KYUNGPOOK Math. J. 54(2014), 271-292
http://dx.doi.org/10.5666/KMJ.2014.54.2.271

Polynomial Unknotting and Singularity Index

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Abstract. We introduce a new method to transform a knot diagram into a diagram of an unknot using a polynomial representation of the knot. Here the unknotting sequence of a knot diagram with least number of crossing changes can be realized by a family of polynomial maps. The number of singular knots in this family is defined to be the *singularity index* of the diagram. We show that the singularity index of a diagram is always less than or equal to its unknotting number.

1. Introduction

Distinguishing knots up to ambient isotopy is still a major task amongst knot theorists. Many knot invariants have been invented for this purpose. They are either numbers or polynomials or some algebraic structures. However, an invariant is useful only if it is computable. *Unknotting number* of a knot is one such knot invariant. This is a number, that can be easily defined (cf. [4]), but is hard to compute. For a very long period of time knot theorists had no general method for finding the unknotting number of knots except when its value is 1. Using the techniques of 4- dimensional topology Peter Kronheimer and Tromasz Mrowka [5] proved a 40-year old conjecture due to John Milnor [3] regarding the unknotting number of all torus knots of type (p, q), which says that *unknotting number for all torus knots of type* (p, q) *(p and q being co-prime) is* $(p-1)(q-1)/2$. This was later proved by Rasmussen using Khovanov Homology [6]. For other classes of knots nothing very significant can be said.

In this paper, we have made an attempt to compute the unknotting number of knot diagrams using their polynomial representations. We know that the classical knots in S^3 can be realized as one point compactification of some proper and smooth embedding of $\mathbb{R}$ in $\mathbb{R}^3$ and a three dimensional plot of such embedding, when viewed on a suitable plane, represents a knot diagram of an open knot. It has been proved that, given any open knot diagram, one can choose an embedding of the form $t \mapsto (f(t), g(t), h(t))$, where $f(t)$, $g(t)$ and $h(t)$ are real polynomials, that represnts

Received April, 2012; revised September 10, 2012; accepted March 19, 2013.
2010 Mathematics Subject Classification: Primary 57M25; Secondary 14P25.
Key words and phrases: unknotting number, double point, immersion.

this diagram. Using such a polynomial representation we define an unknotting operation in a knot diagram with the hope that it will help in the computation of unknotting number. We prove (in Proposition 2.8) that *given a polynomial knot* $\phi : \mathbb{R} \to \mathbb{R}^3$*, there exists a continuous one parameter family of polynomial maps* P_s *which are immersions from* $\mathbb{R}$ *to* $\mathbb{R}^3$ *such that* $P_0 = \phi$ *and* P_R *represents an unknot for some* R *and the projections of each* P_s *for* $0 \leq s \leq R$ *are identical.* Thus the given knot is transformed into an unknot by making the changes in the nature of crossing. During this homotopy each time when we pass through a singular knot, the knot type changes. We observe that, this provides us the exact information regarding the order in which the crossings are switched. Each switch in a crossing corresponds to some P_s which is not an embedding, i.e., is a singular knot. As the deformation takes place inside a compact set the number of singular knots in this deformation is finite and we define it to be the *polynomial singularity index* of the polynomial knot. The minimum of all polynomial singularity index over all polynomial representations representing a specific knot diagram is defined as the *singularity index* of that knot diagram. The minimal singularity index of all knot diagrams of a knot K is defined as the *singularity index* of the knot K. The singularity index of a knot is difficult to compute. However the singularity index of a knot diagram can be computed easily by the method described here. We prove that *the singularity index of a knot diagram is always less than or equal to its unknotting number.*

This paper is organized as follows: In Section 2, we provide the basic background regarding polynomial knots which is relevant in the context of this paper. In Section 3, we define the singularity index of a knot and at the end we prove the main theorem. In Section 4, we compute the singularity index of a few knot diagrams using their polynomial representation constructed by us in our earlier work.

2. Polynomial Knots

Definition 2.1: By a polynomial knot we mean an embedding $\varphi : \mathbb{R} \hookrightarrow \mathbb{R}^3$ defined as $\phi(t) = (f(t), g(t), h(t))$, where $f(t), g(t)$ and $h(t)$ are real polynomials.

Definition 2.2: Two polynomial knots φ_0 and φ_1 are said to be *P-isotopic* if there exists a one parameter family $\{p_s, 0 \leq s \leq 1\}$ of polynomial knots (embeddings)such that $p_0 = \varphi_0$ and $p_1 = \varphi_1$. This family $\{p_s, 0 \leq s \leq 1\}$ is called a *P-isotopy* between φ_0 and φ_1.

Definition 2.3: Two polynomial knots φ_0 and φ_1 are said to be *P-regular isotopic* if there exists a one parameter family $\{p_s, 0 \leq s \leq 1\}$ of polynomial knots (embeddings) such that $p_0 = \varphi_0$ and $p_1 = \varphi_1$ and the knot diagrams of each p_s in this family differ from each other by Reidemister Moves (2) and (3) (cf. [4]) only. This family $\{p_s, 0 \leq s \leq 1\}$ is called a *P- regular isotopy* between φ_0 and φ_1.

It is easy to check the following remarks.

Remark 2.4:

1. Given a polynomial knot $t \mapsto (f(t), g(t), h(t))$, up to P-isotopy we can always assume that the degree of $h(t)$ is odd and its leading coefficient is positive.

2. The set of ambient isotopy classes of open knots is in bijective correspondence with the set of P-isotopy classes of polynomial knots.

3. Every polynomial knot is P-isotopic to some polynomial knot defined as $\varphi(t) = (f(t), g(t), h(t))$, where $f(t), g(t)$ and $h(t)$ are real polynomials such that the map $t \mapsto (f(t), g(t))$ from $\mathbb{R}$ to $\mathbb{R}^2$ is a generic immersion, i.e., the projection of ϕ into xy plane is a regular projection, degree of $h(t)$ is odd, leading coefficient of $h(t)$ is positive and for each parametric value t corresponding to a double point in this regular projection $h(t) < 0$ for an under crossing and $h(t) > 0$ for an over crossing. A polynomial knot with this property will be referred to as a *good polynomial knot*.

4. Given a good polynomial knot $t \mapsto (f(t), g(t), h(t))$ there is a naturally associated knot diagram drawn on xy plane.

Definition 2.5: Two polynomial knots φ_0 and φ_1 are said to be *strongly P-regular homotopic* if there exists a one parameter family $\{p_s = (f_s, g_s, h_s),\ 0 \leq s \leq R\}$ of polynomial maps from $\mathbb{R}$ to $\mathbb{R}^3$ such that $p_0 = \varphi_0$ and $p_R = \varphi_1$ and for each $0 \leq s \leq R$, the maps $t \mapsto (f_s(t), g_s(t))$ have the same crossing data, i.e., the pairs (t_1, t_2) for which $f_s(t_1) = f_s(t_2)$ and $g_s(t_1) = g_s(t_2)$ is same for all $s \in [0, R]$.

Thus if two polynomial knots are strongly P-regular homotopic then their diagrams differ from each other only in terms of difference in the nature of crossings i.e., the diagram of the other polynomial knot can be obtained by changing some over crossings in first diagram into the under crossings or vice-versa.

