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WEAK AND STRONG CONVERGENCE OF THREE STEP ITERATION SCHEME WITH ERRORS FOR NON-SELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, weak and strong convergence theorems of three step iteration process with errors are established for two weakly inward and non-self asymptotically nonexpansive mappings in Banach spaces. The results obtained in this paper extend and improve the several recent results in this area.

1. Introduction

Let K be a nonempty subset of a real normed linear space E. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. A mapping $T : K \to K$ is said to be asymptotically nonexpansive [6] if there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \geq 1$. We denote by $F(T)$ the set of fixed points of T.

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The interest and importance of fixed points of nonexpansive mappings stem mainly from the fact that it may be applied in many areas such as imagine recovery, signal processing and equilibrium problems [1,2,13,18].

The class of asymptotically nonexpansive mappings is a natural generalization of the important class of nonexpansive mappings. Goebel and Kirk [6] proved that if K is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping has a fixed point.

In 2003, Chidume, Ofoedu and Zegeye [4] generalized the concept of asymptotically nonexpansive self-mapping and proposed the concept of non-self asymptotically nonexpansive mapping, which is defined as follows:

Definition 1.1. Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A non-self mapping $T : K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n\|x - y\| \tag{1.2}$$

for all $x, y \in K$ and $n \ge 1$.

If T is self-mapping, then P becomes the identity mapping. So, (1.2) reduces to (1.1).

Definition 1.2. Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K and let $I : K \to E$ be a non-self mapping. A non-self mapping $T : K \to E$ is said to be I-asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n\|I(PI)^{n-1}x - I(PI)^{n-1}y\|$$

for all $x, y \in K$ and $n \ge 1$.

In [4], Chidume et al. obtained the strong convergence theorem of fixed points of a non-self asymptotically nonexpansive mapping. In 2006, Wang [20] generalized their work and obtained some strong and weak convergence theorems of common fixed points of a pair of non-self asymptotically nonexpansive mappings in uniformly convex Banach spaces. And authors of [8,12,14,21] also obtained some convergence theorems for such non-self mappings. However, iterative algorithms for approximation fixed points of non-self asymptotically nonexpansive mappings have not been paid too much attention. The main reason is the fact that when T is not a self-mapping, the mapping T^n is nonsensical.

REMARK 1.1. If $T : K \to E$ is an asymptotically nonexpansive mapping in the light of (1.2) and $P : E \to K$ is a nonexpansive retraction, then for all $x, y \in K$, $n \geq 1$, we have

$$\begin{aligned}\|(PT)^n x - (PT)^n y\| &= \|PT(PT)^{n-1}x - PT(PT)^{n-1}y\| \\ &\leq \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \\ &\leq k_n\|x - y\|.\end{aligned}$$

In 2000, Noor [10] introduced a three step iterative sequence and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [5] applied three step iterative sequence for finding the approximate solution of the elastoviscoplasticity problem, eigenvalue problem and liquid crystal theory.

Recently, Zhou et al.[22] studied the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 & \in K, \\ x_{n+1} & = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, \quad \forall n \geq 1, \end{cases}$$

where K is a nonempty closed convex subset of a real normed linear space E which is also a nonexpansive retraction of E with a retraction P and $T_1, T_2 : K \to E$ are two non-self asymptotically nonexpansive mappings with respect to P.

Inspired and motivated by these facts, we will construct a new type of three step iterative sequence with errors for two non-self asymptotically nonexpansive mappings as following:

(1.3)
$$\begin{cases} x_1 & \in K, \\ z_n & = a_n (PI)^n x_n + (1 - a_n - \mu_n) x_n + \mu_n u_n, \\ y_n & = b_n (PT)^n z_n + c_n (PT)^n x_n + (1 - b_n - c_n - \nu_n) x_n + \nu_n v_n, \\ x_{n+1} & = \alpha_n (PI)^n y_n + \beta_n (PI)^n z_n + (1 - \alpha_n - \beta_n - \lambda_n) x_n + \lambda_n w_n, \end{cases}$$

where $T : K \to E$ is a non-self I-asymptotically nonexpansive mapping, $I : K \to E$ is a non-self asymptotically nonexpansive mapping, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\lambda_n\}$, $\{a_n + \mu_n\}$, $\{b_n + c_n + \nu_n\}$

and $\{\alpha_n + \beta_n + \lambda_n\}$ are appropriate sequences in $[0,1]$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K. The purpose of this paper is to introduce and study convergence problem of the three-step iterative sequence with errors for two non-self asymptotically nonexpansive mappings in an uniformly convex Banach space. The results presented in this paper generalize and extend some results in Jeong [7], Takahashi and Tamura [17], K. Nammanee et al. [9] and H. Y. Zhou et al. [22].

