Journal of the Korean Data & Information Science Society
2014, **25**(4), 921–929

http://dx.doi.org/10.7465/jkdi.2014.25.4.921
한국데이터정보과학회지

Estimation for a bivariate survival model based on exponential distributions with a location parameter†

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Received 14 May 2014, revised 17 June 2014, accepted 1 July 2014

Abstract

A bivariate exponential distribution with a location parameter is proposed as a model for a two-component shared load system with a guarantee time. Some statistical properties of the proposed model are investigated. The maximum likelihood estimators and uniformly minimum variance unbiased estimators of the parameters, mean time to failure, and the reliability function of system are obtained with unknown guarantee time. Simulation studies are given to illustrate the results.

Keywords: Bivariate exponential distribution, reliability, uniformly minimum variance unbiased estimator.

1. Introduction

The exponential distribution has been widely used in life testing and reliability. To improve the reliability of a system, it is a common practice to connect several components in parallel. If component failures are mutually statistically independent, the estimation of the system reliability is straightforward. In some situations, however, all the components share the load during the mission time and the failure rate of the surviving components may increase due to increased load when a component fails. Studies in software reliability and fault-tolerant software have found that the reliability of software is a function of the system load as well as the processing time (see Iyer and Rossetti, 1985). In fault-tolerant design where multiple processors share the incoming load, the failure of one processor does increase the workload, and often the failure rate of the surviving processor (see Musa *et al*., 1987). Statistically dependent failures are typical for such systems. To correctly estimate the reliability of such systems, the increase of failure rate of the surviving components has to be considered.

Freund (1961) proposed a bivariate extension of the exponential distribution by allowing the failure rate of the surviving component to be changed after the failure of another component. Freund model can be viewed as a simple load-share model for a two-component system. He was one of the first who introduced a physically motivated model.

Weier (1981) obtained Bayes estimators for the parameters and reliability for the Freund model using a reparametrization. Lin *et al*. (1993) extended the Freund's bivariate model

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to a the trivarite and multivariate models for nonrepairable systems. Kim and Kvam (2004) extended the Freund's model to the general k component case and introduced load-share parameters using a reparameterization as in Weier (1981). They found maximum likelihood estimators of the parameters and derived a likelihood ratio test for testing equality of parameters against the alternative that they are monotone. Singh *et al.* (2008) obtained the maximum likelihood and Bayes estimates of the parameters of a k-components load-sharing parallel system in which some of the components follow an exponential distribution.

Scheuer (1988) obtained the reliability of m-out-of-n system under the assumption that failure of a component changes the failure rates of the surviving components. Shao and Lamberson (1991) considered a Markov model for analyzing the reliability and availability of an n-unit shared-load repairable k-out-of-n system with imperfect switching. Pham (1992) developed the reliability function of two-component shared-load system with imperfect coverage. Liu (1998) presented a generalized model to calculate the reliability of a load-sharing k-out-of-n system for arbitrary distributions. Cha (2009) obtained the maximum likelihood estimates and the modified maximum likelihood estimates of mean time to failure and reliability function for shared load model by using censored system life data. Cho (2012) obtained maximum likelihood estimates for parameters of Freund's model under the system level life testing.

However, in most research works for exponential shared-load model, they have assumed that location parameters are zero. Clearly, this is not always the case (see Lawless, 1982). For this reason in the present paper we suppose that location parameters are non-zero. In this paper, we propose a generalized Freund's model having a location parameter for a two-component shared-load system with a guarantee time and consider statistical inferences on the parameters, system mean time to failure (MTTF) and reliability. We obtain the maximum likelihood estimators (MLE's) and the uniformly minimum variance unbiased estimators (UMVUE's) of the parameters, MTTF, and the reliability.