Let $(f(t), g(t), h(t))$ be a good polynomial knot. Let (s_i, t_i) be the parameters where there is a crossing, i.e., $f(s_i) = f(t_i)$ and $g(s_i) = g(t_i)$. Let $m_i(h) = \frac{|h(s_i)-h(t_i)|}{|s_i-t_i|}$. Each $m_i(h)$ is a positive real number. Given a polynomial knot $t \to (f(t), g(t), h(t))$ we can compare $m_i(h)$ and $m_j(h)$ for each $i \neq j$. Suppose $m_{i_1}(h) < m_{i_2}(h) < \ldots < m_{i_n}(h)$. Then $\{i_1, i_2, \ldots, i_n\}$ defines an order on the set $\{1, 2, \ldots, n\}$. In the next proposition we show that it is possible to attain each order among $m_i(h)$ by choosing a suitable good polynomial representation of a knot diagram.

Proposition 2.6: Let D be a knot diagram of a knot K with n crossings. Let σ be an order on $\{1, 2, \ldots, n\}$. Then there exists a good polynomial knot $t \mapsto (f(t), g(t), h_\sigma(t))$ representing the diagram D. Suppose in this representation

the crossings occur at parametric pairs of values (s_i, t_i), $i = 1, 2, \ldots, n$. Then the $m_i(h_\sigma) = \frac{|h_\sigma(s_i) - h_\sigma(t_i)|}{|s_i - t_i|}$ satisfy the order σ.

Proof: A knot diagram consists of two things, namely,

1. a generic projection into a plane say XY plane which is a plane curve with finitely many real ordinary double points

2. at each double point, over/under crossing information.

First, we only look at the projection. This is a real plane curve with finite number of real ordinary double points. We would like to give a polynomial parametrization for this curve. Suppose this curve has m local extrema in the x direction. Consider the plane curve $C : (x(t), y(t)) = (t^{m+1}, t^d)$ where d is the least positive integer greater than or equal to $[\frac{2n}{m}]$ and co-prime to $m + 1$. This curve C has an isolated singularity at the origin. For such plane curves, there are two important numbers that remain invariant under any formal isomorphisms of plane curves. The first one is known as the Milnor number (cf. [2]) denoted by μ and the other is the δ invariant. For a single component plane curve they satisfy the relation $2\delta = \mu$. The Milnor number number μ is defined as

$$\mu = \dim \frac{\mathbb{C}[x, y]}{(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})}.$$

Thus in this case it turns out that the δ invariant is equal to $\frac{m(d-1)}{2}$. The δ invariant of a plane curve which is singular at origin measures the number of double points that can be created in a neighborhood of the origin. Note that, we have $\delta \geq n$. If $\delta = n$, by a result of real algebraic geometry by A'Campo [1] we can deform the curve C into a new curve $\tilde{C} : (x(t), y(t)) = (t^{m+1} + a_1 t^m + a_2 t^{p-2} + \ldots + a_{m+1}, (1 + b_0)t^d + b_1 t^{d-1} + \ldots + b_d)$ such that $\tilde{C}$ has all $\delta = n$ real nodes near origin. If $\delta = n + r$, then we can use a result of Daniel Pecker [13] to deform the curve $C : (x(t), y(t)) = (t^{m+1}, t^d)$ into a new curve $\tilde{C} : (x(t), y(t)) = (t^{m+1} + a_1 t^m + a_2 t^{p-2} + \ldots + a_{m+1}, (1 + b_0)t^d + b_1 t^{d-1} + \ldots + b_d)$ such that $\tilde{C}$ has n real nodes and r imaginary nodes. By continuity argument, we can choose the coefficients a_is and b_is such that the nodes occur in the order they are in the regular projection of the given knot. Suppose the n crossings occur at the parametric values (s_1, t_1), (s_2, t_2), $\ldots$, (s_n, t_n) and the crossing data in diagram D is such that in the pairs (s_i, t_i) the point corresponding to $t = s_i$ is above the point corresponding to $t = t_i$. Thus for an over crossing $s_i < t_i$ and for an under crossing $t_i < s_i$. Choose real numbers a_i and b_i for $i = 1, 2, \ldots, n$ such that the numbers $\frac{a_i + b_i}{|s_i - t_i|}$ obey the order σ. Here we have $n - 1$ inequations in $2n$ variables and thus we can find infinitely many solutions for a_i and b_i. After choosing a_is and b_is, we consider a polynomial $h_\sigma(t) = t^{2n+1} + \alpha_1 t^{2n} + \alpha_2 t^{2n-1} + \cdots + \alpha_{2n} t$ of degree $2n + 1$. We want to choose the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ such that $h_\sigma(s_i) = a_i$ and $h_\sigma(t_i) = -b_i$. Note that

the values of s_i and t_i are known. Thus these conditions will result into $2n$ linear equations in $2n$ variables $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$. This system of linear equations will be of the form

$$\begin{array}{rcl} s_1^{2n}\alpha_1 + s_1^{2n-1}\alpha_2 \cdots + s_1\alpha_{2n} & = & a_1 - s_1^{2n+1} \\ t_1^{2n}\alpha_1 + t_1^{2n-1}\alpha_2 \cdots + t_1\alpha_{2n} & = & -b_1 - t_1^{2n+1} \\ s_2^{2n}\alpha_1 + s_2^{2n-1}\alpha_2 \cdots + s_2\alpha_{2n} & = & a_2 - s_2^{2n+1} \\ t_2^{2n}\alpha_1 + t_2^{2n-1}\alpha_2 \cdots + t_2\alpha_{2n} & = & -b_2 - t_2^{2n+1} \\ & \vdots & \\ s_n^{2n}\alpha_1 + s_n^{2n-1}\alpha_2 \cdots + s_n\alpha_{2n} & = & a_n - s_n^{2n+1} \\ t_n^{2n}\alpha_1 + t_n^{2n-1}\alpha_2 \cdots + t_n\alpha_{2n} & = & -b_n - t_n^{2n+1} \end{array}$$

The coefficient matrix of this system has determinant

$$s_1t_1s_2t_2\cdots s_nt_n \prod_{i\neq j}(s_i - s_j)(s_i - t_j)(t_i - t_j).$$

Here each s_is and t_is are distinct and also each $s_i \neq t_j$. By perturbing the coefficients of the polynomials $f(t)$ and $g(t)$ we can ensure that each s_i and t_i is nonzero. Thus we can always find a solution for the above system of linear equations. This polynomial $h_\sigma(t)$ will provide the over/under crossing information as well as $m_i(h_\sigma)$ will satisfy the order σ. Also the polynomial $h_\sigma(t)$ has positive leading coefficient and its degree is odd. This completes the proof. □

Our next aim is to show that given a knot diagram of a polynomial knot we can realize an unknotting sequence of knot diagrams by a one parameter family of polynomial knots. For that we will first prove the following:

Lemma 2.7: Let $p(t)$ be an even degree polynomial, with leading coefficient positive, over the field of real numbers. Then there exists some $R > 0$ such that $p(t) + R > 0$ for all $t \in \mathbb{R}$.

