult to justify in many practical problem. This led to the development of alternative models for life time data. A number of distributions such as Weibull and gamma have been extensively used for analyzing life time data, particularly, in those situations where the hazard rate is monotonically increasing or decreasing. But non-monotonicity of the hazard rate has also been observed in many situations. For example, in the course of the study of mortality associated with some of the diseases, the hazard rate initially increases with time and reaches a peak after some finite period of time and then declines slowly. Thus, the need to analyze such data whose hazard rate is non-monotonic was realized and suitable models were proposed. Killer and Kamath (1982), Lin *et al.* (1989) and Dey (2007) advocated the use of inverted exponential distribution as an appropriate model for this situation. Recently Singh *et al.* (2013) proposed Bayes estimators of the parameter and reliability function of inverted exponential distribution under the general entropy loss function for complete, type I and type

[1] Professor, Department of Computer and Data Information, Sangji University, Wonju 220-702, Korea.
[2] Professor, Department of Statistics, Kyungpook National University, Daegu 702-701, Korea.
[3] Corresponding author: Professor, Department of Asset Management, Daegu Haany University, Kyungsan 712-715, Korea. E-mail: wdlee@dhu.ac.kr

II censored samples. Abouammoh and Alshingiti (2009) introduced a generalized version of inverted exponential distribution, and discussed the statistical and reliability properties of this distribution. However the problem of comparison of more than two scale parameters in these distributions has not been considered yet.

The comparison of parameters based on more than two populations has a long history in statistics. In ANOVA, the equality of means more than three normal populations is of interest. The exact F-test plays an important role in the homoscedastic condition. But in the heteroscedastic situation, an approximation test has been developed by many authors. Bertolino *et al.* (2000) developed a Bayesian model selection approach between homoscedastic and heteroscedastic settings.

In reliability study, assume that an experiment is performed to know whether the change of a condition affects the scale parameter of the inverted exponential distribution or not. More than two levels of condition is considered to this experiment. Then we want to know that the change of condition changes the scale parameter of this distribution. At this moment, a model selection problem between homoscedastic and heteroscedastic model arises. A classical test only concerns about null hypothesis of homoscedasticity, and does not matter to heteroscedasticity. But a Bayesian model selection can choose a model using posterior probability between homoscedastic and heteroscedastic model.

In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. The use of proper prior distributions gives the Bayes factor explicitly. If there is much prior information about the parameter, it will be adequate to consider proper prior. But the assumption of prior distribution and its hyper-parameters are still too subjective. So, one needs to study the sensitivity of his results about prior distribution or hyper-parameters.

The objective prior such as Jeffreys' or reference prior is a prior which satisfies certain objective criterion. For example, the reference prior developed by Berger and Bernardo (1989, 1992) is a prior which gives the least information to the posterior distribution. These kinds of objective priors are typically improper and often called as the noninformative priors. When these objective priors are engaged in Bayesian model selection problem, the arbitrary constants in Bayes factor cause a critical problem. To deal with this problem, Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the imaginary training sample in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper prior. O'Hagan (1995) proposed the fractional Bayes factor. To solve the arbitrariness problem in Bayes factor, he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang *et al.*, 2008, 2011; Lee and Kang, 2008). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

In this paper, we propose the objective Bayesian hypothesis testing procedures for the equality of the scale parameters in several inverted exponential distributions based on the Bayes factors. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factors. In Section 3, under the reference prior, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors. In Section 4, simulation study and an example are given.

2. Intrinsic and fractional Bayes factors

Consider $X_i, i = 1, \cdots, k$ are independently distributed random variables according to the inverted exponential with the scale parameter λ_i. Then the probability density functions of the inverted exponential distribution of X_i is given by

$$f(x_i|\lambda_i) = \frac{1}{\lambda_i x_i^2} \exp\left\{-\frac{1}{x_i\lambda_i}\right\}, x_i > 0, \lambda_i > 0. \tag{2.1}$$

The present paper focuses on testing the equality of more than two scale parameters of the inverted exponential distributions.