2. Preliminaries

Let E be a real Banach space with the topological dual space E^*. The modulus of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let $S(E) = \{x \in E : \|x\| = 1\}$. E is said to be smooth if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in S(E)$. A Banach space E is said to satisfy Opial's condition [11] if for each sequence $\{x_n\}$ in E the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in E$ with $y \neq x$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A subset K of E is said to be retract if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is retraction. A mapping $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A mapping $P : E \to K$ is said to be sunny if $P(Px + t(x - Px)) = Px$ whenever $t \geq 0$. For all $x \in K$ we define a set $I_K(x)$ by $I_K(x) = \{x + \lambda(y - x) : \lambda > 0, y \in K\}$. A non-self mapping $T : K \to E$ is said to be inward if $Tx \in I_K(x)$ for all $x \in K$ and T is said to be weakly inward if $Tx \in \overline{I_K(x)}$ for all $x \in K$.

In order to prove our main results, we shall make use of the following lemmas.

Lemma 2.1. ([9]) *Let E be a uniformly convex Banach space and $B_r = \{x \in E : \|x\| \leq r, r > 0\}$. Then there exists a continuous strictly* *increasing convex function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$ *such that*

$$\|\lambda x + \mu y + \nu z + \kappa w\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 + \kappa\|w\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z, w \in B_r$ *and* $\lambda, \mu, \nu, \kappa \in [0,1]$ *with* $\lambda + \mu + \nu + \kappa = 1$.

LEMMA 2.2. ([19]) *Let* $\{a_n\}$, $\{b_n\}$, $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} \delta_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

LEMMA 2.3.([15]) *Let* E *be a real smooth Banach space and let* K *be a nonempty closed convex subset of* E *with* P *as a sunny nonexpansive retraction and let* $T : K \to E$ *be a mapping satisfying weakly inward condition. Then* $F(PT) = F(T)$.

LEMMA 2.4. ([16]) *Let* E *be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in* E. *Let* $q_1, q_2 \in E$ *be such that* $\lim_{n\to\infty} \|x_n - q_1\|$ *and* $\lim_{n\to\infty} \|x_n - q_2\|$ *exist. If* $\{x_{n_k}\}$, $\{x_{n_j}\}$ *are the subsequences of* $\{x_n\}$ *which converge weakly to* $q_1, q_2 \in E$, *respectively. Then* $q_1 = q_2$.

3. Convergence of the iteration scheme

In this section, we shall prove the weak and strong convergence of the iteration scheme (1.3) to approximate a common fixed point for non-self asymptotically nonexpansive mappings T and I.

LEMMA 3.1. *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *and* $\{\nu_n\}$ *be sequences in* $[0,1]$ *such that* $\limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$. *Let* $\{k_n\}$ *and* $\{l_n\}$ *be sequences of real numbers with* $k_n, l_n \geq 1$ *for all* $n \geq 1$, $\lim_{n\to\infty} k_n = 1$ *and* $\lim_{n\to\infty} l_n = 1$. *Then there exists a positive integer* N *and* $\gamma \in (0,1)$ *such that* $a_n c_n l_n^2 k_n < \gamma$ *for all* $n \geq N$.

Proof. From $\limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$ we have that there exists a positive integer $N_1 > 0$ and $\delta \in (0,1)$ such that

$$a_n c_n \leq c_n \leq b_n + c_n + \nu_n < \delta, \quad \forall n \geq N_1.$$

Let $\delta_1 \in (0,1)$ with $\delta_1 > \delta$. Since $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} l_n = 1$, we have that there exists a positive integer $N \geq N_1$ such that

$$l_n^2 k_n - 1 < \frac{1}{\delta_1} - 1, \quad \forall n \geq N.$$

Thus we obtain

$$l_n^2 k_n < \frac{1}{\delta_1}, \quad \forall n \geq N.$$

Put $\gamma = \frac{\delta}{\delta_1}$. Then we have $a_n c_n l_n^2 k_n < \gamma$ for all $n \geq N$. This completes the proof. □

Lemma 3.2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E with P as a nonexpansive retraction. Let $T : K \to E$ be a I-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $I : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{l_n\} \subset [1, \infty)$ such that $l_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. Suppose that $\{x_n\}$ is the sequence defined by (1.3) satisfying the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$.

