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# A note on nonparametric density deconvolution by weighted kernel estimators†

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**Abstract**

Recently Hazelton and Turlach (2009) proposed a weighted kernel density estimator for the deconvolution problem. In the case of Gaussian kernels and measurement error, they argued that the weighted kernel density estimator is a competitive estimator over the classical deconvolution kernel estimator. In this paper we consider weighted kernel density estimators when sample observations are contaminated by double exponentially distributed errors. The performance of the weighted kernel density estimators is compared over the classical deconvolution kernel estimator and the kernel density estimator based on the support vector regression method by means of a simulation study. The weighted density estimator with the Gaussian kernel shows numerical instability in practical implementation of optimization function. However the weighted density estimates with the double exponential kernel has very similar patterns to the classical kernel density estimates in the simulations, but the shape is less satisfactory than the classical kernel density estimator with the Gaussian kernel.

*Keywords*: Deconvolution, kernel density estimator, support vector regression, weighted kernel density estimator.

## 1. Introduction

In this paper we consider nonparametric density estimation of a random variable when we observe contaminated data instead of true data. The problem of contaminated data with noise exists in many different fields such as biostatistics, chemistry and public health. See for example Stefanski and Carroll (1990) or Carroll *et al.* (1995). This deconvolution problem of interest can be stated as follows. Let $X$ and $Z$ be independent random variables with density functions $f(x)$ and $q(z)$, respectively, where $f(x)$ is unknown and $q(z)$ is known. We observe a univariate random sample $Y_1, Y_2, \cdots, Y_n$ from a density $g(y)$, where $Y_i = X_i + Z_i, i = 1, 2, \cdots, n$. The objective is to estimate the density function $f(x)$ where $g(y)$ is the convolution of $f(x)$ and $q(z)$, $g(y) = (f * q)(y) = \int_{-\infty}^{\infty} f(y-z)q(z)dz$.

The most popular approach to this deconvolution problems has been to estimate $f(x)$ by a kernel estimator and Fourier transform (e.g. Carroll and Hall, 1988; Stefanski and Carroll, 1990; Liu and Taylor, 1989; Fan, 1991). While kernel density estimation is widely

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considered as the most popular approach to density deconvolution, other alternatives have been proposed such as spline based procedures, wavelet methods, EM algorithm, Fourier series and transforms and support vector methods. See for example Mendelsohn and Rice (1982), Koo and Park (1996), Pensky and Vidakovic (1999), Eggermont and LaRiccia (1997), Hall and Qiu (2005), Lee and Taylor (2008) and Lee (2012).

The asymptotic properties of the kernel density estimator in deconvolution problems depend strongly on the error distribution. Following the work of Fan (1991), two types of error distributions can be considered: ordinary smooth and supersmooth distributions. Normal, mixture normal, and Cauchy distributions are supersmooth, that is, the Fourier transform $\widetilde{q}(\xi)$ $(= \int_{-\infty}^{\infty} e^{-i\xi z} q(z)dz)$ of $q(z)$ satisfies

$$d_0|\xi|^{-\beta_0}\exp(-|\xi|^{\beta}/\gamma) \leq |q(\xi)| \leq d_1|\xi|^{-\beta_1}\exp(-|\xi|^{\beta}/\gamma) \text{ as } \xi \to \infty,$$

for some positive constants $d_0, d_1, \beta, \gamma$ and constants $\beta_0$ and $\beta_1$. Gamma and double exponential distributions are ordinary smooth, that is, the Fourier transform $\widetilde{q}(\xi)$ of $q(z)$ satisfies

$$d_0|\xi|^{-\beta} \leq |q(\xi)| \leq d_1|\xi|^{-\beta} \text{ as } \xi \to \infty,$$

for some positive constants $d_0, d_1, \beta$. Thus in the classical deconvolution literature, normal and double exponential distributions have been typically selected and investigated as error distributions.