2. Bivariate exponential distribution and its properties

Let X and Y represent the lifetimes of components C_1 and C_2 of a two-component parallel system. Suppose $X \sim E(\alpha^{-1}, \mu)$ and $Y \sim E(\beta^{-1}, \mu)$, where $E(\alpha^{-1}, \mu)$ denotes the two-parameter exponential distribution with probability density function(pdf) $g(x) = \alpha \exp\{-\alpha(x-\mu)\}, x > \mu$. Further, assume that the failure rate of component $C_2(C_1)$ changes to $\beta'(\alpha')$ from $\beta(\alpha)$ if $C_1(C_2)$ fails first, and the location parameter μ does not change. Applying the approach of Freund (1961), we obtain the joint density of (X, Y)

$$f(x,y) = \begin{cases} \alpha\beta' \cdot \exp -\beta'(y-\mu) - (\alpha+\beta-\beta')(x-\mu), \ y > x > \mu, \\ \alpha'\beta \cdot \exp -\alpha'(x-\mu) - (\alpha+\beta-\alpha')(y-\mu), \ x > y > \mu, \end{cases} \tag{2.1}$$

which is a bivariate extension of the two-parameter exponential distribution and does not allow the simultaneous failures. For convenience, we denote $(X, Y) \sim BVE\ (\alpha, \beta, \alpha', \beta', \mu)$ if a random vector (X, Y) is distributed as a BVE with the joint density function (2.1). Of course when μ=0, (2.1) coincides with Freund model. Freund has given some basic properties for $BVE(\alpha, \beta, \alpha', \beta', 0)$. Modifying his derivations, the moment generating function of BVE

$(\alpha, \beta, \alpha', \beta', \mu)$ is given by

$$\begin{aligned} m(s,t) &= \frac{\alpha+\beta}{1-(s+t)/(\alpha+\beta)}\left(\frac{\beta}{1-s/\alpha'}+\frac{\alpha}{1-t/\beta'}\right)\exp\left\{\mu(s+t)\right\} \\ &= (\alpha+\beta)\left\{1+\frac{s+t}{\alpha+\beta}+\left(\frac{s+t}{\alpha+\beta}\right)^2+\cdots\right\} \\ &\times\left\{\beta\left[1+\frac{s}{\alpha'}+\left(\frac{s}{\alpha'}\right)^2+\cdots\right]+\alpha\left[1+\frac{t}{\beta'}+\left(\frac{t}{\beta'}\right)^2+\cdots\right]\right\} \\ &\times\left\{1+\mu s+\frac{(\mu s)^2}{2!}+\cdots\right\}\left\{1+\mu t+\frac{(\mu t)^2}{2!}+\cdots\right\}. \end{aligned}$$

Selecting the appropriate coefficients of this power series in s and t, it follows that

$$E[X]=\frac{\alpha'+\beta}{\alpha'(\alpha+\beta)}+\mu,\ E[Y]=\frac{\alpha+\beta'}{\beta'(\alpha+\beta)}+\mu,$$

$$Var[X]=\frac{\alpha'^2+2\alpha\beta+\beta^2}{\alpha'^2(\alpha+\beta)^2},\ Var[Y]=\frac{\alpha^2+2\alpha\beta+\beta'^2}{\beta'^2(\alpha+\beta)^2},\ Cov(X,Y)=\frac{\alpha'\beta'-\alpha\beta}{\alpha'\beta'(\alpha+\beta)^2}.$$

Integrating (2.1), in turn, with respect to y and x, we obtain that

$$f_X(x)=\frac{(\alpha-\alpha')(\alpha+\beta)}{\alpha+\beta-\alpha'}\exp\left\{-(\alpha+\beta)(x-\mu)+\frac{\alpha'\beta}{\alpha+\beta-\alpha'}\exp\left\{-\alpha'(x-\mu)\right\}\right\},$$
$$x>\mu,\ \alpha+\beta\neq\alpha',$$

$$f_Y(y)=\frac{(\beta-\beta')(\alpha+\beta)}{\alpha+\beta-\beta'}\exp\left\{-(\alpha+\beta)(y-\mu)+\frac{\alpha'\beta}{\alpha+\beta-\beta'}\exp\left\{-\beta'(y-\mu)\right\}\right\},$$
$$y>\mu,\ \alpha+\beta\neq\beta',$$

which are mixtures of two 2-parameter exponential distributions.

