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Derivation of the Fisher Information Matrix for 4-Parameter Generalized Gamma Distribution Using Mathematica

Tae Ryong Park†

Abstract

Fisher information matrix plays an important role in statistical inference of unknown parameters. Especially, it is used in objective Bayesian inference where we calculate the posterior distribution using a noninformative prior distribution, and also in an example of metric functions in geometry. To estimate parameters in a distribution, we can use the Fisher information matrix. The more the number of parameters increases, the more its matrix form gets complicated. In this paper, by using Mathematica programs we derive the Fisher information matrix for 4-parameter generalized gamma distribution which is used in reliability theory.

Key words : Fisher Information Matrix, Noninformative Prior Distribution, Reliability Theory, 4-Parameters Generalized Gamma Distribution

1. Introduction

4-parameter generalized gamma distribution is frequently used in reliability theory. To estimate four parameters, two methods can be considered. One is to use Jeffrey[1] prior distribution using the Fisher information matrix, and the other is to estimate the parameters from a posterior distribution by inducing non-informative prior distribution such as reference prior distribution etc. Thus, it is very important to obtain the Fisher information matrix. However, its form of the Fisher information matrix becomes complicated as the number of parameters increases. In order to solve this complexity, this paper uses Mathematica program to derive the Fisher information matrix for 4-parameter generalized gamma distribution. Generalized gamma distribution has been shown in Hargar and Bain[2], Lienhard and Meyer[3], Harter[4], etc. The probability density function of 4-parameter generalized gamma distribution is defined as follows,

$$f(x|a,b,c,d) = \frac{c}{b\Gamma(d)}\left(\frac{x-a}{b}\right)^{cd-1} e^{-\left(\frac{x-a}{b}\right)^{c}}, \qquad (1.1)$$
$$here, \quad x > a,\ -\infty < a < \infty,\ c > 0,\ b > 0,\ d > 0$$

where a is location parameter, b is scale parameter, c is shape parameter, and d is index parameter.

Fisher information matrix estimating these 4 parameters at the same time is shown in Lehman and Casella[5], Berger[6,7], and Casella and Berger[8] etc.

When $L(\theta|x)$ denotes a likelihood function, Fisher information matrix is as follows:

$$(I_{jk}) = \left(-E\left[\frac{\partial^2 \ln L(\theta)}{\partial\theta_j \partial\theta_k}\right]\right) \qquad (1.2)$$

where $\theta = (\theta_1, \cdots, \theta_p),\ j,\ k = 1, 2, \cdots, p$

In this study, we will discuss the derivation of the Fisher information matrix for 4-parameter generalized gamma distribution by using Mathematica commands by Wolfram[9]. In section 2, required functions and Mathematica commands are introduced, and the form of the Fisher information matrix for 4-parameter generalized gamma distribution is presented. The actual derivation process is explained in section 3, and in section 4 a couple of examples are described and the paper is concluded.

Department of Computer Engineering, Seokyeong University, Seoul, Korea.
†Corresponding author : trpark@skuniv.ac.kr
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2. Required Functions, Mathematica Commands, and the Form of Fisher Information Matrix for 4-Parameter Generalized Gamma Distribution

2.1. Functions Required to Derive the Fisher Information Matrix for 4-Parameter Generalized Gamma Distribution

Functions used to derive the Fisher information matrix for 4-parameter generalized gamma distribution are as follows:

(1) Gamma Function : $\Gamma(x)=\int_0^\infty u^{x-1}e^{-u}du$

(2) First-order and second-order derivatives for gamma function :

$$\Gamma'(x)=\int_0^\infty u^{x-1}\ln u e^{-u}du$$
$$\Gamma''(x)=\int_0^\infty u^{x-1}(\ln u)^2 e^{-u}du$$

(3) PolyGamma Function :

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}, \text{ then } \Gamma'(x)=\Gamma(x)\psi(x)$$