Suppose that hypotheses H_1, H_2,$\cdots$, H_q are under consideration, with the data $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ under hypothesis H_i. The parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis H_i,$i = 1, 2, \cdots, q$. Then the posterior probability of the hypothesis H_i being true is

$$P(H_i|\mathbf{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji}\right)^{-1}, \tag{2.2}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.3}$$

The B_{ji} can be interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. As mentioned in previous section, the noninformative priors π_i^N such as the uniform prior, Jeffreys' prior or the reference prior are typically improper. Hence the use of noninformative prior π_i^N in (2.3) causes the B_{ji} to contain unspecified constants. The idea of Berger and Pericchi (1996) to overcome this problem is to use a part of data as a training sample. O'Hagan (1995) proposed the use of fraction of likelihood function to cancel arbitrary constants in (2.3). For details, see Berger and Pericchi (1996) for intrinsic Bayes factor and O'Hagan (1995) for fractional Bayes factor.

Now, we will introduce the brief definition of intrinsic Bayes factor and fractional Bayes factor. Let $\mathbf{x}(l)$ denote the part of the data to be used as a training sample and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \cdots, q, \tag{2.4}$$

where $m_i^N(\cdot)$ is a marginal distribution under the noninformative priors π_i^N.

In view (2.4), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor $B_{ji}(l)$ with the remainder of the data $\mathbf{x}(-l)$ using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.5}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and training samples $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L}\sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.6}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.7}$$

where ME indicates the median for all the training sample Bayes factors.

Therefore we can also calculate the posterior probability of H_i using (2.2), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.6) and (2.7), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.8}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (See O'Hagan (1995, 1997) and the discussion by Berger and Mortera in O'Hagan (1995)).

3. Bayesian hypothesis testing procedures

Let $X_{ij}, i = 1, \cdots, k, j = 1, \cdots, n_i$, denote observations from the inverted exponential distribution with the scale parameter λ_i. Then likelihood function is given by

$$f(\mathbf{x}|\lambda_1, \cdots, \lambda_k) = \prod_{i=1}^{k} \frac{1}{\lambda_i^{n_i} \prod_{j=1}^{n_i} x_{ij}^2} \exp\left\{ -\sum_{j=1}^{n_i} \frac{1}{\lambda_i x_{ij}} \right\}, \tag{3.1}$$

where $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_k)$, $\mathbf{x}_i = (x_{i1}, \cdots, x_{in_i})$ and $\lambda_i > 0, i = 1, \cdots, k$.

We are interested in testing the hypotheses $H_1 : \lambda_1 = \lambda_2 = \cdots = \lambda_k$ versus $H_2 : \lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_k$ based on the fractional Bayes factor and the intrinsic Bayes factors.

3.1. Bayesian hypothesis testing procedure based on the fractional Bayes factor

From (3.1) the likelihood function under the hypothesis $H_1 : \lambda_1 = \lambda_2 = \cdots = \lambda_k \equiv \lambda$ is

$$L_1(\lambda|\mathbf{x}) = \lambda^{-n} \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-2} \right] \exp\left\{ -\sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{1}{x_{ij}} \lambda^{-1} \right\}, \tag{3.2}$$

where $n = n_1 + \cdots + n_k$. And under the hypothesis H_1, the reference prior for λ is

$$\pi_1^N(\lambda) \propto \lambda^{-1}. \tag{3.3}$$

Then from the likelihood (3.2) and the reference prior (3.3) , the element $m_1^b(\mathbf{x})$ of the FBF under H_1 is given by

$$\begin{aligned} m_1^b(\mathbf{x}) &= \int_0^\infty L_1^b(\lambda|\mathbf{x}) \pi_1^N(\lambda) d\lambda \\ &= \Gamma[bn] \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-2b} \right] \left[\sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{b}{x_{ij}} \right]^{-bn}. \end{aligned} \tag{3.4}$$

For the hypothesis H_2, the reference prior for $(\lambda_1, \cdots, \lambda_k)$ is

$$\pi_2^N(\lambda_1, \cdots, \lambda_k) \propto \prod_{i=1}^{k} \lambda_i^{-1}. \tag{3.5}$$

The likelihood function under the hypothesis H_2 is

$$L_2(\lambda_1, \cdots, \lambda_k|\mathbf{x}) = \prod_{i=1}^{k} \frac{1}{\lambda_i^{n_i} \prod_{j=1}^{n_i} x_{ij}^2} \exp\left\{ -\sum_{j=1}^{n_i} \frac{1}{\lambda_i x_{ij}} \right\}. \tag{3.6}$$

Thus from the likelihood (3.6) and the reference prior (3.5), the element $m_2^b(\mathbf{x})$ of FBF under H_2 is given as follows.