If $\alpha+\beta=\alpha'$, then

$$f_X(x)=\left\{\alpha+\beta(\alpha+\beta)(x-\mu)\right\}\exp\left\{-(\alpha+\beta)(x-\mu)\right\},\ x>\mu$$

and if $\alpha+\beta=\beta'$

$$f_Y(x)=\left\{\beta+\alpha(\alpha+\beta)(y-\mu)\right\}\exp\left\{-(\alpha+\beta)(y-\mu)\right\},\ y>\mu,$$

which are mixtures of exponential and gamma distributions with the common location parameter μ. Generally, these marginal distributions are not two-parameter exponential. However, in the special cases where $\alpha=\alpha'$ and $\beta=\beta'$, X and Y are independently distributed as $E(1/\alpha,\mu)$ and $E(1/\beta,\mu)$, respectively. When two components are installed in parallel, the system reliability at mission time t is

$$\begin{aligned}R(t) =& P[\max(X,Y) > t]\\ =& \int_{\mu}^{t} P(T_2 > t|T_1 = t) f_X(t_1) dt_1 + \int_{\mu}^{t} P(T_1 > t|T_2 = t) f_Y(t_2) dt_2 + P(T_1 > t, T_2 > t)\\ =& \frac{\alpha}{\alpha+\beta-\beta'} \exp\{-\beta'(t-\mu)\} + \frac{\beta}{\alpha+\beta-\alpha'} \exp\{-\alpha'(t-\mu)\}\\ &+ \left(1 - \frac{\alpha}{\alpha+\beta-\beta'} - \frac{\beta}{\alpha+\beta-\alpha'}\right) \exp\{-(\alpha+\beta)(t-\mu)\}, \qquad (2.2)\end{aligned}$$

for $\alpha+\beta \neq \alpha'$ and $\alpha+\beta \neq \beta'$. In the case of $\alpha+\beta = \alpha'$ and $\alpha+\beta \neq \beta'$ one obtains from (2.2)

$$R(t) = \frac{\alpha}{\alpha+\beta-\beta'} \exp\{-\beta'(t-\mu)\} + \left(\frac{\beta-\beta'}{\alpha+\beta-\beta} + \beta(t-\mu)\right) \exp\{-(\alpha+\beta)(t-\mu)\}. \qquad (2.3)$$

If $\alpha+\beta \neq \alpha'$ and $\alpha+\beta = \beta'$, then

$$R(t) = \frac{\beta}{\alpha+\beta-\alpha'} \exp\{-\alpha'(t-\mu)\} + \left(\frac{\alpha-\alpha'}{\alpha+\beta-\beta'} + \alpha(t-\mu)\right) \exp\{-(\alpha+\beta)(t-\mu)\}. \qquad (2.4)$$

If $\alpha+\beta = \alpha' = \beta'$, then

$$R(t) = \{1 + (\alpha+\beta)(t-\mu)\} \exp\{-(\alpha+\beta)(t-\mu)\}. \qquad (2.5)$$

When $\mu = 0$, (2.2), (2.3), (2.4) and (2.5) coincide with (2-11), (2-12), (2-13), and (2-14) of Lin *et al.* (1993), respectively. (Note that the parametric conditions in Lin *et al.* (1993) include typographical errors. For example, in Equation (2-12), "$\alpha + \triangle\beta_{01} = 0$ and $\triangle\alpha_{01} + \beta \neq 0$" should be corrected to "$\alpha + \triangle\beta_{01} \neq 0$ and $\triangle\alpha_{01} + \beta = 0$". In Equation (2-13), "$\beta + \triangle\alpha_{01} = 0$ and $\triangle\beta_{01} + \alpha \neq 0$" should be corrected to "$\beta + \triangle\alpha_{01} \neq 0$ and $\triangle\beta_{01} + \alpha = 0$").

And the mean time to failure of the system, denoted by M, is given by

$$\begin{aligned}M =& \int_0^{\infty} R(t)dt = \int_0^{\mu} 1dt + \int_{\mu}^{\infty} R(t)dt\\ =& \frac{1}{\alpha+\beta}\left(1 + \frac{\alpha}{\beta'} + \frac{\beta}{\alpha'}\right) + \mu. \qquad (2.6)\end{aligned}$$

