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AN ADDITIVE FUNCTIONAL INEQUALITY

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Abstract. In this paper, we solve the additive functional inequality

$$\|f(x)+f(y)+f(z)\| \le \|\rho f\left(s(x+y+z)\right)\|,$$

where s is a nonzero real number and ρ is a real number with $|\rho|<3$.

Moreover, we prove the Hyers-Ulam stability of the above additive functional inequality in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [12] concerning the stability of group homomorphisms. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [9] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In [5], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \le \|f(xy)\| \tag{1}$$

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then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [10]. Gilányi [6] and Fechner [3] proved the Hyers-Ulam stability of the functional inequality (1).

In Section 2, we solve the additive functional inequality

$$\|f(x) + f(y) + f(z)\| \le \|\rho f\,(s(x + y + z))\|\,, \tag{2}$$

and prove the Hyers-Ulam stability of the additive functional inequality (2).

Park, Cho and Han [8] investigated the additive functional inequalities for the case $\rho = s = 1$, and the case $\rho = 2$ and $s = \frac{1}{2}$.

Throughout this paper, let X be a normed space with norm $\|\cdot\|$ and Y a Banach space with norm $\|\cdot\|$. Assume that s is a nonzero real number and that ρ is a real number with $|\rho| < 3$.

2. The additive functional inequality (2)

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies*

$$\|f(x) + f(y) + f(z)\| \le \|\rho f(s(x + y + z))\| \tag{3}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (3), we get

$$\|3f(0)\| \le \|\rho f(0)\|.$$

So $f(0) = 0$.

Letting $z = -x$ and $y = 0$ in (3), we get

$$\|f(x) + f(-x)\| \le \|\rho f(0)\| = 0$$

for all $x \in X$. Hence $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = -x - y$ in (3), we get

$$\|f(x) + f(y) + f(-x - y)\| \le \|\rho f(0)\| = 0$$

for all $x, y \in X$. So $f(x)+f(y) = -f(-x-y) = f(x+y)$ for all $x, y \in X$, as desired. □

Corollary 2.2. *If a mapping $f : X \to Y$ satisfies*

$$f(x) + f(y) + f(z) = \rho f\,(s(x + y + z)) \tag{4}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Now, we prove the Hyers-Ulam stability of the additive functional inequality (2) in Banach spaces.

THEOREM 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \|f(x) + f(y) + f(z)\| &\leq \|\rho f(s(x+y+z))\| \\ &+ \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2 + 3 \cdot 2^r}{2^r - 2}\theta\|x\|^r \tag{6}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (5), we get $f(0) = 0$.
Letting $y = -x$ and $z = 0$ in (5), we get

$$\|f(x) + f(-x)\| \leq 2\theta\|x\|^r$$

for all $x \in X$. So

$$\|f(2x) + f(-2x)\| \leq 2 \cdot 2^r\theta\|x\|^r \tag{7}$$

for all $x \in X$.
Letting $y = x$ and $z = -2x$ in (5), we get

$$\|2f(x) + f(-2x)\| \leq (2 + 2^r)\theta\|x\|^r \tag{8}$$

for all $x \in X$. It follows from (7) and (8) that

$$\|2f(x) - f(2x)\| \leq (2 + 3 \cdot 2^r)\theta\|x\|^r \tag{9}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \frac{2 + 3 \cdot 2^r}{2^r}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \frac{2 + 3 \cdot 2^r}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\theta\|x\|^r \end{aligned} \tag{10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (10) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (10), we get (6).

It follows from (5) that

$$\begin{aligned} \|h(x) + h(y) + h(z)\| &= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x}{2^n}\right) + f\left(\frac{y}{2^n}\right) + f\left(\frac{z}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 2^n \left\| \rho f\left(s\frac{x+y+z}{2^n}\right)\right\| \\ &+ \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}(\|x\|^r + \|y\|^r + \|z\|^r) \\ &= \|\rho h\left(s(x+y+z)\right)\| \end{aligned}$$

for all $x, y, z \in X$. So

$$\|h(x) + h(y) + h(z)\| \le \|\rho h\left(s(x+y+z)\right)\|$$

for all $x, y, z \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (6). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= 2^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\| \\ &\le 2^n \left(\left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|\right) \\ &\le \frac{2(2+3\cdot 2^r)2^n}{(2^r-2)2^{nr}}\theta\|x\|^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (6). □

Theorem 2.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (5). Then there exists a unique additive mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2+3\cdot 2^r}{2-2^r}\theta\|x\|^r \tag{11}$$

for all $x \in X$.

Proof. It follows from (9) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2 + 3 \cdot 2^r}{2} \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \\ &\leq \frac{2 + 3 \cdot 2^r}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \theta \|x\|^r \end{aligned} \tag{12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (12) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (12), we get (11).

The rest of the proof is similar to the proof of Theorem 2.3. □

By the triangle inequality, we have

$$\begin{aligned} &\|f(x) + f(y) + f(z)\| - \|\rho f(s(x + y + z))\| \\ &\leq \|f(x) + f(y) + f(z) - \rho f(s(x + y + z))\| . \end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the additive functional equation (4) in Banach spaces.

Corollary 2.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x) + f(y) + f(z) - \rho f(s(x + y + z))\| \\ &\leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \tag{13}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $h : X \to Y$ satisfying (6).

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (13). *Then there exists a unique additive mapping $h : X \to Y$ satisfying* (11).

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