(4) First-order PolyGamma Function :

$$\psi'(x)=\left(\frac{\Gamma'(x)}{\Gamma(x)}\right)' =\frac{\Gamma''(x)\Gamma(x)-(\Gamma'(x))^2}{(\Gamma(x))^2} =\frac{\Gamma''(x)\Gamma(x)-(\Gamma(x))^2(\psi(x))^2}{(\Gamma(x))^2}=\frac{\Gamma''(x)}{\Gamma(x)}-(\psi(x))^2$$

Using functions defined above, we can see the second derivative for gamma function is

$$\Gamma''(x)=\Gamma(x)\{\psi'(x)+(\psi(x))^2\}$$

2.2. Mathematica Commands Required for the Fisher Information Matrix

Mathematica commands required for Fisher information matrix for 4-parameter generalized gamma distribution are as follows:

(1) D[f[x], x]: First-order derivative for f(x)
(2) D[D[f[x], x], x]: Second-order derivative for f(x)
(3) D[f[x, y], x]: Partial derivative of x for f(x, y)
(4) D[f[x, y], y]: Partial derivative of y for f(x, y)
(5) D[D[f[x, y], x], y]:
Second-order partial derivative of x and y for f(x, y)
(6) Integrate[f[x, y], x]:
Indefinite integration of x for f(x, y)
(7) Integrate[f[x, y], {x, a, b}]:
Definite integration for f(x,y) in interval [a, b]

If we get a likelihood function of equation (1.1) in section 1 and Mathematica commands above, Fisher information matrix is represented as follows, by using equation (1.2) :

$$(I_{jk})=-E\begin{bmatrix}\frac{\partial^2\ln L(\theta|x)}{\partial a^2}, & \frac{\partial^2\ln L(\theta|x)}{\partial a\partial b}, & \frac{\partial^2\ln L(\theta|x)}{\partial a\partial c}, & \frac{\partial^2\ln L(\theta|x)}{\partial a\partial d}\\ & \frac{\partial^2\ln L(\theta|x)}{\partial b^2}, & \frac{\partial^2\ln L(\theta|x)}{\partial b\partial c}, & \frac{\partial^2\ln L(\theta|x)}{\partial b\partial d}\\ & & ,\frac{\partial^2\ln L(\theta|x)}{\partial c^2}, & \frac{\partial^2\ln L(\theta|x)}{\partial c\partial d}\\ symmetry & & & \frac{\partial^2\ln L(\theta|x)}{\partial d^2}\end{bmatrix}$$
$$j,k=1,2,3,4 \tag{2.1}$$

where $\ln L(\theta|x)$ is a log-likelihood function, which will be defined in the following section 3, and $\theta=(a,b,c)$. Therefore, we need to calculate 10 elements of the matrix above to obtain the Fisher information matrix. The matrix is completed by getting second-order partial derivatives of each log-likelihood function above and computing expectation values.

3. Calculation and Derivation of the Fisher Information matrix for 4-Parameter Generalized Gamma Distribution

Let's say that observation values x have the 4-parameter generalized gamma probability density function as shown in equation (1.1). Log-likelihood function in equation (1.1) is denoted as $L(\theta|x)=\ln(a,b,c,d|x)$, more simply $\ln L(a,b,c,d)$ or $\ln L$. Thus, log-likelihood function is represented using equation (1.1) as follows

$$\ln L(a,b,c,d)=\ln c-\ln b-\ln\Gamma(d) +(cd-1)\left(\ln\left(\frac{x-a}{b}\right)\right)-\left(\frac{x-a}{b}\right)^c$$

In Mathematica command, log-likelihood function is as follows:

```
lnL[a_,b_,c_d_,x]:=Log[c]-Log[b]-Log
[Gamma[d]]+(cd-1)Log[(x-a)/b]-((x-a)/b)^c
lnL[a, b, c, d, x]
```
$$-\left(\frac{-a+x}{b}\right)^c - \mathrm{Log}[b] + \mathrm{Log}[c] + (-1+cd)(-Log[b] + Log[-a+x]) - Log[\Gamma[d]]$$