The asymptotic theory in two types of error distributions (e.g. Carroll and Hall, 1988; Stefanski and Carroll, 1990; Fan, 1991; Wand, 1998) shows that the optimal rate of convergence for supersmooth error distributions is logarithmic and hence very slow. However, for ordinary smooth error distributions reasonably good algebraic rates are obtained and hence much faster. For example, Wand (1998) show that if $f(x)$ is a mixture normal distribution, then the rate of convergence of the mean integrated squared error (MISE) of the classical deconvolution kernel estimator is $n^{-4/5}$ for error-free data, $(\log n)^{-1}$ for Gaussian error and $n^{-4/9}$ for double exponential error.

Recently Hazelton and Turlach (2009) proposed a weighted kernel density estimator for the deconvolution problem. One attraction of the weighted kernel density estimator is that it will take non-negative values. They also showed that if the optimal weighting scheme $\omega_i$ $(= f(Y_i)/g(Y_i))$ was known, then the estimator would have MISE of an asymptotic order of $n^{-4/5}$. However, as the authors indicate, this rate is not achievable in practice because $\omega_i$ $(= f(Y_i)/g(Y_i))$ is a function of the unknown target density $f(x)$. In cases with the Gaussian kernel and measurement error, they showed that the estimator has the simple expression and argued that the weighted kernel density estimator is a competitive estimator over the classical deconvolution kernel estimator through a simulation study.

In this paper weighted density estimators based on the Gaussian kernel and the double exponential kernel are considered when sample observations are contaminated by double exponentially distributed errors. The performance of the weighted kernel density estimators is compared over the classical deconvolution kernel estimator and kernel density estimators based on the support vector regression method by means of a simulation study.

## 2. Weighted kernel density estimators

Hazelton and Turlach (2009) proposed a weighted kernel density estimator for the deconvolution problem. In the case of Gaussian kernels and measurement error, the weighted kernel density estimators is given by

$$\widehat{f}_\omega(x) = \frac{1}{n}\sum_{i=1}^{n} \omega_i K_h(x - Y_i),$$

where $\sum_{i=1}^n \omega_i = n,\ \omega_i \ge 0,\ i = 1, 2, \cdots, n,\ K_h(x) = (\sqrt{2\pi}\sigma_h)^{-1} e^{-x^2/2\sigma_h^2}$. The unknown weight vector $\omega$ will be chosen so as to minimize the objective function

$$\begin{aligned} Q(\omega) &= \int_{-\infty}^{\infty} (\widehat{f}_\omega * q(y) - \widehat{g}(y))^2 dy, \\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i\omega_j \phi_{\sqrt{2}\lambda}(Y_i - Y_j) + \sum_{i=1}^{n}\sum_{j=1}^{n} \phi_{h\sqrt{2}}(Y_i - Y_j) - 2\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i \phi_\nu (Y_i - Y_j)\right) \end{aligned}$$

where $\phi_\lambda(x) = (\sqrt{2\pi}\sigma_\lambda)^{-1} e^{-x^2/2\sigma_\lambda^2},\ \phi_h(x) = (\sqrt{2\pi}\sigma_h)^{-1} e^{-x^2/2\sigma_h^2},\ \sigma_\lambda^2 = \sigma_h^2 + \sigma_Z^2,\ \sigma_\nu^2 = \sigma_Z^2 + 2\sigma_h^2$, and $\widehat{g}(y) = \frac{1}{n}\sum_{i=1}^n K_h(y - Y_i)$.

In cases with the Gaussian kernel and measurement error, they showed that the weighted kernel density estimation can lead to tangible improvements in performance over the classical deconvolution kernel estimator by numerical tests.

In case of a double exponentially distributed error, the weighted kernel density estimator with the Gaussian kernel (Lee, 2010) is given by

$$\widehat{f}_\omega(x) = \frac{1}{n}\sum_{i=1}^{n} \omega_i K_h(x - Y_i) \tag{2.1}$$

where $\sum_{i=1}^n \omega_i = n,\ \omega_i \ge 0, i = 1, 2, \cdots, n,\ K_h(x) = (\sqrt{2\pi}\sigma_h)^{-1} e^{-x^2/2\sigma_h^2}$. The unknown weight vector $\omega$ will be chosen so as to minimize the objective function