3. Estimation of parameters and reliability

In this section, we consider the model (2.1) where $\alpha = \beta = 1/\theta$ and $\alpha' = \beta' = 1/\theta'$. Then from (2.1), we obtain the joint density of (X, Y)

$$f(x,y) = \frac{1}{\theta\theta'} \cdot \exp[-\frac{2}{\theta}\{\min(x,y) - \mu\} - \frac{1}{\theta'}\{\max(x,y) - \min(x,y)\}], \qquad (3.1)$$

where $\min(x,y) > \mu,\ \theta > 0,\ \theta' > 0$. And from (2.2) and (2.5)

$$R(t) = \begin{cases} \dfrac{2\theta'}{2\theta'-\theta}\exp\left\{-\dfrac{t-\mu}{\theta'}\right\} - \dfrac{\theta}{2\theta'-\theta}\exp\left\{-\dfrac{2(t-\mu)}{\theta}\right\}, \text{ if } \theta \neq 2\theta', & (3.2a) \\ \left\{1+\dfrac{2(t-\mu)}{\theta}\right\}\exp\left\{-\dfrac{2(t-\mu)}{\theta}\right\}, \text{ if } \theta = 2\theta', & (3.2b) \end{cases}$$

and from (2.6), we obtain the MTTF

$$M = \theta/2 + \theta' + \mu. \tag{3.3}$$

If $(X_i, Y_i),\ i = 1, 2, \cdots, n$, is a random sample of size n from (3.1), then the likelihood function based on this is given by

$$L(\mu, \theta, \theta') = \prod_{i=1}^{n} f(x_i, y_i) = (\theta\theta')^{-n}\exp\left\{-\frac{2(t_1 - n\mu)}{\theta} - \frac{t_2}{\theta'}\right\},\ z_i > \mu, \tag{3.4}$$

where $t_1 = \sum_{i=1}^{n} z_i,\ t_2 = \sum_{i=1}^{n} w_i$, and where $z_i = \min(x_i, y_i)$, $w_i = \max(x_i, y_i) - \min(x_i, y_i)$. (A lower case letter will denote the realized value of a random variable denoted by the corresponding capital letter.) In the case of $\theta \neq 2\theta'$, the MLE's of μ, θ and θ' are given, respectively, by

$$\widehat{\mu} = Z_{(1)},\ \widehat{\theta} = (2T_1 - 2nZ_{(1)})/n,\ \widehat{\theta}' = T_2/n \tag{3.5}$$

where $Z_{(1)} = \min(Z_1, Z_2, \cdots, Z_n)$. In the case of $\theta = 2\theta'$, letting $\theta' = \theta/2$ in (3.4), we obtain the MLE's of μ and θ as

$$\widehat{\mu} = Z_{(1)},\ \widehat{\theta} = (T_1 + T_2 - nZ_{(1)})/n. \tag{3.6}$$

Employing the invariance property of MLE and the expressions (3.2) and (3.3), the MLE's $\widehat{R}(t)$ and $\widehat{M}$ of $R(t)$ and MTTF can be obtained.

We now find the UMVUE's of parameters, MTTF and reliability function. Suppose that $\theta \neq 2\theta'$. From (3.4), it is readily seen that $(Z_{(1)}, T_1, T_2)$ is a complete sufficient for (μ, θ, θ'). And from (3.1), it is well known (Lawless (1982)) that Z_i and W_i are independently distributed as $E(\theta/2, \mu)$ and $E(\theta', 0)$, respectively. Then it is easily seen that

$$\frac{2n(Z_{(1)-}\mu)}{\theta/2} \sim \chi^2(2),\ \frac{2(T_1 - nZ_{(1)})}{\theta/2} \sim \chi^2(2n-2), \text{ and } \frac{2T_2}{\theta'} \sim \chi^2(2n). \tag{3.7}$$

In the case of $\theta = 2\theta'$ it is easily seen that $(Z_{(1),}T_1 + T_2)$ is a complete sufficient for (μ, θ), and

$$\frac{2(T_1 + T_2 - nZ_{(1)})}{\theta/2} \sim \chi^2(4n-2). \tag{3.8}$$

Therefore, from (3.7) and (3.8), we have the following results.

Theorem 3.1 (UMVUE's of parameters and MTTF when $\mu > 0$)
1) If $\theta \neq 2\theta'$, then the UMVUE's of μ, θ and θ' are

$$\widetilde{\mu} = Z_{(1)} - (T_1 - nZ_{(1)})/n(n-1), \widetilde{\theta} = 2(T_1 - nZ_{(1)})/(n-1), \widetilde{\theta}' = T_2/n.$$

2) If $\theta = 2\theta'$, then the UMVUE's of μ and θ are

$$\widetilde{\mu} = Z_1 - (T_1 + T_2 - nZ_{(1)})/n(2n-1), \widetilde{\theta} = 2(T_1 + T_2 - nZ_{(1)})/(2n-1).$$