Second-order partial derivative of log-likelihood function for a is as follows:

$$\frac{\partial^2 \ln L}{\partial a^2} = \frac{1-cd}{(x-a)^2} - \frac{c(c-1)}{b}\left(\frac{x-a}{b}\right)^{c-1}$$

Mathematica program for this is as follows:

```
D[D[InL[a, b, c, d, x], a], a]
```
$$-\frac{-1+cd}{(-a+x)^2} - \frac{(-1+c)\, c\left(\frac{-a+x}{b}\right)^{-2+c}}{b^2}$$

Therefore, if we calculate the element in the first column of the first row of the Fisher matrix shown in equation (2.1) in section 2.2,

$$-E\left[\frac{\partial^2 \ln L}{\partial a^2}\right] = -E\left[\frac{1-cd}{(X-a)^2} + \frac{c(1-c)}{b^2}\left(\frac{X-a}{b}\right)^{c-2}\right]$$
$$= \frac{1-cd}{b^2}E\left[\left(\frac{b}{X-a}\right)^2\right] + \frac{c(1-c)}{b^2}E\left[\left(\frac{X-a}{b}\right)^{c-2}\right]$$

Here, if $z = \frac{x-a}{b}$, then the following is established.

$$E\left[\left(\frac{b}{X-a}\right)^2\right] = E\left[\frac{1}{Z^2}\right] = \int_0^\infty \frac{c}{\Gamma(d)} z^{cd-1} e^{-z^c} dz$$
$$= \int_0^\infty \frac{1}{\Gamma(d)} u^{d-\frac{2}{c}-1} e^{-u} du$$
$$= \frac{\Gamma\left(d-\frac{2}{c}\right)}{\Gamma(d)}, \text{ where } d > \frac{2}{c} \ (cd > 2).$$

The following can be obtained the same way.

$$E\left[\left(\frac{X-a}{b}\right)^{c-2}\right] = E[Z^{c-2}] = \frac{\left(d-\frac{2}{c}\right)\Gamma\left(d-\frac{2}{c}\right)}{b^2\Gamma(d)}.$$

Therefore,

$$-E\left[\frac{\partial^2 \ln L}{\partial a^2}\right] = \frac{(c^2 d - 2c + 1)\Gamma\left(d-\frac{2}{c}\right)}{b^2\Gamma(d)}, \text{ where } cd > 2.$$

If we calculate the element in the second column of the first row the same way,

$$\frac{\partial^2 \ln L}{\partial a \partial b} = -\frac{c^2}{b^2}\left(\frac{x-a}{b}\right)^{c-1}$$

Mathematica program for this equation is as follows:

```
D[D[InL[a, b, c, d, x], a], b]
```
$$-\frac{(-1+c)\,c(-a+x)\left(\frac{-a+x}{b}\right)^{-2+c}}{b^3} - \frac{c\left(\frac{-a+x}{b}\right)^{-1+c}}{b^2}$$

By using same as before method, the third element of the first low is

$$-E\left[\frac{\partial^2 \ln L}{\partial a \partial b}\right] = \frac{c(cd-1)\Gamma\left(d-\frac{1}{c}\right)}{b^2\Gamma(d)}$$

Thus, the element in the third column of the first row is

$$\frac{\partial^2 \ln L}{\partial a \partial c} = -\frac{d}{x-a} + \frac{1}{b}\left(\frac{x-a}{b}\right)^{c-1} + \frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1}\ln\frac{x-a}{b}$$

Mathematica program for this is as follows:

```
D[D[lnL[x,a,b,c], a], c]
```
$$-\frac{d}{-a+x} + \frac{\left(\frac{-a+x}{b}\right)^{-1+c}}{b} + \frac{c\left(\frac{-a+x}{b}\right)^{-1+c}\mathrm{Log}\left[\frac{-a+x}{b}\right]}{b}$$

