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DEGREE OF APPROXIMATION BY PERIODIC NEURAL NETWORKS

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Abstract. We investigate an approximation order of a continuous 2π-periodic function by periodic neural networks. By using the De La Vallée Poussin sum and the modulus of continuity, we obtain a degree of approximation by periodic neural networks.

1. Introduction

Approximation by neural networks has been investigated by many researchers [2, 3, 4, 5, 7, 8] because it has been widely applied in engineering such as robotics, signal processing and etc.

A neural network has three layers: input layer, hidden layer and output layer.

A mathematical expression of a neural network with one hidden layer is defined by

$$\sum_{i=1}^{n} c_i \sigma(a_i x + b_i), \tag{1.1}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a univariate activation function.

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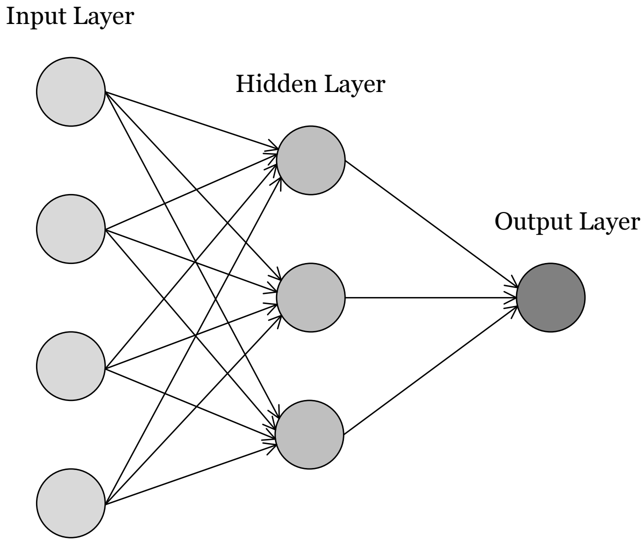

FIGURE 1. Neural network with a single hidden layer

A large class of functions such as the Gaussian function $\sigma(x) = e^{-x^2}$, the squashing function $\sigma(x) = (1+e^{-x})^{-1}$, the generalized multiquadrics $\sigma(x) = (1+x^2)^{\alpha}, \alpha \notin \mathbb{Z}$ and the thin plate splines $\sigma(x) = |x|^{2q-1}, q \in \mathbb{N}$ are used as activation functions.

In papers [4, 5, 7, 8] related to the neural network approximation, the approximation of non-periodic continuous functions defined on a compact set by neural networks has been investigated.

The periodic neural network approximation was suggested in [2, 3]. In fact, Guanzhen [2] suggested a periodic neural network approximation based on scattered nodes. On the other hand, Hahm and Hong [3] proved the approximation capability of continuous 2π-periodic functions by periodic neural networks.

In this paper, by using the De La Vallée Poussin sum and the modulus of continuity, we compute an actual periodic neural network approximation order of a continuous 2π-periodic function.

2. Preliminaries

We use the same notations in [1, 6]. Note that the class of all trigonometric polynomials of order at most n is denoted by $\mathbb{T}_n$. An element in

$\mathbb{T}_n$ has a complex expression of the form

$$\sum_{|j|\le n} l_j e^{ijx}, \tag{2.1}$$

where $l_j \in \mathbb{C}$. For a continuous 2π-periodic function f, the norm of f is given by

$$\|f\|_\infty = \sup\{|f(x)| : x \in [-\pi, \pi]\}.$$

For $n \in \mathbb{N}$, we define

$$E_n^*(f) = \inf_{T\in\mathbb{T}_n} \|f - T\|_\infty. \tag{2.2}$$

The quantity $E_n^*(f)$ measures the best approximation of a continuous 2π-periodic function f by trigonometric polynomials in $\mathbb{T}_n$. We write, for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$c_k(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt \tag{2.3}$$

and

$$S_n(f,x) = \sum_{|k|\le n} c_k(f)e^{ikx}, \tag{2.4}$$

where $c_k(f)$ and $S_n(f)$ denote the Fourier coefficient of f and the nth partial sum of Fourier series of f, respectively. It is clear that

$$S_n(h) = h \tag{2.5}$$

for a trigonometric polynomial h of degree m with $m \le n$. Moreover for $n \in \mathbb{N}$, the Fejér sum $F_n(f)$ of f is defined by

$$F_n(f,x) = \frac{1}{n}\sum_{k=1}^{n} S_k(f,x) \tag{2.6}$$

and the De La Vallée Poussin sum $V_n(f)$ of f is given by

$$V_n(f,x) = \frac{1}{n}\sum_{k=n+1}^{2n} S_k(f,x). \tag{2.7}$$

It is well known that $F_n(f) \in \mathbb{T}_n$, $\|F_n(f)\|_\infty \le \|f\|_\infty$ and $V_n(f) \in \mathbb{T}_{2n}$.