Proof: Since the degree of $p(t)$ is even and the leading coefficient is positive we can find a closed interval $[a, b]$ such that $p(t) > 0$ for all t outside $[a, b]$. Thus, all the points where $p(t)$ can be negative lies inside the closed interval $[a, b]$. Consider the function $q : [a, b] \to \mathbb{R}$ defined as $q(t) = |p(t)|$. q is a continuous function on a compact set and hence will attain its maxima. Let M be the maximum value of $q(t)$ over the interval $[a, b]$. Let $R > M$. Clearly $R > 0$. Consider $p(t) + R$. For all t outside $[a, b]$ we certainly have $p(t) + R$ positive. For $t \in [a, b]$ we have $p(t) \leq q(t) < R$. Now for those $t \in [a, b]$ for which $p(t) > 0$ there is no problem. Let $t_0 \in [a, b]$ be such that $p(t_0) < 0$ say $p(t_0) = -r$. Then $q(t_0) = r$ and $R > r$. Thus $R - r > 0$,

i.e., $p(t_0)+R > 0$. Thus we have proved that $p(t)+R$ is always positive for all $t \in \mathbb{R}$.

Proposition 2.8: Every polynomial knot is strongly P-regular homotopic to a polynomial unknot.

Proof: Let $\varphi : \mathbb{R} \hookrightarrow \mathbb{R}^3$ be a polynomial knot defined as $\varphi(t) = (f(t), g(t), h(t))$. We can assume that φ is a good polynomial knot, i.e., the map $t \mapsto (f(t), g(t))$ is an immersion and the polynomial $h(t)$ has positive coefficient with $deg(h(t))$ is odd. For each $s \in \mathbb{R}$ consider a family of maps $\Phi_s : \mathbb{R} \hookrightarrow \mathbb{R}^3$ as $\Phi_s(t) = (f(t) + s, g(t) + s, h(t) + s^2t)$. The proposition now follows from the following two claims.

Claim 1. For each $s \in \mathbb{R}$ the map $\phi_s(t) = (f(t) + s, g(t) + s, h(t) + s^2t)$ is an immersion and the maps $t \mapsto (f(t) + s, g(t) + s)$ have the same crossing data as that of $t \mapsto (f(t), g(t))$.

For each s the derivative map is $\phi'_s(t) = (f'(t), g'(t), h'(t) + s^2)$ and as $\phi(t) = (f(t), g(t), h(t))$ is an embedding with $t \mapsto (f(t), g(t))$ an immersion it follows that $t \mapsto (f(t)+s, g(t)+s)$ is an immersion. Also by solving algebraically for crossings we can see that they have the same crossing data as that of $t \mapsto (f(t), g(t))$. This proves Claim 1.

Claim 2. There exists some real number $R \geq 0$ such that for $s \geq R$ the maps $\phi_s : \mathbb{R} \hookrightarrow \mathbb{R}^3$ represent the trivial knot.

Consider the map $t \mapsto (f(t) + s, g(t) + s, h(t) + s^2t)$. If $h'(t)$ is always positive, i.e., $h(t)$ is monotonically increasing then the knot given by $t \mapsto (f(t), g(t), h(t))$ is itself a trivial knot and hence there is nothing to prove as we can take $R = 0$.

Consider the case when $h'(t)$ is not always positive. Now, here we have assumed that the degree of the polynomial $h(t)$ is odd. Let $h_1(t) = h'(t)$. Here $h_1(t)$ is an even degree polynomial with positive leading coefficient. By Lemma 2.7, we can find some $M > 0$ such that $h_1(t) + M > 0$ for all $t \in \mathbb{R}$. Consider $h_s(t) = h(t) + s^2t$. Let $R = \sqrt{M}$. Then for $s \geq R$ we have $h'_s(t) > 0$ i.e., $h_s(t)$ is monotonically increasing for $s \geq R$. Hence the knot given by $t \mapsto (f(t) + s, g(t) + s, h(t) + s^2t)$ is a trivial knot for $s \geq R$. This proves Claim 2. □

This proof demonstrates that we have a continuous map $\Phi : \mathbb{R} \times [0, R] \longrightarrow \mathbb{R}^3$ such that $\Phi(t, 0) = (f(t), g(t), h(t))$, is the given knot and $\Phi(t, R) = (f(t) + R, g(t) + R, h(t) + R^2t)$ is a trivial knot and for each $s \in [0, R]$ the map $\Phi(t, s) = (f(t) + s, g(t) + s, h(t) + s^2t)$ is an immersion. The values of s for which $\Phi(t, s) = (f(t)+s, g(t)+s, h(t)+s^2t)$ fails to be an embedding are called the singular values.

Remark 2.9:

1. Let D be a knot diagram with n crossings. Then for each order σ among $\{1, 2, \ldots, n\}$ by Proposition 2.6, we have a polynomial knot $P_\sigma : t \to (f(t), g(t), h_\sigma(t))$ representing the diagram D in which $m_i(h_\sigma)$ (defined above) satisfy the order σ. Now, for each P_σ by Proposition 2.8, there exists a real number $R_\sigma > 0$ such that the maps defined by $t \to (f(t) + s, g(t) + s, h_\sigma(t) + s^2 t)$ are unknots for $s \geq R_\sigma$.

2. By Proposition 2.8 it follows that if the given polynomial knot is non trivial then we can obtain a polynomial unknot with the same crossing data whose diagram is obtained by switching some of the crossings of the given knot from over crossing to under crossing or vice versa. As it is a continuous deformation, for some finite number of values of $s \in [0, R]$ the maps $\Phi(-, s) : \mathbb{R} \longrightarrow \mathbb{R}^3$ must be " singular knots " , i.e., must have double points.

3. Singularity Index

Definition 3.1: Given a knot diagram D_K the least number of crossing changes required to convert it into a knot diagram of an unknot is called the unknotting number of that diagram denoted by $u(D_K)$.

Definition 3.2: The unknotting number of a knot K is defined as the minimal number of crossing changes required among all possible diagrams of K to be able to convert it into the unknot. It is a knot invariant and is denoted by u(K).

Definition 3.3: Let a polynomial knot ϕ be defined by $\phi(t) = (f(t), g(t), h(t))$, with say n crossings. Let R be the least positive real number such that the map $\Phi_s : \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by $t \mapsto (f(t) + s, g(t) + s, h(t) + s^2 t)$ represents a trivial knot for $s \geq R$. Then the minimum number of singular values, i.e., the values of $s \in [0, R]$ for which the map Φ_s is a singular knot, is defined as the Singularity index of the polynomial knot ϕ denoted by SI_ϕ.

Definition 3.4: Let D be a knot diagram with n crossings. We define the Singularity index $SI(D)$ of the diagram D as:

$$SI(D) = \min\{SI_\phi | \phi \text{ is a polynomial knot that represents } D\}.$$

Remark 3.5: For each order σ on $\{1, 2, \ldots, n\}$ let $P_\sigma : t \to (f(t), g(t), h_\sigma(t))$ denote a polynomial knot that represents the diagram D and for which the $m_i(h_\sigma)$ defined in Proposition 2.6 satisfy the order σ. It is easy to check that

$$SI(D) = \min\{SI_{P_\sigma}\}.$$

Since there are finitely many σ we can easily compute $SI(D)$ for a given knot diagram D.