$$\begin{aligned} Q(\omega) &= \int_{-\infty}^{\infty} (\widehat{f}_\omega * q(y) - \widehat{g}(y))^2 dy \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{\omega_i}{2\sqrt{2\pi}\sigma_h\sigma_z} e^{-(x-Y_i)^2/2\sigma_h^2 - |y-x|/\sigma_z} dx - \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_h} e^{-(y-Y_i)^2/2\sigma_h^2}\right)^2 dy \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i\omega_j \frac{e^{\sigma_h^2/\sigma_z^2}}{4\sigma_z^2 n^2} \int_{-\infty}^{\infty} \left(e^{-(y-Y_i)/\sigma_z} \Phi\left(\frac{y - Y_i - \sigma_h^2/\sigma_z}{\sigma_h}\right) + e^{(y-Y_i)/\sigma_z}\left(1 - \Phi\left(\frac{y - Y_i + \sigma_h^2/\sigma_z}{\sigma_h}\right)\right)\right) \\ &\times \left(e^{-(y-Y_j)/\sigma_z} \Phi\left(\frac{y - Y_j - \sigma_h^2/\sigma_z}{\sigma_h}\right) + e^{(y-Y_j)/\sigma_z}\left(1 - \Phi\left(\frac{y - Y_j + \sigma_h^2/\sigma_z}{\sigma_h}\right)\right)\right) dy \\ &- \sum_{i=1}^{n} \omega_i \frac{e^{\sigma_h^2/2\sigma_z^2}}{\sigma_z n^2} \int_{-\infty}^{\infty} \left\{\left(e^{-(y-Y_i)/\sigma_z} \Phi\left(\frac{y - Y_i - \sigma_h^2/\sigma_z}{\sigma_h}\right) + e^{(y-Y_i)/\sigma_z}\left(1 - \Phi\left(\frac{y - Y_i + \sigma_h^2/\sigma_z}{\sigma_h}\right)\right)\right)\right. \\ &\times \left.\left(\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_h} e^{-(y-Y_i)^2/2\sigma_h^2}\right)\right\} dy + \int_{-\infty}^{\infty} \left\{\frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_h} e^{-(y-Y_i)^2/2\sigma_h^2}\right\}^2 dy \end{aligned} \tag{2.2}$$

where $\widehat{g}(y) = \frac{1}{n}\sum_{i=1}^n K_h(y - Y_i)$ and $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ .

As equation (2.2) indicates, the coefficients are not available in closed form and hence a numerical method is needed :

$$\begin{aligned}&\text{minimize}_\omega \quad \frac{1}{2}\omega' H\omega + f'\omega,\\ &\text{subject to} \quad \sum_{i=1}^{n}\omega_i = n, \omega_i \geq 0, i = 1, 2, \cdots, n.\end{aligned}$$

In practical implementation of calculation of integrals and optimization, numerical instability is faced and computational time (running on a 3.2 GHz PC with 3.4 GB RAM) rapidly increases as the sample size increases. The solution in the quadratic programming to minimize quadratic function subject to constraints is unstable. As it is indicated in Delaigle and Gijbels (2007), more adequate numerical procedure in calculating integrals and optimizing objective functions is needed. Thus, only a very limited figure is presented in Section 3.

In this section, we propose another weighted kernel density estimator such that the coefficients are available in closed form. It is well known that in kernel density estimation the choice of kernel is not crucial, but the choice of bandwidth is important. Thus we try to evaluate a weighted kernel density estimator based on the double exponential kernel in this deconvolution problem. Let

$$\widehat{f}_\omega(x) = \frac{1}{n}\sum_{i=1}^{n}\omega_i K_h(x - Y_i) \tag{2.3}$$

where $\omega_i \geq 0$, $\sum_{i=1}^n \omega_i = n$, $K_h(x) = (2\sigma_h)^{-1}e^{-|x|/\sigma_h}$. Then the unknown weight vector $\omega$ will be chosen so as to minimize the objective function