If $F = F(T) \cap F(I) \neq \phi$, then

(1) $\lim_{n\to\infty} \|x_n - q\|$ *exists for all* $q \in F$,

(2) $\lim_{n\to\infty} \|(PT)^n z_n - x_n\| = 0$,

(3) $\lim_{n\to\infty} \|(PI)^n y_n - x_n\| = 0$.

Proof. (1) For any given $q \in F$, by the boundedness of sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$, there exists a constant $M_1 > 0$ such that

$$\max\{\sup_{n\geq 1} \|u_n - q\|^2, \sup_{n\geq 1} \|v_n - q\|^2, \sup_{n\geq 1} \|w_n - q\|^2\} \leq M_1.$$

From (1.3) we obtain

$$\begin{aligned}
\|z_n - q\| &= \|a_n((PI)^n - q) + (1 - a_n - \mu_n)(x_n - q) + \mu_n(u_n - q)\|^2 \\
&\leq a_n\|(PI)^n x_n - q\|^2 + (1 - a_n - \mu_n)\|x_n - q\|^2 + \mu_n\|u_n - q\|^2 \\
&\quad - a_n(1 - a_n - \mu_n)g(\|(PI)^n x_n - x_n)\|) \\
&\leq a_n l_n^2\|x_n - q\|^2 + (1 - a_n - \mu_n)\|x_n - q\|^2 + \mu_n\|u_n - q\|^2 \\
&\leq [1 + a_n(l_n^2 - 1) - \mu_n]\|x_n - q\|^2 + \mu_n M_1. \qquad (3.1)
\end{aligned}$$

Again, from (1.3) and (3.1) we have

$$
\begin{aligned}
&\|y_n - q\|^2 \\
&= \|b_n((PT)^n z_n - q) + c_n((PT)^n x_n - q) \\
&\quad + (1 - b_n - c_n - \nu_n)(x_n - q) + \nu_n(v_n - q)\|^2 \\
&\le b_n\|(PT)^n z_n - q\|^2 + c_n\|(PT)^n x_n - q\|^2 + (1 - b_n - c_n - \nu_n)\|x_n - q\|^2 \\
&\quad + \nu_n\|v_n - q\|^2 - b_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|) \\
&\le b_n k_n^2\|(PI)^n z_n - q\|^2 + c_n k_n^2\|(PI)^n x_n - q\|^2 + (1 - b_n - c_n - \nu_n)\|x_n - q\|^2 \\
&\quad + \nu_n\|v_n - q\|^2 - b_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|) \\
&\le b_n k_n^2 l_n^2\|z_n - q\|^2 + c_n k_n^2 l_n^2\|x_n - q\|^2 + (1 - b_n - c_n - \nu_n)\|x_n - q\|^2 \\
&\quad + \nu_n M_1 - b_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|) \\
&\le b_n k_n^2 l_n^2[(1 + a_n(l_n^2 - 1) - \mu_n)\|x_n - q\|^2 + \mu_n M_1] \\
&\quad + c_n k_n^2 l_n^2\|x_n - q\|^2 + (1 - b_n - c_n - \nu_n)\|x_n - q\|^2 \\
&\quad + \nu_n M_1 - b_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|) \\
&= [b_n k_n^2 l_n^2(1 + a_n(l_n^2 - 1) - \mu_n) + c_n k_n^2 l_n^2 + 1 - b_n - c_n - \nu_n]\|x_n - q\|^2 \\
&\quad + b_n k_n^2 l_n^2 \mu_n M_1 + \nu_n M_1 - b_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|).
\end{aligned} \tag{3.2}
$$