Definition 3.6: The minimum value of all $SI(D)$, minimum taken over all knot diagrams that represent a knot K is defined as the *singularity index* of the knot K and is denoted by $SI(K)$.

Remark 3.7:

1. The singularity index SI(K) of a knot is a knot invariant.
2. The singularity index of the unknot is zero.
3. For any nontrivial knot K, $SI(K) \geq 1$.

Theorem 3.8: *The singularity index of a knot diagram is less than or equal to its unknotting number.*

Proof. Let K be a knot. Let " $\phi : \mathbb{R} \to \mathbb{R}^3$ defined by $\phi(t) = (f(t), g(t), h(t))$" be a polynomial representation of K and D_K is the knot diagram of K represented by ϕ. From Proposition 2.8 ϕ is strongly-P-regular homotopic to a polynomial unknot. Let R be the smallest real number such that the polynomial knot ϕ_K^R defined by $\phi_K^R(t) = (f(t) + R, g(t) + R, h(t) + R^2 t)$ has a diagram $D_{\phi^R(K)}$ of an unknot. Clearly $D_{\phi^R(K)}$ is obtained by making some crossing changes in D_K. Let (s_i, t_i), $s_i < t_i$ $i = 1, 2, \ldots, N$ be the parametric values for which the crossings occur, i.e., $f(s_i) = f(t_i)$ and $g(s_i) = g(t_i)$. Suppose for some (s_i, t_i) we have $h(s_i) < h(t_i)$, i.e, (s_i, t_i) is an under crossing. Then $h(s_i) + s^2 s_i = h(t_i) + s^2 t_i$ will not have any real solution. On the other hand if for some (s_j, t_j) we have $h(s_j) > h(t_j)$ then the equation $h(s_j) + s^2 s_j = h(t_j) + s^2 t_j$ will have a unique real solution $s_0 \in [0, R]$. In fact $s_0^2 = \frac{h(t_1)-h(t_2)}{t_2-t_1}$. If we take $s > s_0$ then in the embedding $\phi_s : t \to (f(t) + s, g(t) + s, h(t) + s^2 t)$ we will have $h(s_j) + s^2 s_j < h(t_j) + s^2 t_j$ which means (s_j, t_j) is an under crossing in this embedding. Thus in the transformation from ϕ to ϕ_s the crossing (s_j, t_j) has been switched from an over crossing to an under crossing. If our polynomial embedding represents a symmetric diagram of the knot in the sense that if $(-s_j, t_j)$ is a crossing the $(-t_j, s_j)$ is also a crossing, then in this transformation for $s > s_0$ both the crossings corresponding to $(-s_j, t_j)$ and $(-t_j, s_j)$ will switch from over to under crossing. Thus, in the P-regular homotopy inside the interval $[0, R]$ each singular knot indicates at least one crossing change in the diagram $D_{\phi(K)}$ (in case of symmetric diagrams there may be two crossing changes taking place). Thus the number of singular knots before arriving at a diagram of an unknot is less than or equal to the number of crossing changes occurred before getting a knot diagram of an unknot. Thus the singularity index of the diagram D_K is less than or equal to the unknotting number $u(D_K)$ of the diagram D_K. This completes the proof. □

4. Examples

In this section, we compute the polynomial singularity index for few polynomial knots. This gives an estimate for the singularity index of these knots. We can compare the singularity index with the unknotting numbers.

1. Let us consider a polynomial representation of the *trefoil knot* which is the torus knot of type $(2,3)$ as $t \mapsto (t^3-3t, t^4-4t^2, t^5-10t)$. The knot diagram associated to this representation is shown in Figure 1.

   Let us compute the singularity index of this polynomial knot. Consider the immersion $(t,s) \mapsto (t^3-3t+s, t^4-4t^2+s, t^5-10t+s^2t, s)$. We can see that for $s^2 > 10$, i.e., for $s > \sqrt{10}$ the maps $t \mapsto (t^3-3t+s, t^4-4t^2+s, t^5-10t+s^2t)$ are trivial knots. To compute the double points of the immersion $(t,s) \mapsto (t^3-3t+s, t^4-4t^2+s, t^5-10t+s^2t, s)$ we consider $(t_1^3-3t_1+s_1, t_1^4-4t_1^2+s_1, t_1^5-10t_1+s_1^2t_1, s_1) = (t_2^3-3t_2+s_2, t_2^4-4t_2^2+s_2, t_2^5-10t_2+s_2^2t_2, s_2)$. Then clearly $s_1 = s_2$. Then $(t_1^3-3t_1+s, t_1^4-4t_1^2+s) = (t_2^3-3t_2+s, t_2^4-4t_2^2+s)$ implies that $(t_1^3-3t_1, t_1^4-4t_1^2) = (t_2^3-3t_2, t_2^4-4t_2^2)$, which upon solving gives us three pairs of values for (t_1,t_2) which are $(-1.93185, 0.517638)$, $(-1.73205, 1.73205)$ and $(-0.517638, 1.93185)$. Thus the pairs (t_1,s) and (t_2,s) may give us double points for some values of s. For finding these values of s we find the roots of the equation $(t_1^5-10t_1+s^2t_1)-(t_2^5-10t_2+s^2t_2) = 0$ for the above pairs of (t_1,t_2). For $(t_1,t_2) = (-1.93185, 0.517638)$ the above equation turns out to be

$$-2.44939205 - 2.449488000s^2 = 0$$

   which is never zero for any real value of s. For $(t_1,t_2) = (-1.73205, 1.73205)$ the equation becomes

$$3.464158140 - 3.464100000s^2 = 0$$

   which has one real solution namely $s = 1.000008392$ in the interval $[0, \sqrt{10}]$. Similarly again for $(t_1,t_2) = (-0.517638, 1.93185)$ the equation is

$$-2.44939205 - 2.449488000s^2 = 0$$

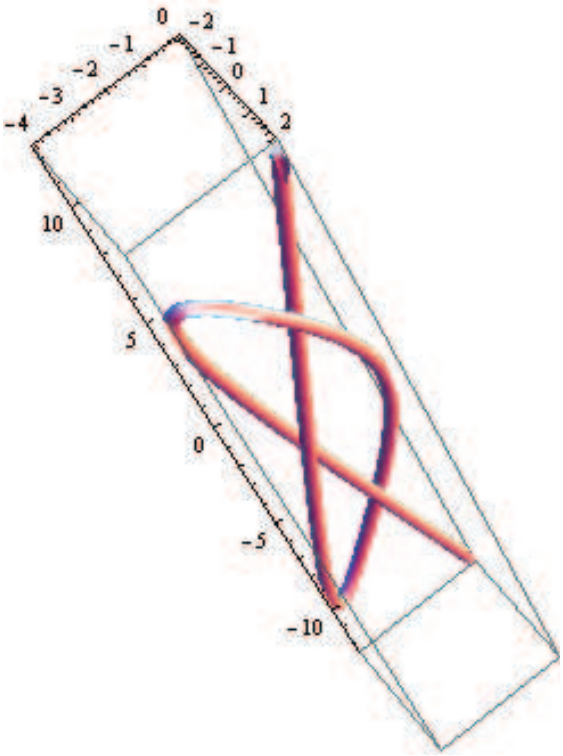

Figure 1

which is never zero for any real value of s. Thus the map $(t, s) \mapsto (t^3 - 3t + s, t^4 - 4t^2 + s, t^5 - 10t + s^2t, s)$ has one double point corresponding to $(-1.73205, 1.000008329)$ and $(1.73205, 1.000008329)$. Thus the singularity index of the polynomial knot $t \mapsto (t^3-3t, t^4-4t^2, t^5-10t)$ is 1. This is same as the unknotting number of Trefoil. We can clearly see that for $s > 1.00008392$ the polynomial knot $t \mapsto ((t^3 - 3t + s, t^4 - 4t^2 + s, t^5 - 10t + s^2t)$ has diagram of an unknot as shown in Figure 2.