$$\begin{aligned} m_2^b(\mathbf{x}) &= \int_0^\infty \cdots \int_0^\infty L_2^b(\lambda_1, \cdots, \lambda_k|\mathbf{x}) \pi_2^N(\lambda_1, \cdots, \lambda_k) d\lambda_1 \cdots d\lambda_k \\ &= \left[\prod_{i=1}^{k} \prod_{j=1}^{n_i} x_{ij}^{-2b} \right] \prod_{i=1}^{k} \Gamma[bn_i] \left[\sum_{j=1}^{n_i} \frac{b}{x_{ij}} \right]^{-bn_i} \end{aligned} \tag{3.7}$$

Therefore the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{\prod_{i=1}^k \Gamma[n_i]}{\Gamma[n]} \frac{\prod_{i=1}^k \left[\sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-n_i}}{\left[\sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-n}}. \tag{3.8}$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x})}{m_2^b(\mathbf{x})} = \frac{\Gamma[bn]}{\prod_{i=1}^k \Gamma[bn_i]} \frac{\left[\sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-bn}}{\prod_{i=1}^k \left[\sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-bn_i}}. \tag{3.9}$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{\Gamma[bn]\prod_{i=1}^k \Gamma[n_i]}{\Gamma[n]\prod_{i=1}^k \Gamma[bn_i]} \cdot \frac{\left[\sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-bn} \prod_{i=1}^k \left[\sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-n_i}}{\left[\sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-n} \prod_{i=1}^k \left[\sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-bn_i}}. \tag{3.10}$$

Note that the FBF of H_2 versus H_1 has a closed form.

3.2. Bayesian hypothesis testing procedure based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor is computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2, respectively. The marginal density of $(X_{l_1}, \cdots, X_{l_k})$, is finite for all $1 \le l_i \le n_i, i = 1, \cdots, k$ under each hypothesis. Thus we conclude that any training sample of size k is a minimal training sample.

The marginal density $m_1^N(x_{l_1}, \cdots, x_{l_k})$ under H_1 is given by

$$\begin{aligned} m_1^N(x_{l_1}, \cdots, x_{l_k}) &= \int_0^\infty f(x_{l_1}, \cdots, x_{l_k}|\lambda)\pi_1^N(\lambda)d\lambda \\ &= \Gamma[k]\left[\prod_{i=1}^k x_{l_i}^{-2}\right]\left[\sum_{i=1}^k x_{l_i}^{-1}\right]^{-k}. \end{aligned}$$

And the marginal density $m_2^N(x_{l_1}, \cdots, x_{l_k})$ under H_2 is given by

$$\begin{aligned} m_2^N(x_{l_1}, \cdots, x_{l_k}) &= \int_0^\infty \cdots \int_0^\infty f(x_{l_1}, \cdots, x_{l_k}|\lambda_1, \cdots, \lambda_k)\pi_2^N(\lambda_1, \cdots, \lambda_k)d\lambda_1 \cdots d\lambda_k \\ &= \left[\prod_{i=1}^k x_{l_i}^{-2}\right]\left[\prod_{i=1}^k x_{l_i}^{-1}\right]^{-1}. \end{aligned}$$

Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{\prod_{i=1}^k \Gamma[n_i]}{\Gamma[n]} \frac{\prod_{i=1}^k \left[\sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-n_i}}{\left[\sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}^{-1}\right]^{-n}} \left[\frac{1}{n_1 \cdots n_k} \sum_{l_1=1}^{n_1} \cdots \sum_{l_k=1}^{n_k} \frac{T_1(x_{l_1}, \cdots, x_{l_k})}{T_2(x_{l_1}, \cdots, x_{l_k})}\right], \tag{3.11}$$

where

$$T_1(x_{l_1}, \cdots, x_{l_k}) = \Gamma[k] \left[\sum_{i=1}^{k} x_{l_i}^{-1} \right]^{-k} \text{ and } T_2(x_{l_1}, \cdots, x_{l_k}) = \left[\prod_{i=1}^{k} x_{l_i}^{-1} \right]^{-1}.$$

Also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{\prod_{i=1}^{k} \Gamma[n_i]}{\Gamma[n]} \frac{\prod_{i=1}^{k} \left[\sum_{j=1}^{n_i} x_{ij}^{-1} \right]^{-n_i}}{\left[\sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}^{-1} \right]^{-n}} \cdot ME \left[\frac{T_1(x_{l_1}, \cdots, x_{l_k})}{T_2(x_{l_1}, \cdots, x_{l_k})} \right]. \tag{3.12}$$