From now on, the letters c, c_1 and c_2 in this paper denote positive constants which are independent of f and their values may be different at different occurrences.

3. Main results

In order to obtain an approximation order by periodic neural networks, we introduce Jackson's theorem for periodic functions. Natanson especially stated Jackson's theorem for continuous 2π-periodic functions in [6]. His result is the following Theorem.

THEOREM 3.1. *For a continuous 2π-periodic function f and $n \in \mathbb{N}$, there exists $T \in \mathbb{T}_n$ such that*

$$\|f - T\|_\infty \le c_1 \omega(f, \frac{1}{n}), \tag{3.1}$$

where ω is the modulus of continuity and c_1 is an absolute positive constant.

Therefore, Theorem 3.1 implies that

$$E_n^*(f) \le c_1 \omega(f, \frac{1}{n}) \tag{3.2}$$

for a continuous 2π-periodic function f on $[-\pi, \pi]$ and $\omega(f, \frac{1}{n}) \to 0$ as $n \to \infty$ by the definition of the modulus of continuity.

Note that the De La Vallée Poussin sum $V_n(f)$ of 2π-periodic function f has the following properties.

LEMMA 3.2. *For $n \in \mathbb{N}$, we have*
(1) $V_n(h) = h$ for any trigonometric polynomial $h \in \mathbb{T}_m$ with $m \le n$.
(2) $\|V_n(f)\|_\infty \le 3\|f\|_\infty$ for a continuous 2π-periodic function f.

Proof. (1) By (2.5), we have $S_k(h) = h$ for $n+1 \le k \le 2n$ and $h \in \mathbb{T}_m$ with $m \le n$. Hence

$$V_n(h) = \frac{1}{n} \sum_{k=n+1}^{2n} S_k(h) = \frac{1}{n} \sum_{k=n+1}^{2n} h = h.$$

(2) Since $V_n(f) = 2F_{2n}(f) - F_n(f)$ for $n \in \mathbb{N}$, we have

$$\|V_n(f)\|_\infty \le 2\|F_{2n}(f)\|_\infty + \|F_n(f)\|_\infty \le 3\|f\|_\infty.$$

Thus we complete the proof. □

We obtain the following result from Theorem 3.1 and Lemma 3.2.

THEOREM 3.3. *Let $n \in \mathbb{N}$ and let f be a continuous 2π-periodic function. Then*

$$\|f - V_n(f)\|_\infty \leq c_2\omega(f, \frac{1}{n}), \tag{3.3}$$

where $V_n(f)$ is the De La Vallée Poussin sum of f and c_2 is an absolute constant.

Proof. By Theorem 3.1, there exists a trigonometric polynomial $T^* \in \mathbb{T}_n$ such that

$$\|f - T^*\|_\infty \leq c_1\omega(f, \frac{1}{n}), \tag{3.4}$$

where ω is the modulus of continuity and c_1 is an absolute positive constant. By Lemma 3.2, we have $V_n(T^*) = T^*$ and $\|V_n(T^* - f)\|_\infty \leq 3\|f - T^*\|_\infty$. Hence

$$\begin{aligned} \|f - V_n(f)\|_\infty &= \|f - T^* + T^* - V_n(f)\|_\infty \\ &\leq \|f - T^*\|_\infty + \|T^* - V_n(f)\|_\infty \\ &= \|f - T^*\|_\infty + \|V_n(T^* - f)\|_\infty \\ &= \|f - T^*\|_\infty + 3\|T^* - f\|_\infty \\ &= 4\|f - T^*\|_\infty \\ &\leq 4c_1\omega(f, \frac{1}{n}) \\ &:= c_2\omega(f, \frac{1}{n}), \end{aligned} \tag{3.5}$$

where c_2 is an absolute constant. □

Since $V_n(f) \in \mathbb{T}_{2n}$ for $n \in \mathbb{N}$, we rewrite

$$V_n(f, x) = \sum_{|j| \leq 2n} l_j(f)e^{ijx}, \tag{3.6}$$

where $l_j(f) \in \mathbb{C}$.

Now we have to show e^{ijx} in (3.6) is approximated arbitrarily well by periodic neural networks. Using a Riemann sum, we obtained the following lemma in [3].

LEMMA 3.4. *Let σ be a continuous 2π-periodic function with $\Gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(t)e^{-it}dt \neq 0$. Then, for a given $j \in \mathbb{Z}$,*

$$\|e^{ij\cdot} - \frac{1}{m\Gamma}\sum_{k=1}^{m} e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty \to 0 \tag{3.7}$$

as $m \to \infty$.

Lemma 3.4 gives us the following.