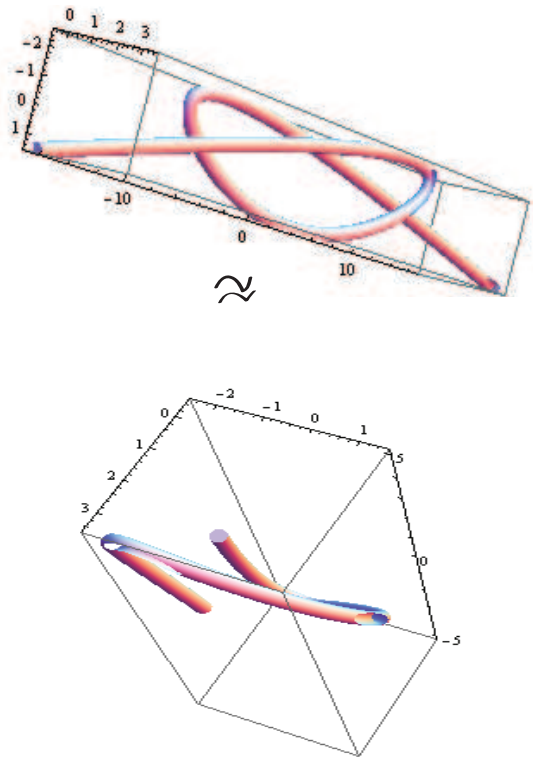

Figure 2

2. Consider a polynomial representation of the *figure eight knot* given by $t \mapsto (t(t-2)(t+2), (t-2.1)(t+2.1)t^3, -12.8064t + 22.4679t^3 - 8.90928t^5 + t^7)$. It has a knot diagram as shown in Figure 3.

   First we find that the double points in the projection $t \to t(t-2)(t+2), (t-2.1)(t+2.1)t^3)$ occur at $(t_1, t_2) = (-2.25, 1.57), (-2.10, .221), (-1.57, 2.25)$ and $(-.221, 2.10)$. Consider the deformation $\phi_s = t \mapsto (t(t-2)(t+2)+s, (t-2.1)(t+2.1)t^3+s, -12.8064t+22.4679t^3-8.90928t^5+t^7+s^2t)$. we see that for $s \geq 1.5$ each ϕ_s is an unknot (Figure 4). For each (t_1, t_2) we write down the equations to find the singular value. In this case the $(t_1, t_2) = (-2.25, 1.57)$ and $(-1.57, 2.25)$ will give identical equations with no real solution. Similarly the other two pairs give identical equation with one real solution $s = 1.48$ and hence there is only one singular knot corresponding to $s = 1.48$. Thus the singularity index of this diagram is 1 which is same as its unknotting number.

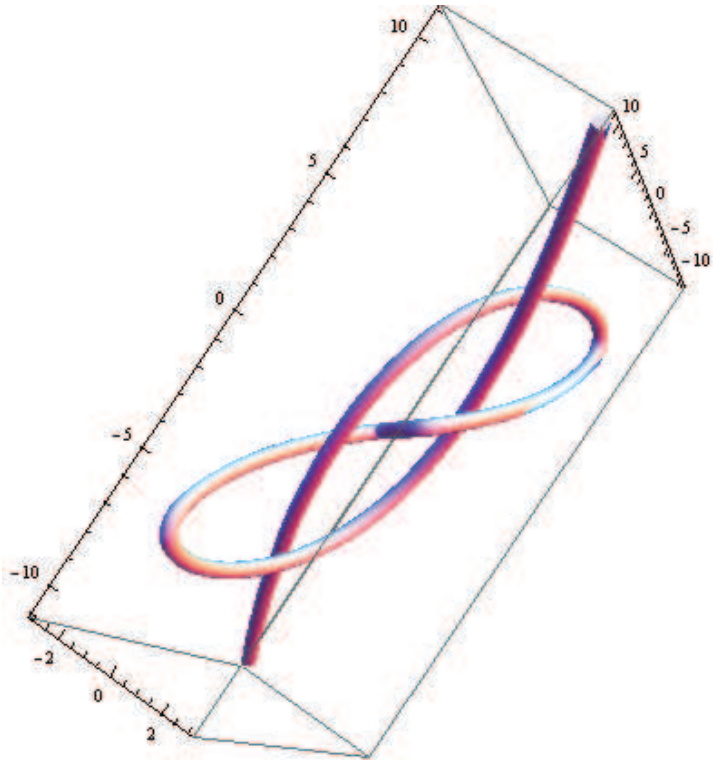

Figure 3

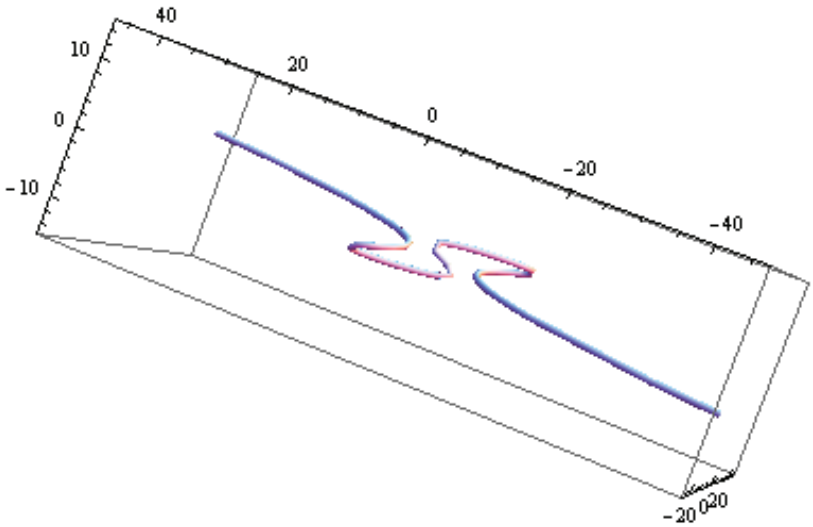

Figure 4

3. Consider the knot 5_1 in the Rolfsen's table. We have a polynomial representation $t \mapsto (t^3-4t, (t^2-1.2)(t^2-2.25)(t^2-3.9)(t^2-4.85), 120.76616719796145t-162.68665250562293t^3 + 75.13982708378909t^5 - 14.48957349775025t^7 + t^9)$ which has a knot diagram as shown in Figure 5.