Therefore we have

$$
\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|\alpha_n((PI)^n y_n - q) + \beta_n((PI)^n z_n - q) \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - q) + \lambda_n(w_n - q)\|^2 \\
&\le \alpha_n\|(PI)^n y_n - q\|^2 + \beta_n\|(PI)^n z_n - q\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 \\
&\quad + \lambda_n\|w_n - q\|^q - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|(PI)^n y_n - x_n\|) \\
&\le \alpha_n l_n^2\|y_n - q\|^2 + \beta_n l_n^2\|z_n - q\|^2 + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 \\
&\quad + \lambda_n\|w_n - q\|^2 - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|(PI)^n y_n - x_n\|)
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha_n l_n^2[b_n k_n^2 l_n^2(1 + a_n(l_n^2 - 1) - \mu_n)\|x_n - q\|^2 \\
&\quad + (c_n k_n^2 l_n^2 + 1 - b_n - c_n - \nu_n)\|x_n - q\|^2 + b_n k_n^2 l_n^2 \mu_n M_1 + \nu_n M_1 \\
&\quad - b_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|)] \\
&\quad + \beta_n l_n^2[(1 + a_n(l_n^2 - 1) - \mu_n)\|x_n - q\|^2 + \mu_n M_1] \\
&\quad + (1 - \alpha_n - \beta_n - \lambda_n)\|x_n - q\|^2 + \lambda_n\|w_n - q\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|(PI)^n y_n - x_n\|) \\
&\leq [1 + c_n l_n^2 \alpha_n(k_n^2 l_n^2 - 1) + \alpha_n(l_n^2 - 1) + \beta_n(l_n^2 - 1) + a_n l_n^2 \beta_n(l_n^2 - 1) \\
&\quad + a_n b_n k_n^2 l_n^4 \alpha_n(l_n^2 - 1) + b_n l_n^2 \alpha_n(k_n^2 l_n^2 - 1)]\|x_n - q\|^2 \\
&\quad + b_n k_n^2 l_n^4 \alpha_n \mu_n M_1 + l_n^2 \alpha_n \nu_n M_1 + l_n^2 \beta_n \mu_n M_1 + \lambda_n M_1 \\
&\quad - b_n l_n^2 \alpha_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|) \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|(PI)^n y_n - x_n\|) \\
&= [1 + c_n l_n^2 \alpha_n\{(k_n^2 - 1)(l_n^2 - 1) + (k_n^2 - 1) + (l_n^2 - 1)\} + \alpha_n(l_n^2 - 1) \\
&\quad + \beta_n(l_n^2 - 1) + a_n l_n^2 \beta_n(l_n^2 - 1) + a_n b_n k_n^2 l_n^4 \alpha_n(l_n^2 - 1) \\
&\quad + b_n l_n^2 \alpha_n\{(k_n^2 - 1)(l_n^2 - 1) + (k_n^2 - 1) + (l_n^2 - 1)\}]\|x_n - q\|^2 \\
&\quad + b_n k_n^2 l_n^4 \alpha_n \mu_n M_1 + l_n^2 \alpha_n \nu_n M_1 + l_n^2 \beta_n \mu_n M_1 + \lambda_n M_1 \\
&\quad - b_n l_n^2 \alpha_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|) \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|(PI)^n y_n - x_n\|) \\
&\leq [1 + (k_n^2 - 1)(c_n l_n^2 \alpha_n + b_n l_n^2 \alpha_n) + (l_n^2 - 1)(c_n l_n^2 \alpha_n + \alpha_n + \beta_n + a_n l_n^2 \beta_n \\
&\quad + a_n b_n k_n^2 l_n^4 \alpha_n + b_n l_n^2 \alpha_n) + (k_n^2 - 1)(l_n^2 - 1)(c_n l_n^2 \alpha_n + b_n l_n^2 \alpha_n)]\|x_n - q\|^2 \\
&\quad + b_n k_n^2 l_n^4 \alpha_n \mu_n M_1 + l_n^2 \alpha_n \nu_n M_1 + l_n^2 \beta_n \mu_n M_1 + \lambda_n M_1 \\
&\quad - b_n l_n^2 \alpha_n(1 - b_n - c_n - \nu_n)g(\|(PT)^n z_n - x_n\|) \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n - \lambda_n)g(\|(PI)^n y_n - x_n\|).
\end{aligned} \tag{3.3}$$