Therefore, if we rewrite $-E\left[\frac{\partial^2 \ln L}{\partial a \partial b}\right] = \frac{c(cd-1)\Gamma\left(c-\frac{1}{c}\right)}{b^2\Gamma(d)}$ using the equation above,

$$-E\left[\frac{\partial^2 \ln L}{\partial a \partial c}\right] = \frac{d}{b}E\left[\frac{b}{X-a}\right] - \frac{1}{b}E\left[\left(\frac{X-a}{b}\right)^{c-1}\right]$$
$$- \frac{c}{b}E\left[\left(\frac{X-a}{b}\right)^{c-1}\ln\left(\frac{X-a}{b}\right)\right]$$

Then,

$E\left[\frac{b}{X-a}\right] = \frac{\Gamma\left(d-\frac{1}{c}\right)}{\Gamma(d)}$ and,

$E\left[\left(\frac{X-a}{b}\right)^{c-1}\right] = \left(d-\frac{1}{c}\right)\frac{\Gamma\left(d-\frac{1}{c}\right)}{\Gamma(d)}$ as before, and if we use $\Gamma'(a+1) = \Gamma(a) + a\Gamma'(a)$,

$$E\left[\left(\frac{X-a}{b}\right)^{c-1}\ln\left(\frac{X-a}{b}\right)\right] = E[Z^{c-1}\ln Z]$$
$$= \int_0^\infty z^{c-1}\ln z \frac{c}{\Gamma(d)} z^{cd-1} e^{-z^c} dz$$

$$= \frac{1}{\Gamma(d)} \int_0^{\infty} e^{-u} u^{d-\frac{1}{c}} \frac{1}{c} \ln u\, du = \frac{1}{c\Gamma(d)} \Gamma'\left(d - \frac{1}{c} + 1\right)$$
$$= \frac{1}{c\Gamma(d)} \left[\Gamma\left(d - \frac{1}{c}\right) + \left(d - \frac{1}{c}\right) \Gamma'\left(d - \frac{1}{c}\right) \right]$$

Mathematica program for this is as follows:

```
Integrate[E^(-u)u^(d- 1/c), {u, 0, Infinity}]
Γ[1 - 1/c + d]
```

Therefore,

$$-E\left[\frac{\partial^2 \ln L}{\partial a \partial c}\right] = \frac{(1-c)\Gamma\left(d-\frac{1}{c}\right) - (cd-1)\Gamma'\left(d-\frac{1}{c}\right)}{bc\Gamma(d)}$$
$$= \frac{\Gamma\left(d-\frac{1}{c}\right)}{bc\Gamma(d)} \left\{(1-c) - (cd-1)\psi\left(d-\frac{1}{c}\right)\right\}$$

The element in the fourth column of the first row is $\frac{\partial^2 \ln L}{\partial a \partial d} = -\frac{c}{x-a}$, and Mathematica program for this is as follows:

```
D[D[lnL[a, b, c, d, x], a], d]
-c/(-a+x)
```

Also, same as before method, the element of the fourth column of the first low is

$$-E\left[\frac{\partial^2 \ln L}{\partial a \partial d}\right] = \frac{c}{b} E\left[\frac{b}{X-a}\right] = \frac{c}{b} \frac{\Gamma\left(d-\frac{1}{c}\right)}{\Gamma(d)}.$$

The element of second column of the second row for fisher informantion matrix is

$$\frac{\partial^2 \ln L}{\partial b^2} = \frac{c}{b^2}\left[d - (c+1)\left(\frac{x-a}{b}\right)^c\right].$$

Mathmatica program for this is as follows :

```
D[D[lnL[a, b, c, d, x], b], b]
1/b^2 + (-1+cd)/b^2 - ((-1+c) c(-a+x)^2 ((-a+x)/b)^(-2+c))/b^4
- (2c(-a+x)((-a+x)/b)^(-1+c))/b^3
```

Therefore, following equation is established.

$$-E\left[\frac{\partial^2 \ln L}{\partial b^2}\right] = -\frac{cd}{b^2} + \frac{c(c+1)}{b} E\left[\left(\frac{X-a}{b}\right)^c\right].$$