$$\begin{aligned}
Q(\omega) &= \int_{-\infty}^{\infty} (\widehat{f}_\omega * q(y) - \hat{g}(y))^2 dy\\
&= \int_{-\infty}^{\infty}\left[\left\{\int_{-\infty}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\omega_i K_h(x-Y_i)\right)\times\frac{1}{2\sigma_z}e^{-|y-x|/\sigma_z}dx\right\} - \frac{1}{n}\sum_{i=1}^{n}K_h(y-Y_i)\right]^2 dy\\
&= \int_{-\infty}^{\infty}\left[\left\{\int_{-\infty}^{\infty}\frac{1}{4n\sigma_h\sigma_z}\sum_{i=1}^{n}\omega_i e^{-|x-Y_i|/\sigma_h}e^{-|y-x|/\sigma_z}dx\right\} - \frac{1}{2n\sigma_h}\sum_{i=1}^{n}e^{-|y-Y_i|/\sigma_h}\right]^2 dy\\
&= \frac{1}{16n^2\sigma_h^2\sigma_z^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\omega_i\omega_j\left\{\int_{-\infty}^{\infty}e^{-|t+(y-Y_i)|/\sigma_h}e^{-|t|/\sigma_z}dt\right\}\\
&\quad\times\left\{\int_{-\infty}^{\infty}e^{-|t+(y-Y_j)|/\sigma_h}e^{-|t|/\sigma_z}dt\right\}dy + \frac{1}{4n^2\sigma_h^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}e^{-|y-Y_i|/\sigma_h}e^{-|y-Y_j|/\sigma_h}dy\\
&\quad - \frac{1}{4n^2\sigma_h^2\sigma_z^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\omega_i e^{-|t+(y-Y_i)|/\sigma_h}e^{-|t|/\sigma_z}dt\right\}\times e^{-|y-Y_j|/\sigma_h}dy\\
&= \frac{1}{16n^2\sigma_h^2\sigma_z^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i\omega_j\left\{e^{-|Y_i-Y_j|/\sigma_h}\left(\frac{\sigma_h + |Y_i - Y_j|}{\phi^2} + \frac{2(\lambda+\nu)}{\nu\lambda\phi\psi}\right)\right.\\
&\quad\left.+e^{-|Y_i-Y_j|/\sigma_z}\left(\frac{\sigma_z + |Y_i - Y_j|}{\psi^2} + \frac{2(\nu-\lambda)}{\nu\lambda\phi\psi}\right)\right\} + \frac{1}{4n^2\sigma_h^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma_h + |Y_i - Y_j|)e^{-|Y_i-Y_j|/\sigma_h}\\
&\quad - \frac{1}{4n^2\sigma_h^2\sigma_z^2}\sum_{i=1}^{n}\omega_i\sum_{j=1}^{n}\left\{e^{-|Y_i-Y_j|/\sigma_h}\left(\frac{\sigma_h + |Y_i - Y_j|}{\phi} + \frac{\lambda+\nu}{\nu\lambda\psi}\right) + \frac{\nu-\lambda}{\nu\lambda\psi}e^{-|Y_i-Y_j|/\sigma_z}\right\}
\end{aligned} \tag{2.4}$$

where $q(z) = (2\sigma_z)^{-1}e^{-|z|/\sigma_z}$, $\widehat{g}(y) = \frac{1}{n}\sum_{i=1}^n K_h(y - Y_i)$ and $\lambda = 1/\sigma_z + 1/\sigma_h$, $\nu = 1/\sigma_z - 1/\sigma_h$, $1/\phi = 1/\lambda + 1/\nu$, $1/\psi = 1/\lambda - 1/\nu$.

Thus optimizing $Q(\omega)$ in (2.4) under the constraints that the weights are non-negative and sum to $n$ leads to a quadratic programming problem:

$$\begin{aligned} &\text{minimize}_\omega \quad \frac{1}{2}\omega' H\omega + f'\omega, \\ &\text{subject to} \quad \sum_{i=1}^{n} \omega_i = n, \omega_i \geq 0, \ i = 1, 2, \cdots, n. \end{aligned}$$

Then, we obtain

$$\widehat{f}_{\widehat{\omega}}(x) = \frac{1}{n}\sum_{i=1}^{n} \widehat{\omega}_i K_h(x - Y_i), \text{ where } \hat{\omega} = \arg\min_{\omega} Q(\omega).$$