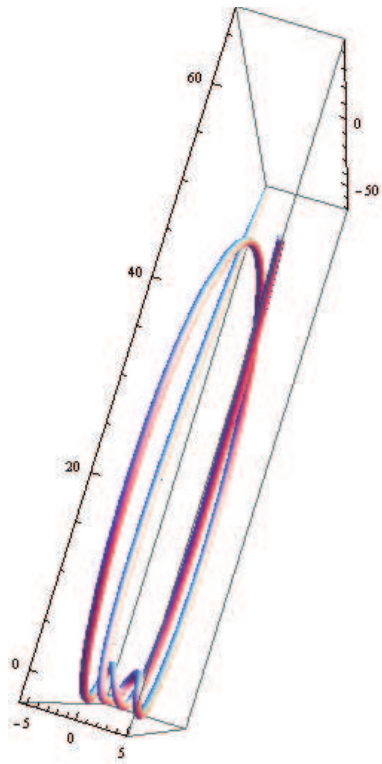

Figure 5

It has five crossings corresponding to pair of parameters: $(-2.29267, .906039)$, $(-2.23355, 1.62515)$, $(-2, 2)$, $(-1.62515, 2.2355)$ and $(-.906039, 2.29267)$. At each pair if we solve for s which makes the map $t \mapsto (t^3-4t+s, (t^2-1.2)(t^2-2.25)(t^2-3.9)(t^2-4.85)+s, 120.76616719796145t-162.68665250562293t^3+75.13982708378909t^5-14.48957349775025t^7+t^9+s^2t)$ a singular knot we obtain just one solution $s = 1.21393$. Thus the singularity index is 1. However if we plot the diagram for $s > 1.21393$ we see that two crossings corresponding to $(-2.23355, 1.62515)$ and $(-1.62515, 2.2355)$ are switched (Figure 6 (A))from over crossing to under crossing and it is an unknot as shown in Figure 6 (B). In this example the singularity index of the diagram is less than the unknotting number.

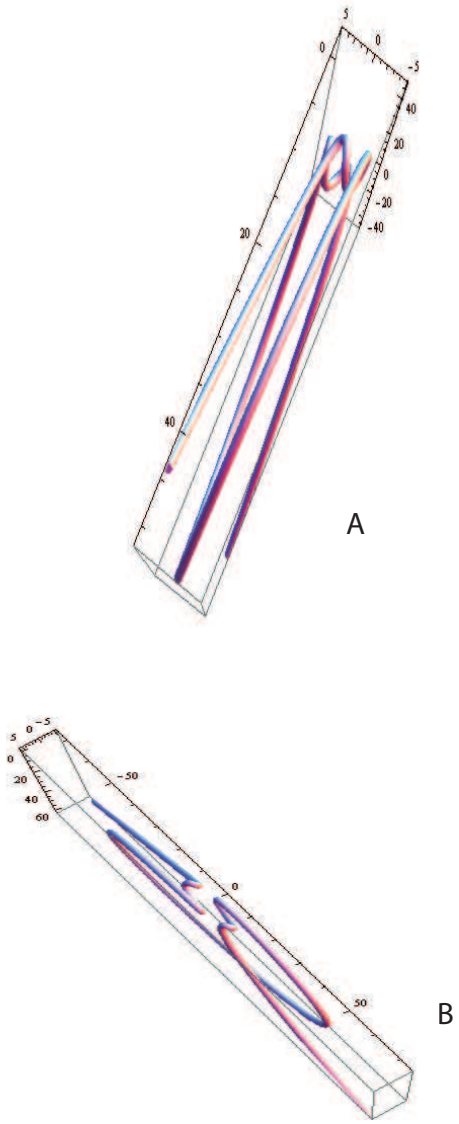

Figure 6

In the above examples there was just one solution for s if we compute at all the crossings which makes ϕ_s a singular knot. In the next example we see that the singularity index may be smaller than the total number of values of s for which ϕ_s is a singular knot. This is due to the fact that the knot diagram for ϕ_s for $s > s_i$ may represent an unknot for $s < s_{i+1}$ where s_i and s_{i+1} are two consecutive singular values.

4. Consider the knot 5_2 from Rolfsen's table. We have a polynomial representation $t \mapsto ((t-2)(t+4)(t^2-9), t(t^2-6)(t^2-15), t(t+3.9)(t+3.5)(t+2)(t+1)(t-2)(t-2.8)(t-3.2)(t-3.65))$ which gives a knot diagram as in Figure 7.

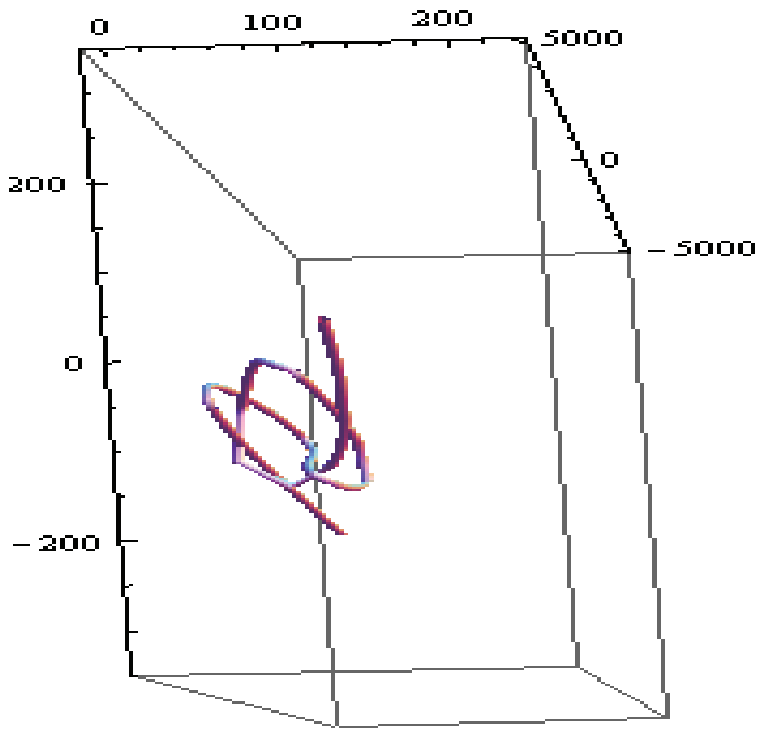

Figure 7

Here if we perform the calculations for singular values we obtain two solutions namely 9.69 and 15.12. However, the knot diagram for ϕ_s for $9.69 < s < 15.12$ represents an unknot if we perform Reidemeister Moves as shown in Figure 8.

Here, we note that in this polynomial knot $m_4(h) < m_2(h)$. In case we had a polynomial representation of the same diagram with $m_2 < m_4$ then the polynomial singularity index will be 2. Considering the minimum, the singularity index of this diagram is 1.