4. Numerical studies

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of $(\lambda_1, \cdots, \lambda_k)$ and $(n_1, \cdots, n_k)$. In particular, for fixed $(\lambda_1, \cdots, \lambda_k)$, we take 1,000 independent random samples of $\mathbf{X}_i$ with sample sizes n_i from the inverted exponential distributions with $\lambda_i, i = 1, \cdots, k$, respectively. We want to test the hypotheses $H_1 : \lambda_1 = \lambda_2 = \cdots = \lambda_k$ versus $H_2 : \lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_k$. The posterior probabilities of H_1 being true are computed assuming equal prior probabilities.

Tables 4.1, 4.2 and 4.3 show the results of the averages and the standard deviations in parentheses of posterior probabilities. In Tables 4.1, 4.2 and 4.3, $P^F(\cdot)$, $P^{AI}(\cdot)$ and $P^{MI}(\cdot)$ are the posterior probabilities of the hypothesis H_1 being true based on FBF, AIBF and MIBF, respectively. From Tables 4.1, 4.2 and 4.3, the FBF, the AIBF and the MIBF accept the hypothesis H_1 when the values of λ_2 are close to values of λ_1, whereas reject the hypothesis H_1 when the values of λ_2 are far from values of λ_1. Also the AIBF and the MIBF give a similar behavior for all sample sizes. However the FBF favors the hypothesis H_2 than the AIBF and the MIBF.

Table 4.1 The averages and the standard deviations (in parentheses) of posterior probabilities when $k = 2$

λ_1	λ_2	(n_1, n_2)	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$
1.0	0.2	5,5	0.250 (0.210)	0.272 (0.238)	0.274 (0.231)
		5,10	0.160 (0.204)	0.172 (0.224)	0.169 (0.218)
		10,10	0.083 (0.137)	0.096 (0.157)	0.093 (0.151)
		10,20	0.036 (0.107)	0.041 (0.119)	0.039 (0.114)
1.0	0.5	5,5	0.530 (0.182)	0.586 (0.205)	0.576 (0.200)
		5,10	0.531 (0.225)	0.575 (0.242)	0.562 (0.238)
		10,10	0.506 (0.242)	0.565 (0.259)	0.550 (0.254)
		10,20	0.453 (0.277)	0.497 (0.293)	0.481 (0.288)
1.0	0.8	5,5	0.612 (0.131)	0.676 (0.146)	0.661 (0.144)
		5,10	0.647 (0.148)	0.699 (0.156)	0.683 (0.155)
		10,10	0.661 (0.153)	0.727 (0.157)	0.710 (0.158)
		10,20	0.690 (0.172)	0.740 (0.173)	0.723 (0.173)
1.0	1.0	5,5	0.623 (0.119)	0.686 (0.133)	0.670 (0.133)
		5,10	0.653 (0.138)	0.704 (0.147)	0.689 (0.147)
		10,10	0.687 (0.136)	0.751 (0.139)	0.735 (0.140)
		10,20	0.723 (0.138)	0.773 (0.137)	0.757 (0.138)
1.0	1.5	5,5	0.590 (0.150)	0.652 (0.167)	0.638 (0.163)
		5,10	0.623 (0.170)	0.673 (0.181)	0.659 (0.179)
		10,10	0.606 (0.194)	0.671 (0.203)	0.655 (0.201)
		10,20	0.622 (0.218)	0.672 (0.224)	0.656 (0.222)
1.0	2.0	5,5	0.523 (0.189)	0.578 (0.213)	0.567 (0.207)
		5,10	0.526 (0.216)	0.572 (0.231)	0.559 (0.225)
		10,10	0.498 (0.244)	0.555 (0.262)	0.541 (0.257)
		10,20	0.464 (0.252)	0.512 (0.265)	0.496 (0.261)
1.0	5.0	5,5	0.253 (0.215)	0.276 (0.244)	0.277 (0.238)
		5,10	0.198 (0.190)	0.218 (0.209)	0.219 (0.202)
		10,10	0.087 (0.150)	0.100 (0.173)	0.097 (0.167)
		10,20	0.047 (0.096)	0.054 (0.109)	0.052 (0.105)