3) The UMVUE of MTTF is

$$\widetilde{M} = (T_1 + T_2)/n.$$

Theorem 3.2 Let $U = \min(T_1, T_2),\ V = \max(T_1, T_2)$, and $S = T_1 + T_2$. Then, for $n > 2$, the UMVUE of $R(t)$ is given by
1) Case of $\theta \neq 2\theta'$;

$$\widetilde{R}(t) = \frac{(t_2 + z_{(1)} - t)^{n-1}}{t_2^{n-1}} + \frac{A_1}{n-2}\int_0^{t-z_{(1)}} (t_2 - s)^{n-2}(t_1 + z_{(1)} - t + s)^{n-2} ds,$$
$$\text{if } z_{(1)} < t < \min(t_1, t_2) + z_{(1)},$$
$$\widetilde{R}(t) = \frac{A_1}{n-2}\int_0^{t_2} (t_2 - s)^{n-2}(t_1 + z_{(1)} - t + s)^{n-2} ds, \text{ if } t_2 + z_{(1)} < t \leq t_1 + z_{(1)},$$
$$\widetilde{R}(t) = \frac{(t_2 + z_{(1)} - t)^{n-1}}{nt_2^{n-1}} + \frac{A_1}{n-1}\int_0^{t_1} (t_1 - s)^{n-3}(t_2 + z_{(1)} - t + s)^{n-1} ds,$$
$$\text{if } t_1 + z_{(1)} < t \leq t_2 + z_{(1)},$$
$$\widetilde{R}(t) = \frac{A_1 B(n, n-2)}{n-1}(t_1 + t_2 + z_{(1)} - t)^{2n-3}, \text{ if } \max(t_1, t_2) + z_{(1)} < t \leq t_1 + t_2 + z_1,$$

2) Case of $\theta \neq 2\theta'$;

$$\begin{aligned}\widetilde{R}(t) =& 1, \text{ if } t < z_{(1)} \\ =& \frac{2(n-1)^2}{n}\left(\frac{t - z_{(1)}}{s}\right)\left(1 - \frac{t - z_{(1)}}{s}\right)^{2n-3} + \frac{n-1}{n}\left(1 - \frac{t - z_{(1)}}{s}\right)^{2n-2}, \text{ if } z_{(1)} \leq t \leq z_{(1)} + s \\ =& 0, \textit{ otherwise}.\end{aligned}$$

Proof of 1: Let ξ_1 be the one of the observations in $Z_i,\ i = 1, 2, \cdots, n$, and $W_1 = \max(X_1, Y_1) - \min(X_1, Y_1)$. Then $I(W_1 + \xi_1 > t)$ is an unbiased estimator of $R(t)$, where $I(\cdot)$ denotes the usual indicator function. Since $(T_1, T_2, Z_{(1)})$ is complete sufficient statistic, the conditional expectation $E[I(W_1 + \xi_1 > t)|T_1 = t_1, T_2 = t_2, Z_{(1)} = z_{(1)}]$ is the UMVUE. By noting that $(\xi_1, T_1, Z_{(1)})$ and (W_1, T_2) are independent, the conditional joint density $f(w_1, \xi_1|t_1, t_2, z_{(1)})$ of (w_1, ξ_1) given $(T_1, T_2, Z_{(1)})$ is the product of two conditional distributions $g(\xi_1|t_1, z_{(1)})$ and $h(w_1|t_2)$. Applying the approach of Laurent (1963), we obtain the conditional density of ξ_1 given $(T_1, Z_{(1)})$ as

$$g(\xi_1|t_1, z_{(1)}) = \begin{cases} 1/n, \xi_1 = z_{(1)}, \\ (n-1)(n-2)(t_1 + z_{(1)} - \xi_1)^{n-3}/nt_1^{n-2}, z_{(1)} < \xi_1 \leq t_1 + z_{(1)}, \\ 0, \text{ otherwise.} \end{cases} \quad (3.9)$$

And the conditional density of w_1 given T_2 is easily obtained as

$$h(w_1|t_2) = \begin{cases} (n-1)(t_2 - w_1)^{n-2}/t_2^{n-1}, \ 0 < w_1 < t_2, \\ 0, \text{otherwise}. \end{cases} \tag{3.10}$$