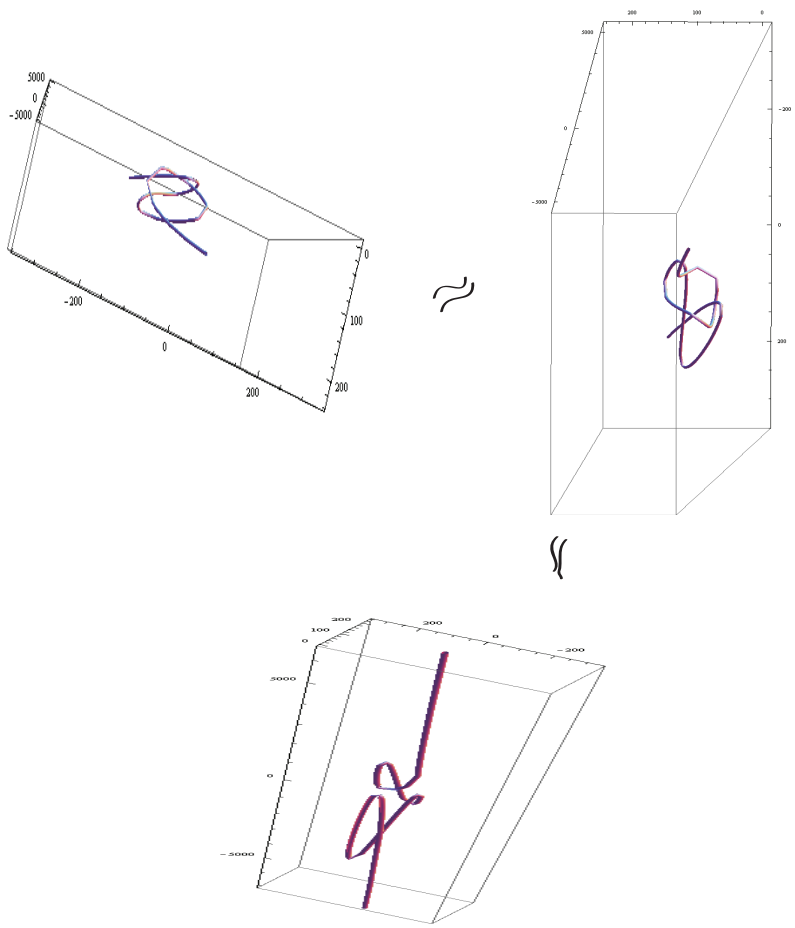

Figure 8

5. Consider the polynomial knot $t \mapsto ((t^2 - 12)(t^2 - 11), t(t^2 - 21)(t^2 -$

$7), -48278.6t + 46195.t^3 - 12497.7t^5 + 1336.22t^7 - 61.0393t^9 + t^{11})$. This represents the long 6_2 knot on Rolfsen's table. A three dimensional plot from maple is shown in Figure 9.

In this knot the crossings occur at parametric pair of values in order at $(-4.73,\ -.778)$, $(-4.58, 4.58)$, $(-4.36, 1.990)$, $(-2.64, 2.64)$, $(-1.990, 4.36)$ and $(.77, 4.73)$. In this there are only two singular knots corresponding to the second crossing at $(-4.58, 4.58)$ and the fourth crossing at $(-2.64, 2.64)$. Both of them are over crossings. Here we can check that $m_2 < m_4$. The singular knots corresponding to $(-4.58, 4.58)$ and $(-2.64, 2.64)$ are at $s = 42.58$ and $s = 92.73$ respectively. The embedding $\phi_s(t) = ((t^2 - 12)(t^2 - 11) + s, t(t^2 - 21)(t^2 - 7) + s, -48278.6t + 46195.t^3 - 12497.7t^5 + 1336.22t^7 - 61.0393t^9 + t^{11} + s^2t)$ for $42.58 < s < 92.73$ has the second crossing corresponding to $(-4.58, 4.48)$ switched from under crossing to over crossing as shown in Figure 10. This can be shown to be equivalent to unknot. Thus essentially there is just one singular knot before the diagram transforms into a diagram of an unknot. Hence the polynomial singularity index of this polynomial knot is 1 and therefore the singularity index of 6_2 knot is 1 which is same as its unknotting number.

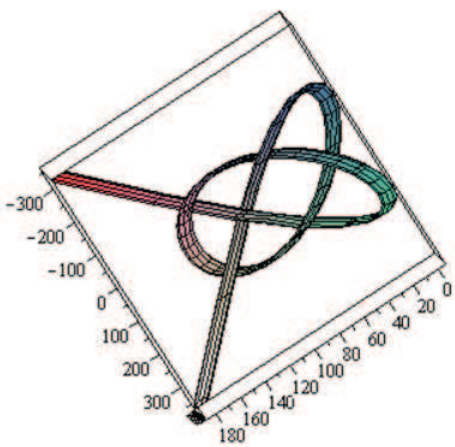

Figure 9

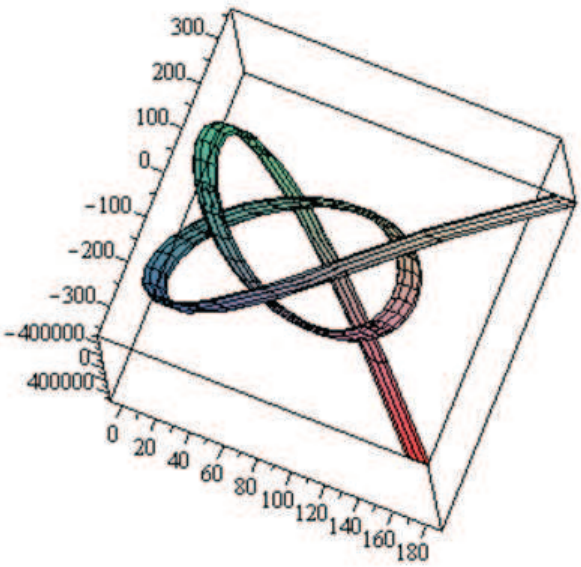

Figure 10

6. Let us consider a polynomial knot $t \mapsto (t^3 - 17t, t^2(t^2 - 18)(t^2 - 4.15)(t^2 - 22.09), t(t^2 - (4.6)^2)(t^2 - (4.35)^2)(t^2 - (4.18)^2)(t^2 - 9)(t^2 - (1.8)^2)(t^2 - (0.75)^2)) = (f(t), g(t), h(t))$ (say), which represents a knot diagram of the 7_4 **knot** in the Rolfsen's knot table, shown in Figure 11.