## 3. Simulation and discussion

In this section we compare the performance of three different deconvolution density estimators when measurement errors are double exponential: weighted kernel density estimators, classical kernel density estimators, and support vector kernel density estimators based on the support vector regression method. Target distributions are selected from distribution functions used in Hazelton and Turlach (2009). The classical kernel density estimator with the Gaussian kernel (Pensky and Vidakovic, 1999) is evaluated as

$$\begin{aligned} \widehat{f}(x) &= \frac{1}{2\pi n}\sum_{j=1}^{n}\int_{-\infty}^{\infty} e^{i\xi(x-Y_j)}\widetilde{K}(\sigma_h\xi)/\widetilde{q}(\xi)\ d\xi \\ &= \frac{1}{n\sqrt{2\pi}\sigma_h}\sum_{j=1}^{n} e^{-0.5\left(\frac{x-Y_j}{\sigma_h}\right)^2}\left[1 - \frac{\sigma_Z^2}{\sigma_h^2}\left(\left(\frac{x-Y_j}{\sigma_h}\right)^2 - 1\right)\right] \end{aligned} \tag{3.1}$$

where $\tilde{K}(\sigma_h\xi) = e^{-0.5\sigma_h^2\xi^2}$ and $q(z) = (2\sigma_z)^{-1}e^{-|z|/\sigma_z}$.

The support vector kernel density estimator with the Gaussian kernel (Mukherjee and Vapnik, 1999; Lee, 2010) is evaluated as

$$\begin{aligned} \widehat{f}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{n}\omega_j\widetilde{K_h}(Y_j,\xi)e^{i\xi x}/\widetilde{q}(\xi)\ d\xi \\ &= \sum_{j=1}^{n}\frac{\omega_j}{\sqrt{2\pi}\sigma_h}e^{-(x-Y_j)^2/2\sigma_h^2} + \sigma_z^2\sum_{j=1}^{n}\frac{\omega_j}{\sqrt{2\pi}\sigma_h^3}e^{-(x-Y_j)^2/2\sigma_h^2} \\ &\quad - \sigma_z^2\sum_{j=1}^{n}\frac{\omega_j}{\sqrt{2\pi}\sigma_h^5}(x-Y_j)^2 e^{-(x-Y_j)^2/2\sigma_h^2} \end{aligned} \tag{3.2}$$

where the unknown weight vector $\omega$ will be estimated by solving the following quadratic programming problem and applying the equation $\omega = \Gamma_h^{-1}R(\alpha - \alpha^*)$ :

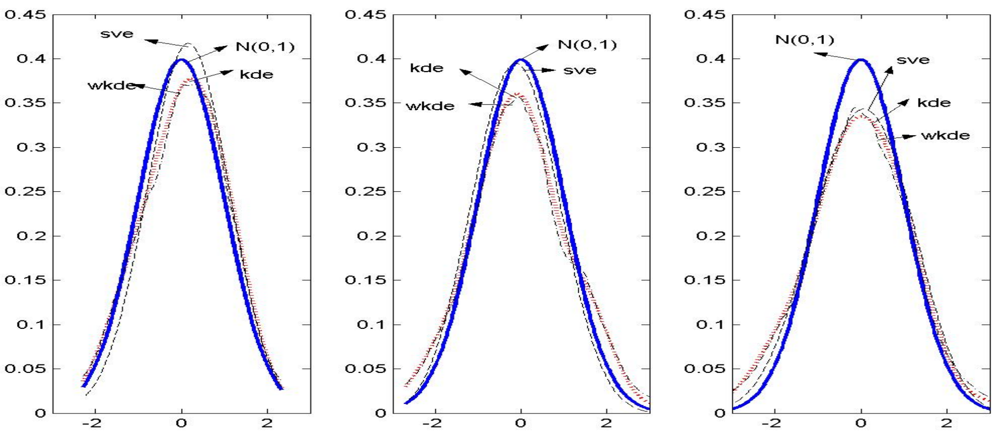
$$\text{minimize } \frac{1}{2}(\alpha-\alpha^*)^t R^t \Gamma_h^{-1} R(\alpha-\alpha^*) - \sum_{i=1}^{n} Y_i(\alpha_i-\alpha_i^*) + \epsilon \sum_{i=1}^{n}(\alpha_i+\alpha_i^*)$$
$$\text{subject to } 0 \le \alpha_i^*,\ \alpha_i \le C,\ i=1,\cdots,n$$

where $\Gamma_h = [K_h(Y_i,Y_j)]_{n\times n}$, $K_h(Y_i,Y_j) = (\sqrt{2\pi}\sigma_h)^{-1}e^{-(Y_i-Y_j)^2/2\sigma_h^2}$, $R^t = [r_{ij}]_{n\times n}$, $r_{ij} = \int_{-\infty}^{Y_i} K_h(Y_j,y)dy$.