Table 4.2 The averages and the standard deviations (in parentheses) of posterior probabilities when $k = 3$

λ_1	λ_2	λ_3	(n_1, n_2, n_3)	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$
1.0	0.5	0.2	5,5,5	0.308 (0.248)	0.375 (0.290)	0.411 (0.293)
			5,5,10	0.212 (0.247)	0.251 (0.283)	0.271 (0.291)
			10,10,10	0.139 (0.200)	0.188 (0.247)	0.204 (0.256)
			10,10,20	0.060 (0.145)	0.079 (0.177)	0.086 (0.185)
1.0	0.7	0.3	5,5,5	0.439 (0.260)	0.522 (0.291)	0.554 (0.286)
			5,5,10	0.381 (0.285)	0.441 (0.317)	0.464 (0.318)
			10,10,10	0.323 (0.279)	0.409 (0.314)	0.432 (0.315)
			10,10,20	0.190 (0.251)	0.238 (0.289)	0.251 (0.296)
1.0	0.9	0.7	5,5,5	0.682 (0.184)	0.775 (0.184)	0.790 (0.173)
			5,5,10	0.712 (0.196)	0.786 (0.194)	0.799 (0.186)
			10,10,10	0.755 (0.197)	0.838 (0.177)	0.849 (0.169)
			10,10,20	0.762 (0.224)	0.828 (0.208)	0.837 (0.202)
1.0	1.0	1.0	5,5,5	0.712 (0.171)	0.804 (0.172)	0.817 (0.163)
			5,5,10	0.743 (0.184)	0.816 (0.178)	0.828 (0.170)
			10,10,10	0.795 (0.172)	0.870 (0.149)	0.879 (0.141)
			10,10,20	0.837 (0.150)	0.894 (0.126)	0.901 (0.119)
1.0	1.5	2.0	5,5,5	0.612 (0.226)	0.704 (0.239)	0.724 (0.228)
			5,5,10	0.644 (0.233)	0.723 (0.235)	0.742 (0.225)
			10,10,10	0.636 (0.260)	0.730 (0.254)	0.745 (0.246)
			10,10,20	0.612 (0.280)	0.693 (0.276)	0.708 (0.270)
1.0	2.0	3.0	5,5,5	0.507 (0.253)	0.595 (0.279)	0.622 (0.272)
			5,5,10	0.484 (0.275)	0.561 (0.295)	0.590 (0.290)
			10,10,10	0.409 (0.294)	0.504 (0.321)	0.527 (0.321)
			10,10,20	0.345 (0.295)	0.419 (0.324)	0.437 (0.324)
1.0	3.0	5.0	5,5,5	0.329 (0.255)	0.397 (0.295)	0.429 (0.296)
			5,5,10	0.275 (0.248)	0.332 (0.282)	0.364 (0.286)
			10,10,10	0.131 (0.195)	0.177 (0.242)	0.193 (0.253)
			10,10,20	0.084 (0.154)	0.114 (0.190)	0.127 (0.200)

Table 4.3 The averages and the standard deviations (in parentheses) of posterior probabilities when $k = 5$