From (3.9) and (3.10), the conditional joint density of (w_1, ξ_1) given $(T_1, T_2, Z_{(1)})$ is

$$\begin{aligned} f(w_1, \xi_1 | t_1, t_2, z_{(1)}) &= g(\xi_1 | t_1, z_{(1)}) \cdot h(w_1|t_2) \\ &= \begin{cases} A_1(t_1 + z_{(1)} - \xi_1)^{n-3}(t_2 - w_1)^{n-2}, \ \text{if } z_{(1)} < \xi_1 \le t_1 + z_{(1)}, \ 0 < w_1 < t_2, \\ A_2(t_2 - w_1)^{n-2}, \ \text{if } z_{(1)} = \xi_1, \ 0 < w_1 < t_2. \end{cases} \end{aligned}$$

According to the Rao-Blackwell-Lehmann-Scheffe' theorem, the UMVUE of $R(t)$ is given by

$$\widetilde{R}(t) = \iint I(w_1 + \xi_1 > t) f(w_1, \xi_1 | t_1, t_2, z_{(1)}) dw_1 d\xi_1. \tag{3.11}$$

The stated results can be obtained by evaluating the integral of (3.11). □

Proof of 2: We now find the UMVUE of $R(t)$ when $\theta = 2\theta'$. Let $S = T_1 + T_2 - nZ_{(1)}$. Let ξ_1 be the one of the observations in $Z_i, i = 1, 2 \cdots, n$, from $E(\theta, \mu)$ and ξ_2 be the one of the observations in $W_i, i = 1, 2, \cdots, n$, from $E(\theta, 0)$. The conditional distribution of (ξ_1, ξ_2) given $(S, Z_{(1)}) = (s, z_{(1)})$ is given by (see Appendix)

$$p(\xi_1, \xi_2 | s, z_{(1)}) = \begin{cases} 1/n, \xi_1 = z_{(1)}, \\ A_3(s + z_{(1)} - \xi_1 - \xi_2)^{2n-4}, \ z_{(1)} < \xi_1 < z_{(1)} + s - \xi_2, 0 < \xi_2 < s, \\ 0, \ \text{otherwise}, \end{cases} \tag{3.12}$$

where $A_3 = 2(n-1)^2(2n-3)/ns^{2n-2}$. Thus the UMVUE $\widetilde{R}(t) = E[I(\xi_1 + \xi_2 > t)|S = s, Z_{(1)} = z_{(1)}]$ of $R(t)$ can be obtained by evaluating the integral

$$\widetilde{R}(t) = \iint I(\xi_1 + \xi_2 > t) p(\xi_1, \xi_2 | s, z_{(1)}) d\xi_1 d\xi_2.$$ □

4. Simulations

In this sub-section, we compare the relative bias and the mean square error(MSE) of the MLE and UMVUE of $R(t)$ for moderate sample sizes n=(10, 20, 30) through Monte Carlo simulation. Estimates of the relative bias and MSE were obtained from 1,000 trials with $\mu = (0, 5)$, $\theta = 10$, and $\theta' = (2.5, 5.0, 7.5)$. In Tables 1 and 2, the relative bias(in percentage) of MLE(or UMVUE) is calculated from 100× ($\widehat{R}(t)$ or $\widetilde{R}(t) - R(t))/R(t)$, and REF represents the relative efficiency of UMVUE over MLE, that is, REF=MSE ($\widehat{R}(t)$) /MSE ($\widetilde{R}(t)$). Although $\widetilde{R}(t)$ is known to be unbiased, its estimated relative bias is recorded as a check on the computations.

In general, as expected, the estimated absolute relative bias of UMVUE is found to be considerably smaller than the estimated absolute relative bias of MLE. The estimated absolute relative bias and MSEs of the two estimators tend to decrease as n increases. Furthermore, the MSEs of both estimates appear to be nearly equal.