The following Figures 3.1-3.3 show plots of classical kernel density estimates (3.1), weighted kernel density estimates with the double exponential kernel (2.3) and support vector kernel density estimates (3.2) when 100 points are randomly generated respectively from a target distribution $f(x)$ and a noise distribution, double exponential distribution $q(z)$ with mean zero. Each figure is selected among 30 randomly generated data sets. We simulated when the measurement error variance is set at low $(= \mathrm{var}(Z)/\mathrm{var}(X) = 0.1)$, moderate $(= \mathrm{var}(Z)/\mathrm{var}(X) = 0.25)$, and high levels $(= \mathrm{var}(Z)/\mathrm{var}(X) = 0.5)$ as shown in Hazelton and Turlach (2009). The target distribution $f(x)$ is shown in bold line.

In kernel density estimation the choice of kernel is not crucial, but the choice of bandwidth is very important. In the following figures, a rule of thumb bandwidth, $\sigma_h = (5\sigma_Z^4/n)^{1/9}$, is used as an initial value (Fan, 1991; Wang and Wang, 2011). MATLAB 6.5 is used in the simulation and Gunn's program (Gunn, 1998) is used for the support vector kernel density estimates,

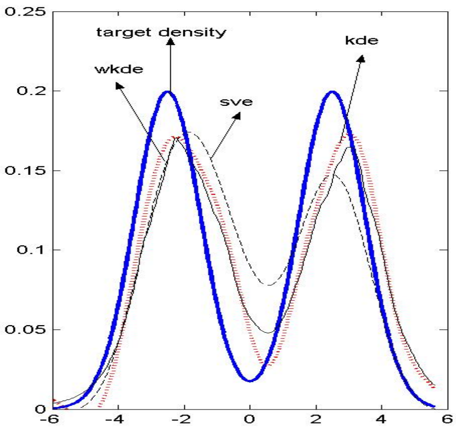
Figure 3.1 presents the simulation study when the target distribution is the standard normal probability distribution $f(x)$. The parameters $(= \sigma_h)$ of classical and kernel density estimates corresponding to variance ratios of 0.1, 0.25 and 0.5 are 0.6, 0.6 and 0.75 respectively. The parameters $(= \sigma_h)$ of the support vector kernel density estimates corresponding to variance ratios of 0.1, 0.25 and 0.5 are 0.9, 0.95, 0.95 respectively and $\epsilon = 0.05$, $C = \infty$ are used.



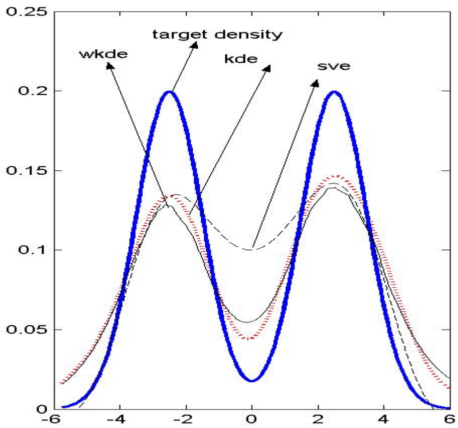

(a) var$(Z)$/var$(X)$=0.1 (b) var$(Z)$/var$(X)$=0.25 (c) var$(Z)$/var$(X)$=0.5

**Figure 3.1** The simulation study when target density $f(x)$ is $N(0,1)$

Figure 3.2 presents the simulation study when the target distribution is the symmetric bimodal density $0.5N(-2.5,1)+0.5N(2.5,1)$. The parameters $(=\sigma_h)$ of classical and weighted kernel density estimates corresponding to variance ratios of 0.1 and 0.25 are 0.75, 1.1 respectively. The parameters $(=\sigma_h)$ of the support vector kernel density estimates corresponding to variance ratios of 0.1 and 0.25 is 1.05, 1.1 respectively and $\epsilon = 0.05,\ C = \infty$ are used.