Also, same as before method, we can see that the following equations is correct.

$$E\left[\left(\frac{X-a}{b}\right)^c\right] = d \text{ and, } -E\left[\frac{\partial^2 \ln L}{\partial b^2}\right] = \frac{c^2 d}{b^2}.$$

The element of third column of the second low is

$$\frac{\partial^2 \ln L}{\partial b \partial c} = -\frac{d}{b} + \frac{1}{b}\left(\frac{x-a}{b}\right)^c + \frac{c}{b}\left(\frac{x-a}{b}\right)^c \ln\left(\frac{x-a}{b}\right), \text{ and}$$

Mathematica program for this is as follows :

```
D[D[lnL[a, b, c, d, x], b], c]
-d/b + ((-a+x)((-a+x)/b)^(-1+c))/b^2
+ (c(-a+x)((-a+x)/b)^(-1+c) Log[(-a+x)/b])/b^2
```

Thus,

$$-E\left[\frac{\partial^2 \ln L}{\partial b \partial c}\right] = \frac{d}{b} - \frac{1}{b} E\left[\left(\frac{X-a}{b}\right)^c\right] - \frac{c}{b} E\left[\left(\frac{X-a}{b}\right)^c \ln \frac{X-a}{b}\right]$$
$$E\left[\left(\frac{X-a}{b}\right)^c\right] = d,$$
$$E\left[\left(\frac{X-a}{b}\right)^c \ln \frac{X-a}{b}\right] = \int_0^{\infty} \frac{1}{\Gamma(d)} u^d e^{-u} \frac{1}{c} \ln u\, du$$
$$= \frac{\Gamma'(d+1)}{c\Gamma(d)} = \frac{\Gamma(d) + d\Gamma'(d)}{c\Gamma(d)}$$

Therefore,

$$-E\left[\frac{\partial^2 \ln L}{\partial b \partial c}\right] = -\frac{1}{b}\left[1 + d\frac{\Gamma'(d)}{\Gamma(d)}\right] = -\frac{1}{b}\left[1 + d\psi(d)\right]$$

By the equation $\frac{\partial^2 \ln L}{\partial b \partial d} = -\frac{c}{b}$, the element in the fourth column of the second low is $-E\left[\frac{\partial^2 \ln L}{\partial b \partial d}\right] = \frac{c}{b}$.

By using above result, Mathematica program for this is as follows:

```
D[D[lnL[a, b, c, d, x], b], d]
-c/b
```

The element in the third column of the third row is

$$\frac{\partial^2 \ln L}{\partial c^2} = -\frac{1}{c^2} - \left(\frac{x-a}{b}\right)^c \ln\left(\frac{x-a}{b}\right)^2.$$

Mathematica proram for this is as follows:

```
D[D[lnL[a, b, c, d, x], c], c]
```
$$-\frac{1}{c^2} - \left(\frac{-a+x}{b}\right)^c \mathrm{Log}\left[\frac{-a+x}{b}\right]^2$$

Therefore, following equation is established.

$$-E\left[\frac{\partial^2 \ln L}{\partial c^2}\right] = \frac{1}{c^2} + E\left[\left(\frac{X-a}{b}\right)^c \left(\ln\frac{X-a}{b}\right)^2\right].$$

Here, we can calculate the following equation.

$$E\left[\left(\frac{X-a}{b}\right)^c \left(\ln\frac{X-a}{b}\right)^2\right] = E[Z^c(\ln Z)^2]$$
$$= \int_0^\infty z^c(\ln z)^2 \frac{c}{\Gamma(d)} Z^{cd-1} e^{-z^c} dz$$
$$= \frac{1}{c^2\Gamma(d)} \int_0^\infty u^d e^{-u} (\ln u)^2 du = \frac{\Gamma''(d+1)}{c^2\Gamma(d)}$$
$$= \frac{2\Gamma'(d) + d\Gamma''(d)}{c^2\Gamma(d)} = \frac{1}{c^2}\left(2\psi(d) + d\psi'(d) + d(\psi(d))^2\right).$$