Table 4.1 Estimates of relative bias, MSE and efficiency ($\theta = 10, \mu = 5, t = 6.0$)

θ'	$R(t)$	n	relative bias of		MSE of		REF
			MLE	UMVUE	MLE	UMVUE	
2.5	0.9671	10	-1.623	0.145	0.0087	0.0095	0.916
		20	-0.910	-0.124	0.0063	0.0062	1.016
		30	-0.724	-0.083	0.0041	0.0040	1.025
5	0.9825	10	-1.679	0.173	0.0098	0.0096	1.021
		20	-1.008	-0.153	0.0067	0.0062	1.081
		30	-0.733	-0.102	0.0044	0.0041	1.073
7.5	0.9881	10	-1.700	0.192	0.0099	0.0097	1.021
		20	-1.063	-0.172	0.0061	0.0056	1.089
		30	-0.789	-0.111	0.0048	0.0031	1.548

Table 4.2 Estimates of relative bias, MSE and efficiency ($\theta = 10, \mu = 0, t = 1.0$)

θ'	$R(t)$	n	relative bias of		MSE of		REF
			MLE	UMVUE	MLE	UMVUE	
2.5	0.9671	10	-1.799	0.238	0.0086	0.0095	0.905
		20	-0.827	-0.124	0.0062	0.0051	1.216
		30	-0.672	-0.062	0.0023	0.0020	1.150
5	0.9825	10	-1.842	0.244	0.0098	0.0097	1.010
		20	-0.885	-0.122	0.0067	0.0063	1.063
		30	-0.692	-0.071	0.0026	0.0021	1.238
7.5	0.9881	10	-1.913	0.253	0.0098	0.0107	0.916
		20	-0.901	-0.142	0.0070	0.0065	1.077
		30	-0.617	-0.091	0.0031	0.0025	1.240

5. Conclusions

In this paper, we proposed a generalized Freund's model having a location parameter for a two-component shared-load system with a non-zero guarantee time. We obtained the MLE's and the UMVUE's of the parameters, MTTF, and the reliability of the model.

Appendix

Derivation of formula (3.12)

Let $Z^* = (Z_1^*, Z_2^*, \cdots, Z_{n-1}^*)$ be the subsample obtained by deleting ξ_1 from Z_i, $i = 1, 2, \cdots, n$, and $W^* = (W_1^*, W_2^*, \cdots, W^*{}_{n-1})$ be the subsample obtained by deleting ξ_2 from W_i, $i = 1, 2, \cdots, n$. Finally let

$$Z_{(1)}^* = \min(Z_1^*, Z_2^*, \cdots, Z_{n-1}^*)$$

and

$$S^* = \sum{}_{i=1}^{n-1} Z_i^* + \sum{}_{i=1}^{n-1} W_i^* - (n-1)Z_{(1)}^*.$$

Since $\dfrac{2n(Z_{(1)} - \mu)}{\theta/2} \sim \chi^2(2)$ and $\dfrac{2s}{\theta/2} \sim \chi^2(4n-2)$, we can show that

$$q(s, z_{(1)}) = ns^{2n-2} \exp\left\{-(s + n(z_{(1)} - \mu))/\theta\right\} / \theta^{2n}\Gamma(2n-1). \tag{A.1}$$

Then the joint pdf of $(\xi_1, \xi_2, S^*, Z^*_{(1)})$ is

$$p(\xi_1, \xi_2, s^*, z^*_{(1)}) = f_z(\xi_1) f_w(\xi_2) q(s^*, z^*_{(1)})$$
$$= (n-1) s^{*2n-4} \exp\left\{-(\xi_1 - \mu + \xi_2 + s^* + (n-1)(z^*_{(1)} - \mu))/\theta\right\} / \theta^{2n}\Gamma(2n-3), \quad \text{(A.2)}$$

if $\xi_1 > \mu, \xi_2 > 0, s^* > 0, z^*_{(1)} > \mu$.
Consider the case where $\xi_1 > Z_{(1)}$. In this case $Z^*_{(1)} = Z_{(1)}$, and $S^* = S - (\xi_1 - Z_{(1)}) - \xi_2$. The joint pdf of $(\xi_1, \xi_2, S^*, Z^*_{(1)})$ is, from (A.2),

$$p(\xi_1, \xi_2, s^*, z^*_{(1)}) = \frac{(n-1)}{\theta^{2n}\Gamma(2n-3)} (s + z_{(1)} - \xi_1 - \xi_2)^{2n-4} \exp\left\{-\frac{s + n(z_{(1)} - \mu)}{\theta}\right\}, \quad \text{(A.3)}$$

if $\mu < z_{(1)} < \xi_{(1)} < z_{(1)} + s - \xi_2, 0 < \xi_2 < s, s > 0$. From (A.1) and (A.3), the conditional distribution of (ξ_1, ξ_2) given $(S, Z_{(1)}) = (s, z_{(1)})$ is then given by formula (3.12).

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