(a) var($Z$)/var($X$)=0.1




(b) var($Z$)/var($X$)=0.25

**Figure 3.2** The simulation study when target density $f(x)$ is $0.5N(-2.5,1)+0.5N(2.5,1)$

Figure 3.3 presents the simulation study when the target distribution is the kurtotic density $2/3N(0,1)+1/3N(0,0.04)$. The parameter $(=\sigma_h)$ of classical and weighted kernel density estimates corresponding to variance ratio of 0.25 is 0.5. The parameters $(\sigma_h, \epsilon, C)$ of the support vector kernel density estimate corresponding to variance ratios of 0.25 are 0.9, 0.05 and infinity respectively.

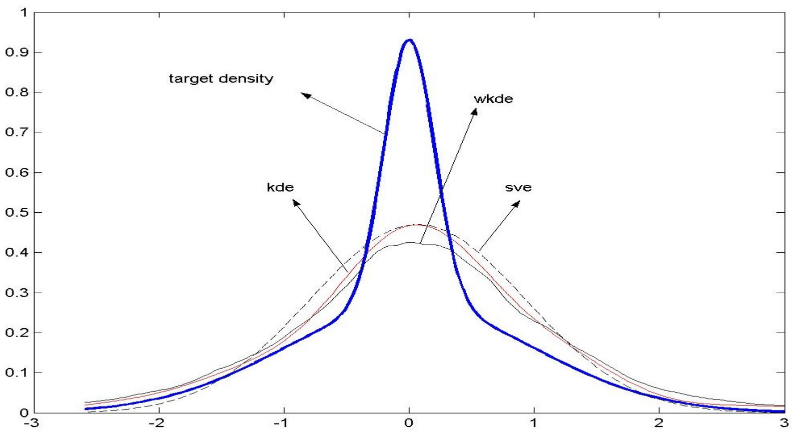



**Figure 3.3** The simulation study when target density $f(x)$ is $2/3N(0,1)+1/3N(0,0.04)$ and var($Z$)/var($X$)=0.25

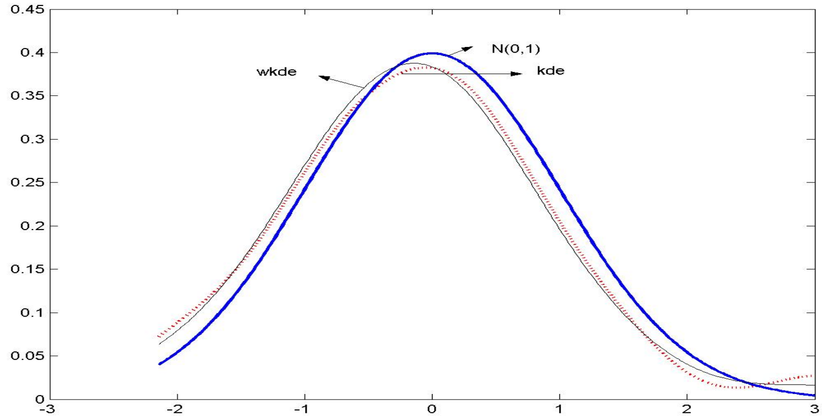



**Figure 3.4** The simulation study when target density $f(x)$ is $N(0,1)$ and var(Z)/var(X)=0.25

Figure 3.4 shows plots of a classical kernel density estimate (3.1) and a weighted kernel density estimate with the Gaussian kernel (2.1) when 40 points are randomly generated. The parameter ($= \sigma_h$) of classical and weighted kernel density estimate corresponding to variance ratios of 0.25 is 0.6. The numerical results for the weighted density estimator with the Gaussian kernel were obtained by using quadprog in the MATLAB package.

As the illustrated figures suggest in the simulation, the weighted density estimates with the double exponential kernel show very similar plots to the classical kernel density estimates even though the shape is less satisfactory than the classical kernel density estimates. The weighted density estimate with the Gaussian kernel show very similar figure to the classical kernel density estimate in Figure 3.4. The classical kernel density estimates show better performance than the support vector kernel density estimates in the mixed normal distributions. However, in the simulation, the support vector kernel density estimates sometimes showed good performance when the classical kernel density estimates face difficulties to figure out the true density. The support vector kernel density estimator is very attractive in the sense that some coefficients are very close to zero.