Since $\{k_n\}$, $\{l_n\}$ are bounded, there exists a constant $M_2 > 0$ such that

$$c_n l_n^2 \alpha_n + b_n l_n^2 \alpha_n < M_2,$$

$$c_n l_n^2 \alpha_n + \alpha_n + \beta_n + a_n l_n^2 \beta_n + a_n b_n k_n^2 l_n^4 \alpha_n + b_n l_n^2 \alpha_n < M_2$$

and

$$b_n k_n^2 l_n^4 \alpha_n M_1 < M_2, \quad l_n^2 \alpha_n M_1 < M_2, \quad l_n^2 \beta_n M_1 < M_2$$

for all $n \geq 1$. By (3.3), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\leq [1 + (k_n^2 - 1)M_2 + (l_n^2 - 1)M_2 + (k_n^2 - 1)(l_n^2 - 1)M_2]\|x_n - q\|^2 \\
&\quad + 2M_2\mu_n + M_2\nu_n + M_1\lambda_n \\
&\quad - b_n l_n^2 \alpha_n (1 - b_n - c_n - \nu_n) g(\|(PT)^n z_n - x_n\|) \\
&\quad - \alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|(PI)^n y_n - x_n\|) \\
&\leq [1 + (k_n^2 - 1)M_2 + (l_n^2 - 1)M_2 + (k_n^2 - 1)(l_n^2 - 1)M_2]\|x_n - q\|^2 \\
&\quad + 2M_2\mu_n + M_2\nu_n + M_1\lambda_n. \qquad (3.4)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ are equivalent to $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^2 - 1) < \infty$, respectively, it follows from Lemma 2.2 and (3.4) that $\lim_{n\to\infty} \|x_n - q\|$ exists.

(2) By (1), there exists a constant $M > 0$ such that $\|x_n - q\|^2 \leq M$ for all $n \geq 1$. From (3.4) we have

$$\begin{aligned}
&\alpha_n l_n^2 b_n (1 - b_n - c_n - \nu_n) g(\|(PT)^n z_n - x_n\|) \\
&\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + M_2\{(k_n^2 - 1) + (l_n^2 - 1) + (k_n^2 - 1)(l_n^2 - 1)\}M \\
&\quad + 2M_2\mu_n + M_2\nu_n + M_1\lambda_n \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g(\|(PI)^n y_n - x_n\|) \\
&\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + M_2\{(k_n^2 - 1) + (l_n^2 - 1) + (k_n^2 - 1)(l_n^2 - 1)\}M \\
&\quad + 2M_2\mu_n + M_2\nu_n + M_1\lambda_n. \qquad (3.6)
\end{aligned}$$

Since $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ and $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$, there exist a positive integer n_0 and $\eta, \delta, \gamma \in (0, 1)$ such that

$$0 < \eta < \alpha_n, \quad 0 < \delta < b_n$$

and

$$\alpha_n + \beta_n + \lambda_n < \gamma < 1, \quad b_n + c_n + \nu_n < \gamma < 1$$

for all $n \geq n_0$. This implies by (3.5) and (3.6) that

$$\eta\delta(1-\gamma)g(\|(PT)^n z_n - x_n\|) \\ \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + M_2\{(k_n^2 - 1) + (l_n^2 - 1) + (k_n^2 - 1)(l_n^2 - 1)\}M \\ + 2M_2\mu_n + M_2\nu_n + M_1\lambda_n \tag{3.7}$$

and

$$\eta(1-\gamma)g(\|(PI)^n y_n - x_n\|) \\ \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + M_2\{(k_n^2 - 1) + (l_n^2 - 1) + (k_n^2 - 1)(l_n^2 - 1)\}M \\ + 2M_2\mu_n + M_2\nu_n + M_1\lambda_n \tag{3.8}$$

for all $n \geq n_0$. It follows from (3.7) and (3.8) that

$$\sum_{n=n_0}^{m} g(\|(PT)^n z_n - x_n\|) \\ \leq \frac{1}{\eta\delta(1-\gamma)}\Big[\sum_{n=n_0}^{m}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) \\ + M_2 M \sum_{n=n_0}^{m}\{(k_n^2 - 1) + (l_n^2 - 1) + (k_n^2 - 1)(l_n^2 - 1)\} \\ + 2M_2\sum_{n=n_0}^{m}\mu_n + M_2\sum_{n=n_0}^{m}\nu_n + M_1\sum_{n=n_0}^{m}\lambda_n\Big] \tag{3.9}$$