THEOREM 3.5. *Let $l_j \in \mathbb{C}$ for $j \in \mathbb{Z}$ and let σ be a continuous 2π-periodic function with $\Gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(t)e^{-it}dt \neq 0$. Then for a given $\epsilon > 0$, there exists a periodic neural network*

$$N_{2n,m}(\sigma, x) := \sum_{|j|\leq 2n}\sum_{k=1}^{m} l_j \frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(jx - \frac{\pi(2k-m)}{m}) \tag{3.8}$$

such that

$$\|\sum_{|j|\leq 2n} l_j e^{ij\cdot} - N_{2n,m}(\sigma,\cdot)\|_\infty < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. Then by Lemma 3.4, there exists $m_j \in \mathbb{N}$ such that

$$\|e^{ij\cdot} - \frac{1}{m_j\Gamma}\sum_{k=1}^{m_j} e^{i\frac{\pi(2k-m_j)}{m_j}}\sigma(j\cdot - \frac{\pi(2k-m_j)}{m_j})\|_\infty < \frac{\epsilon}{2^{j^2+2}(|l_j|+1)} \tag{3.9}$$

for each $j \in \mathbb{Z}$ with $|j| \leq 2n$.

Let $m = \max\{m_j : |j| \leq 2n\}$. Then

$$\begin{aligned}
&\|\sum_{|j|\leq 2n} l_j e^{ij\cdot} - N_{2n,m}(\sigma,\cdot)\|_\infty \qquad (3.10)\\
= \;& \|\sum_{|j|\leq 2n} l_j e^{ij\cdot} - \sum_{|j|\leq 2n}\sum_{k=1}^{m} l_j\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty\\
\leq \;& \sum_{|j|\leq 2n}|l_j|\cdot\|e^{ij\cdot} - \sum_{k=1}^{m}\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty\\
< \;& \epsilon.
\end{aligned}$$

Thus we complete the proof. □

From Theorem 3.3 and Theorem 3.5, we obtain the following theorem that is the main result of this paper.

THEOREM 3.6. *Let $n \in \mathbb{N}$ and let σ be a continuous 2π-periodic function with with $\Gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma(t)e^{-it}dt \neq 0$. Then for a continuous 2π-periodic function f, there exists a neural network*

$$N_{2n,m}(\sigma, x) := \sum_{|j|\leq 2n} \sum_{k=1}^{m} l_j(f)\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(jx - \frac{\pi(2k-m)}{m}) \tag{3.11}$$

such that

$$\|f - N_{2n,m}\|_\infty < c\omega(f, \frac{1}{n}),$$

where $l_j(f)$'s are coefficients of the De La Vallée Poussin sum of f, ω is the modulus continuity and c is an absolute constant.

Proof. Let $\epsilon > 0$ be given. Then by Theorem 3.3, there exists $V_n(f) \in \mathbb{T}_{2n}$ such that

$$\|f - V_n(f)\|_\infty \leq c\omega(f, \frac{1}{n}), \tag{3.12}$$

where c is an absolute constant. Since $V_n(f,x) = \sum_{|j|\leq 2n} l_j(f)e^{ijx}$ for $l_j(f) \in \mathbb{C}$, Theorem 3.4 gives that there exists a neural network

$$N_{2n,m}(\sigma, x) := \sum_{|j|\leq 2n} \sum_{k=1}^{m} l_j(f)\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(jx - \frac{\pi(2k-m)}{m}) \tag{3.13}$$

such that

$$\|V_n(f, \cdot) - N_{2n,m}(\sigma, \cdot)\|_\infty < \epsilon. \tag{3.14}$$

Therefore, from the equations (3.12) and (3.14), we have

$$\begin{aligned} \|f - N_{2n,m}\|_\infty &\leq \|f - V_n(f)\|_\infty + \|V_n(f) - N_{2n,m}\|_\infty \\ &< c\omega(f, \frac{1}{n}) + \epsilon. \end{aligned} \tag{3.15}$$

Since $\epsilon > 0$ is arbitrary, we get

$$\|f - N_{2n,m}\|_\infty \leq c\omega(f, \frac{1}{n}). \tag{3.16}$$

This completes the proof. $\square$

4. Discussions

In [3], we showed a possibility of periodic function approximation by periodic neural networks. One of main topics in the neural network approximation is the complexity problem which shows approximation orders, since density results are easily obtained from complexity results.

In this paper, we obtained a complexity result by periodic neural networks using the De La Vallée Poussin sums, since the De La Vallée Poussin sums have the properties of the nth partial sums of Fourier series and the Fejér sums. Since $V_n(f) \in \mathbb{T}_{2n}$ for a continuous 2π-periodic function f, Theorem 3.3 shows that

$$E^*_{2n}(f) \leq c_1\omega(f, \frac{1}{n}), \tag{4.1}$$

where c_1 is an absolute constant. Thus we used a periodic neural network with $2n$ neurons instead of n neurons in the hidden layer in order to obtain the main result. So, the following question arises. Even if we use a periodic neural network with n neurons in the hidden layer, is the inequality

$$\|f - N_{n,m}\|_\infty \leq c\omega(f, \frac{1}{n}), \tag{4.2}$$

where c is an absolute constant still true? We think that it is much more complicated and will study this in the future.

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