Thus,

$$-E\left[\frac{\partial^2 \ln L}{\partial c^2}\right] = \frac{1}{c^2}\left(1 + 2\psi(d) + d\psi'(d) + d(\psi(d))^2\right).$$

Mathematica program for $\int_0^\infty u^d e^{-u} (\ln u)^2 du$ is as follows:

```
Integrate[ E^(-u) u^dLog[u])^2,{u,0,infinity}]
Γ[1+d](PolyGamma[0,1+d]^2
+PolyGamma[1,1+d])
```

we can see that the element in the fourth column of the third row is

$$\frac{\partial^2 \ln L}{\partial c \partial d} = \ln\frac{x-a}{b} = \ln(x-a) - \ln(b).$$

Mathematica program for this is as follows:

```
D[D[lnL[a, b, c, d, x], c], d]
-Log[b]+Log[-a+x]
```

Also, by using the equation,

$$E[\ln Z] = \int_0^\infty \ln z \frac{c}{\Gamma(d)} z^{cd-1} e^{-z^c} dz$$

$= \frac{1}{c\Gamma(d)} \int_0^\infty u^{d-1} e^{-u} \ln u\, du = \frac{1}{c}\frac{\Gamma'(d)}{\Gamma(d)}$, we can see that the following equation is established.

$$-E\left[\frac{\partial^2 \ln L}{\partial c \partial d}\right] = -E\left[\ln\frac{X-a}{b}\right] = -E[\ln Z] = -\frac{1}{c}\frac{\Gamma'(d)}{\Gamma(d)}$$
$$= -\frac{1}{c}\psi(d).$$

Mathematica program for $\int_0^\infty u^{d-1} e^{-u} \ln u\, du$ is as follows:

```
Integrate[u^(d-1) E^(-u)Log[u], {u,0,infinity}]
Γ[d]PolyGamma[0,d]
```

Finally, the element in the fourth column of the forth row is

$$\frac{\partial \ln L}{\partial d} = -\frac{\Gamma'(d)}{\Gamma(d)} + c\ln\frac{x-a}{b} = -\psi(d) + c\ln\frac{x-a}{b}.$$

Therefore, $\frac{\partial^2 \ln L}{\partial d^2} = -\left(\frac{\Gamma'(d)}{\Gamma(d)}\right)' = -\psi'(d)$ and

$$-E\left[\frac{\partial^2 \ln L}{\partial d^2}\right] = \psi'(d).$$

If we apply the calculation results so far to equation (2.1) in seciton 2, Fisher information matrix for 4-parameter generalized gamma distribution is expressed as follows:

$$(I_{jk}) = \left(-E\left[\frac{\partial^2 \ln L}{\partial\theta_j \partial\theta_k}\right]\right) \quad j,k = 1,2,3,4$$

$$= \begin{bmatrix} \frac{(c^2 d - 2c + 1)\Gamma\left(d - \frac{2}{c}\right)}{b^2\Gamma(d)} & \frac{c(cd-1)\Gamma\left(d - \frac{1}{c}\right)}{b^2\Gamma(d)} & \frac{\Gamma\left(d-\frac{1}{c}\right)}{bc\Gamma(d)}\left\{(1-c) - (cd-1)\psi\left(d - \frac{1}{c}\right)\right\} & \frac{c}{b}\frac{\Gamma\left(d-\frac{1}{c}\right)}{\Gamma(d)} \\ & \frac{c^2 d}{b^2} & -\frac{1}{b}[1 + d\psi(d)] & \frac{c}{b} \\ & & \frac{1}{c^2}\left(1 + 2\psi(d) + d\psi'(d) + d(\psi(d))^2\right) & -\frac{1}{c}\psi(d) \\ & & & \psi'(d) \end{bmatrix}$$