## 4. Concluding remarks

In this paper three different deconvolution density estimators were introduced when the sample observations are contaminated by double exponentially distributed errors. The simulation study conducted is limited. However, it indicates that the classical kernel density estimator with the Gaussian kernel show better performance than the weighted density estimator with the double exponential kernel. It is suggested by Figure 3.4 that the double exponential kernel seems to have affected the shape of the estimates more than we expected in the simulation. Thus we speculate that the weighted kernel density estimates with Gaussian kernel will improve this problem. However, in this paper a simulation study for the weighted density estimate with Gaussian kernel is very limited because numerical instability and computational time trouble for sample sizes above $n = 50$ (running on a 3.2 GHz PC with 3.4 GB RAM) are faced in practical implementation of calculation of integrals and optimization. We speculate that they will be computed relatively quickly and stably through more adequate numerical procedures.

## References


Carroll, R. J. and Hall, P. (1988). Optimal rates of convergence for deconvoluting a density. *Journal of the American Statistical Association*, **83**, 1184-1886.

Carroll, R. J., Ruppert, D. and Stefanski, L. (1995). *Measurement error in nonlinear models*, Chapman and Hall, London.

Dilaigle, A. and Gijbels, I. (2007). Frequent problems in calculating integrals and optimizing objective functions: A case study in density estimation. *Statistics and Computing*, **17**, 349-355.

Eggermont, P. and LaRiccia, V. (1997). Nonlinearly smoothed EM density estimation with augmented smoothing parameter selection for nonparametric deconvolution problems. *Journal of the American Statistical Association*, **92**, 1451-1458.

Fan, J. (1991). On the optimal rates of convergence for nonparametric deconvolution problem. *Annals of Statistics*, **19**, 1257-1272.

Gunn, S. R. (1998). *Support vector machines for classification and regression*, Technical report, University of Southampton, Southampton, UK.

Hall, P. and Qiu, P. (2005). Discrete-transform approach to deconvolution problems. *Biometrika*, **92**, 135-148.

Hazelton, M. L. and Turlach, B. A. (2009). Nonparametric density deconvolution by weighted kernel estimators. *Statistics and Computing*, **19**, 217-228.

Koo, J.-Y. and Park, B. U. (1996). B-spline deconvolution based on the EM algorithm. *Journal of Statistical Computation and Simulation*, **54**, 275-288.

Lee, S. and Taylor, R. L.(2008). A note on support vector density estimation for the deconvolution problem. *Communications in Statistics: Theory and Methods*, **37**, 328-336.

Lee, S. (2010). A Support vector method for the deconvolution problem. *Communications of the Korean Statistical Society*, **17**, 451-457.

Lee, S. (2012). A Note on deconvolution estimators when measurements errors are normal. *Communications of the Korean Statistical Society*, **19**, 517-526.

Liu, M. C. and Taylor, R. L. (1989). A Consistent nonparametric density estimator for the deconvolution problem. *The Canadian Journal of Statistics*, **17**, 427-438.

Louis, T. A. (1991). Using empirical Bayes methods in biopharmaceutical research. *Statistics in Medicine*, **10**, 811-827.

Mendelsohn, J. and Rice, R. (1982). Deconvolution of microfluorometric histograms with B splines. *Journal of the American Statistical Association*, **77**, 748-753.

Mukherjee, S. and Vapnik, V. (1999). Support vector method for multivariate density estimation. *Proceedings in Neural Information Processing Systems*, 659-665.

Pensky, M. and Vidakovic, B. (1999). Adaptive wavelet estimator for nonparametric density deconvolutoin. *Annals of Statistics*, **27**, 2033-2053.

Stefanski, L. and Carroll, R. J. (1990). Deconvoluting kernel density estimators. *Statistics*, **21**, 169-184.

Wang, X. and Wang, B. (2011). Deconvolution estimation in measurement error models: The R package decon. *Journal of Statistical Software*, **39**, Issue 10.