λ_1	λ_2	λ_3	λ_4	λ_5	$(n_1, n_2, n_3, n_4, n_5)$	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$
1.2	1.0	0.5	0.3	0.1	5,5,5,5,5	0.081 (0.150)	0.135 (0.216)	0.201 (0.260)
					5,5,5,10,10	0.015 (0.064)	0.025 (0.094)	0.039 (0.122)
					10,10,10,10,10	0.003 (0.022)	0.007 (0.041)	0.012 (0.057)
					10,10,10,20,20	0.000 (0.002)	0.000 (0.003)	0.000 (0.005)
1.2	1.0	0.6	0.5	0.4	5,5,5,5,5	0.553 (0.309)	0.686 (0.314)	0.757 (0.285)
					5,5,5,10,10	0.525 (0.344)	0.633 (0.348)	0.699 (0.326)
					10,10,10,10,10	0.453 (0.361)	0.588 (0.372)	0.652 (0.357)
					10,10,10,20,20	0.359 (0.368)	0.463 (0.394)	0.518 (0.395)
1.2	1.0	0.8	0.7	0.6	5,5,5,5,5	0.726 (0.250)	0.843 (0.214)	0.887 (0.177)
					5,5,5,10,10	0.767 (0.257)	0.858 (0.221)	0.895 (0.188)
					10,10,10,10,10	0.765 (0.276)	0.869 (0.223)	0.901 (0.191)
					10,10,10,20,20	0.760 (0.300)	0.847 (0.253)	0.880 (0.223)
1.0	1.0	1.0	1.0	1.0	5,5,5,5,5	0.812 (0.188)	0.910 (0.143)	0.938 (0.111)
					5,5,5,10,10	0.870 (0.176)	0.934 (0.136)	0.953 (0.108)
					10,10,10,10,10	0.914 (0.140)	0.965 (0.090)	0.976 (0.071)
					10,10,10,20,20	0.943 (0.124)	0.974 (0.087)	0.982 (0.073)
0.8	1.0	1.2	1.6	1.8	5,5,5,5,5	0.690 (0.267)	0.811 (0.243)	0.860 (0.208)
					5,5,5,10,10	0.716 (0.268)	0.824 (0.235)	0.870 (0.198)
					10,10,10,10,10	0.693 (0.307)	0.813 (0.271)	0.852 (0.246)
					10,10,10,20,20	0.693 (0.311)	0.806 (0.270)	0.848 (0.241)
0.8	1.0	1.2	2.2	2.5	5,5,5,5,5	0.522 (0.320)	0.654 (0.331)	0.727 (0.305)
					5,5,5,10,10	0.506 (0.329)	0.628 (0.336)	0.700 (0.317)
					10,10,10,10,10	0.399 (0.349)	0.535 (0.374)	0.601 (0.366)
					10,10,10,20,20	0.303 (0.324)	0.421 (0.369)	0.484 (0.377)
0.8	1.0	3.0	3.5	5.0	5,5,5,5,5	0.207 (0.252)	0.305 (0.319)	0.394 (0.343)
					5,5,5,10,10	0.162 (0.228)	0.252 (0.294)	0.337 (0.326)
					10,10,10,10,10	0.034 (0.109)	0.065 (0.169)	0.091 (0.201)
					10,10,10,20,20	0.016 (0.069)	0.032 (0.106)	0.048 (0.135)

Example 4.1 This example is taken from Singh *et al.* (2013). The data represents the survival time (in days) of guinea pigs injected with different doses of tubercle bacilli. The regimen number is the common logarithm of the number of bacillary units in 0.5 ml of challenge solution; that is, regimen 6.6 corresponds to 4.0×10^6 bacillary units per 0.5 ml ($\ln(4.0 \times 10^6)$=6.6). Corresponding to regimen 6.6, 72 observations are listed below. To test for equality of scale parameters, we randomly divided this data into three groups. The data sets are given by

Group 1:15, 24, 32, 38, 44, 52, 53, 54, 55, 57, 60, 60, 61, 70, 76, 81, 83, 99, 127, 146, 175, 233, 341, 376
Group 2: 22, 33, 38, 43, 59, 62, 67, 68, 70, 73, 76, 84, 91, 95, 96, 109, 131, 143, 146, 175, 211, 258, 297, 341
Group 3: 12, 24, 32, 34, 48, 54, 56, 58, 58, 60, 60, 63, 65, 65, 72, 75, 85, 87, 98, 110, 121, 129, 258, 263.

We want to test the hypotheses $H_1 : \lambda_1 = \lambda_2 = \lambda_3$ versus $H_2 : \lambda_1 \neq \lambda_2 \neq \lambda_3$. The values of the Bayes factors and the posterior probabilities of H_1 are given in Table 4.4. From the results of Table 4.4, the posterior probabilities based on various Bayes factors give the same answer, and select the hypothesis H_1. The FBF has the smallest posterior probability than any other posterior probabilities based on the AIBF and the MIBF.

Table 4.4 Bayes factor and posterior probabilities of $H_1 : \lambda_1 = \lambda_2 = \lambda_3$

B_{21}^F	$P^F(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{AI}	$P^{AI}(H_1\|\mathbf{x},\mathbf{y})$	B_{21}^{MI}	$P^{MI}(H_1\|\mathbf{x},\mathbf{y})$
0.067	0.933	0.047	0.955	0.050	0.953

5. Concluding remarks

In this paper, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factors for the equality of the scale parameters of several inverted exponential distributions under the reference priors. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations. However the FBF favors the hypothesis H_2 than the AIBF and the MIBF. From our simulation and example, we recommend the use of the FBF than the AIBF and MIBF for practical application in view of its simplicity and ease of implementation.

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