4. Examples and Conclusion

Example 4.1 When index parameter $d=1$ in equation (1.1), probability density function is

$$f(x|a,b,c)=\frac{c}{b}\left(\frac{x-a}{b}\right)^{c-1}\exp\left\{-\left(\frac{x-a}{b}\right)^{c}\right\}$$

Fisher information matrix for this is as follows the same way described in section 3:

$$(I_{jk})=\left(-E\left[\frac{\partial^2 \ln L(\theta)}{\partial\theta_j\partial\theta_k}\right]\right)_{j,k=1,2,3}$$

$$=\begin{bmatrix} \frac{(c-1)^2}{b^2}\Gamma\left(1-\frac{2}{c}\right), & \frac{c(c-1)}{b^2}\Gamma\left(1-\frac{1}{c}\right), & \frac{1}{bc}\Gamma\left(1-\frac{1}{c}\right)-\frac{1}{b}\Gamma'\left(2-\frac{1}{c}\right) \\ & \frac{c^2}{b^2} & -\frac{1}{b}\Gamma'(2) \\ \text{symmetry} & & \frac{1}{c^2}(\Gamma''(2)+1) \end{bmatrix}$$

$$=\begin{bmatrix} \frac{(c-1)^2}{b^2}\Gamma\left(1-\frac{2}{c}\right) & \frac{c(c-1)}{b^2}\Gamma\left(1-\frac{1}{c}\right), & \frac{1-c}{bc}\Gamma\left(1-\frac{1}{c}\right)\left\{1+\psi\left(1-\frac{1}{c}\right)\right\} \\ & \frac{c^2}{b^2} & -\frac{1}{b}(1-\gamma) \\ \text{symmetry} & & \frac{1}{c^2}\left(-2\gamma+\gamma^2+\frac{\pi^2}{6}\right) \end{bmatrix}$$

where, $\gamma=0.577716\ldots$(Euler constant).

In other words, it is a sub-matrix excluding the fourth row and the fourth column from the original matrix when $d=1$. It is consistent with the result of Yang and Baek [10].

Example 4.2 When index parameter $d=1$ and location parameter $a=0$ in equation (1.1), probability density function is

$$f(x|b,c)=\frac{c}{b}\left(\frac{b}{x}\right)^{c-1}\exp\left\{-\left(\frac{x}{b}\right)^{c}\right\}.$$

Fisher information matrix for this is

$$(I_{jk})=\left(-E\left[\frac{\partial^2 \ln L(\theta)}{\partial\theta_j\partial\theta_k}\right]\right)=\begin{bmatrix} \frac{c^2}{b^2} & -\frac{1}{b}\Gamma'(2) \\ -\frac{1}{b}\Gamma'(2) & \frac{1}{c^2}(\Gamma''(2)+1) \end{bmatrix}$$

$$=\begin{bmatrix} \frac{c^2}{b^2} & -\frac{1}{b}(1-\gamma) \\ -\frac{1}{b}(1-\gamma) & \frac{1}{c^2}\left(-2\gamma+\gamma^2+\frac{\pi^2}{6}\right) \end{bmatrix}$$

We can easily confirm that the 3×3sub-matrix excluding the first row and the first column from the original matrix derived in section 3 is same as Fisher information matrix for 3-parameter generalized gamma distribution with $a=0$. The matrix above is established only when $cd>2$. In other words, shape parameter c and index parameter d are limited for density function of four parameter gamma distribution function. This is shown in Jan and Noortwijk [11], where some of the elements of the Fisher information matrix for 4-parameter generalized gamma distribution were calculated, and their result is consistent with that of this paper. When $d=1$, it is Weibull distribution with 3 parameters and is, thus, consistent with Fisher information matrix of Weibull distribution. And besides the calculation problems of the Fisher information matrix in other distributions can be easily resolved using Mathematica programs. This geometric application of the Fisher information matrix is shown in Lee [12] and Kass [13] with related to the statistical geometric curvature. Furthermore, this result can be applied to an objective Bayesian inference for each parameter by deriving non-information prior distribution and posterior distribution.

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