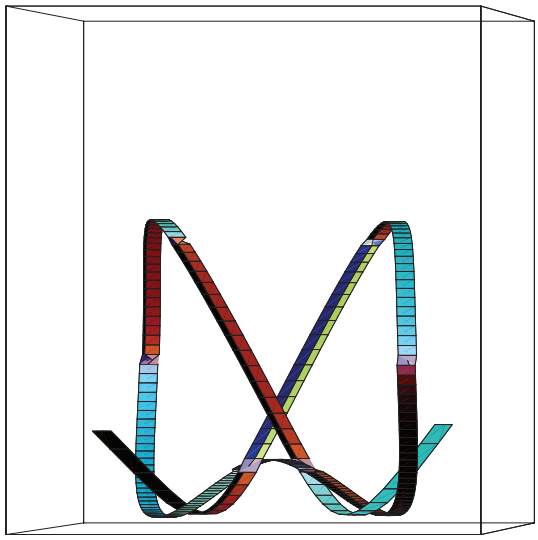

Figure 11

Let us compute the polynomial singularity index of this polynomial knot. Here the maps $t \mapsto (t^3 - 17t + s, t^2(t^2 - 18)(t^2 - 4.15)(t^2 - 22.09) + s, t(t^2 - (4.6)^2)(t^2-(4.35)^2)(t^2-(4.18)^2)(t^2-9)(t^2-(1.8)^2)(t^2-(0.75)^2)+s^2t)$ are trivial knots for $s^2 \geq 999999$, i.e., for $s \geq 999.99$. To compute the double points of the associated immersion, we proceed as in the earlier example and find that the projection $t \mapsto (t^3-17t, t^2(t^2-18)(t^2-4.15)(t^2-22.09))$ has double points for $(t_1, t_2) = (-4.75906, 2.2632)$, $(-4.56165, 1.10036)$, $(-4.25783, 0.284139)$, $(-4.12311, 4.12311)$, $(-2.2632, 4.75906)$, $(-1.10036, 4.56165)$, $(-0.284139, 4.25783)$.

Thus for these values of (t_1, t_2) the equations $(h(t_1) - h(t_2) + s^2(t_1 - t_2) = 0$ are respectively obtained as:

$$\begin{aligned}
-1.005332368 \times 10^6 - 7.02226s^2 &= 0 \qquad (1)\\
1.071471515 \times 10^5 - 5.66201s^2 &= 0 \qquad (2)\\
-42698.78494 - 4.541969s^2 &= 0 \qquad (3)\\
56367.35336 - 8.24622s^2 &= 0 \qquad (4)\\
-1.005332368 \times 10^6 - 7.02226s^2 &= 0 \qquad (5)\\
1.071471515 \times 10^5 - 5.66201s^2 &= 0 \qquad (6)\\
-42698.78494 - 4.541969s^2 &= 0. \qquad (7)
\end{aligned}$$

We see that the equations (1) and (5) are identical and have no real solution. Similarly the equations (3) and (7) are identical and have no real solution.

The equations (2) and (6) are identical with real solution $s = 137.5640644$ inside $[0, 999.99]$. Also the equation (4) has one real solution $s = 82.67731483$ inside $[0, 999.99]$.

Thus in the one parameter family of maps $t \mapsto (t^3 - 17t + s, t^2(t^2 - 18)(t^2 - 4.15)(t^2 - 22.09) + s, t(t^2 - (4.6)^2)(t^2 - (4.35)^2)(t^2 - 4.18)^2)(t^2 - 9)(t^2 - (1.8)^2)(t^2 - (0.75)^2) + s^2 t)$ there are two singular knots corresponding to $s = 82.67731483$ and $s = 137.5640644$. Hence the singularity index of this polynomial knot is 2. In this example we see that in the transformation from this knot diagram to an unknot diagram three crossings have been switched. However, it is known that the unknotting number of 7_4 knot is 2.

7. Let us consider a polynomial knot given by $t \mapsto (t^5 - 5.5t^3 + 4.5t, -7.8375 + 14t^2 - 7.35t^4 + t^6, 10.4337t - 18.5762t^3 + 8.13297t^5 - t^7)$ that represents the knot diagram of the knot 8_{19} as in Rolfsen's table shown in Fig 12. It is a torus knot of type $(3, 4)$.

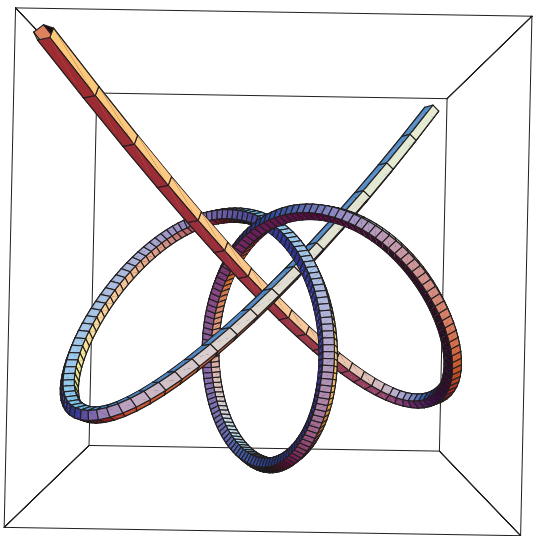

Figure 12

Let us compute the polynomial singularity index of this polynomial knot.

Here the maps $t \mapsto (t^5 - 5.5t^3 + 4.5t + s, -7.8375 + 14t^2 - 7.35t^4 + t^6 + s, -10.4337t + 18.5762t^3 - 8.13297t^5 + t^7 + s^2 t)$ are trivial knots for $s \geq 6$. The double points of the immersion $t \mapsto (t^5 - 5.5t^3 + 4.5t + s, -7.8375 + 14t^2 - 7.35t^4 + t^6 + s)$ are at the parametric values $(t_1, t_2) = (-2.17, 1.24), (-2.15, -.829), (-2.12, 2.06), (-2.06, .54), (-1.24, 2.17), (-2.17, 1.24), (-1, 1), (-.54, 2.06)$ and $(.82, 2.15)$.

Thus for these values of (t_1, t_2) the equations $(h(t_1) - h(t_2) + s^2(t_1 - t_2) = 0$ are respectively obtained as:

$$5.56516 - 3.41s^2 = 0 \quad (8)$$
$$1.89484 - 1.321s^2 = 0 \quad (9)$$
$$-4.32508 - 4.18s^2 = 0 \quad (10)$$
$$-6.45593 - 2.6s^2 = 0 \quad (11)$$
$$5.56516 - 3.41s^2 = 0 \quad (12)$$
$$2.01906 - 2s^2 = 0 \quad (13)$$
$$-6.45593 - 2.6s^2 = 0 \quad (14)$$
$$1.89484 - 1.321s^2 = 0. \quad (15)$$

We see that the equations (10), (11) and (14) do not have any real solutions. Also equation (8) is identical with equation (12) and equation (9) is identical with equation (15) and equation (13) have real solutions. Thus we have only three distinct equations which have real solution, one solution each in the interval $[0, 6]$ giving the singularity index of this polynomial knot to be 3. This is same as the unknotting number of torus knot of type $(3, 4)$ by Milnor's theorem.

5. Conclusion

This method provides us with an algorithm for obtaining a lower bound for unknotting number of a knot diagram if we have polynomial representations of the diagram. Looking at the examples discussed above, it is clear that in this method only the over crossings are switched to the under crossings. Thus if there are k over crossings, there are possible $k!$ ways to order them. For each order σ we can find a polynomial h_σ that realizes the order and in the deformation family corresponding to this representation after each singular knot when a crossing is switched we can examine the diagram and test for unknot by using the invariants that are unknot detectors such as *Khovanov Homology.* Thus we have to test for only $k!$ deformation families. The one with least number of singular knots before arriving at a diagram for trivial knot gives the singularity index and the unknotting sequence. One can write a computer program for it.

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