and

$$\sum_{n=n_0}^{m} g(\|(PI)^n y_n - x_n\|) \\ \leq \frac{1}{\eta(1-\gamma)}\Big[\sum_{n=n_0}^{m}(\|x_n - q\|^2 - \|x_{n+1} - q\|^2) \\ + M_2 M \sum_{n=n_0}^{m}\{(k_n^2 - 1) + (l_n^2 - 1) + (k_n^2 - 1)(l_n^2 - 1)\} \\ + 2M_2\sum_{n=n_0}^{m}\mu_n + M_2\sum_{n=n_0}^{m}\nu_n + M_1\sum_{n=n_0}^{m}\lambda_n. \tag{3.10}$$

Let $m \to \infty$ in (3.9) and (3.10). Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$ are equivalent to $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n^2 - 1) < \infty$, respectively, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, it follows from (3.9) and (3.10) that

$$\sum_{n=n_0}^{\infty} g(\|(PT)^n z_n - x_n\|) < \infty$$

and

$$\sum_{n=n_0}^{\infty} g(\|(PI)^n y_n - x_n\|) < \infty.$$

Hence we obtain

$$\lim_{n\to\infty} g(\|(PT)^n z_n - x_n\|) = 0 \tag{3.11}$$

and

$$\lim_{n\to\infty} g(\|(PI)^n y_n - x_n\|) = 0. \tag{3.12}$$

Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows from (3.11) and (3.12) that

$$\lim_{n\to\infty} \|(PT)^n z_n - x_n\| = 0$$

and

$$\lim_{n\to\infty} \|(PI)^n y_n - x_n\| = 0.$$

This completes the proof. □

Lemma 3.3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T : K \to E$ be a I-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $I : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{l_n\} \subset [1, \infty)$ such that $l_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. Suppose that $\{x_n\}$ is the sequence defined by (1.3) satisfying the following conditions*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$.

If $F = F(T) \cap F(I) \neq \phi$, then

(1) $\lim_{n\to\infty} \|(PI)^n x_n - x_n\| = 0$,

(2) $\lim_{n\to\infty} \|(PT)^n x_n - x_n\| = 0$,

(3) $\lim_{n\to\infty} \|(PI)^n z_n - x_n\| = 0.$

Proof. By Lemma 3.2, we have

$$\lim_{n\to\infty} \|(PT)^n z_n - x_n\| = 0 \tag{3.13}$$

and

$$\lim_{n\to\infty} \|(PI)^n y_n - x_n\| = 0. \tag{3.14}$$

By (1.3) we obtain

$$\begin{aligned} \|x_n - z_n\| &= \|a_n(PI)^n x_n + (1 - a_n - \mu_n)x_n + \mu_n u_n - x_n\| \\ &\leq a_n\|(PI)^n x_n - x_n\| + \mu_n\|u_n - x_n\| \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \|x_n - y_n\| &= \|b_n(PT)^n z_n + c_n(PT)^n x_n + (1 - b_n - c_n - \nu_n)x_n + \nu_n v_n - x_n\| \\ &\leq b_n\|(PT)^n z_n - x_n\| + c_n\|(PT)^n x_n - x_n\| + \nu_n\|v_n - x_n\|. \end{aligned} \tag{3.16}$$

By (3.15) and (3.16), we have

$$\begin{aligned} &\|(PT)^n x_n - x_n\| \\ &\leq \|(PT)^n x_n - (PT)^n z_n\| + \|(PT)^n z_n - x_n\| \\ &\leq k_n\|(PI)^n x_n - (PI)^n z_n\| + \|(PT)^n z_n - x_n\| \\ &\leq k_n l_n\|x_n - z_n\| + \|(PT)^n z_n - x_n\| \\ &\leq k_n l_n\{a_n\|(PI)^n x_n - x_n\| + \mu_n\|u_n - x_n\|\} + \|(PT)^n z_n - x_n\| \\ &= a_n k_n l_n\|(PI)^n x_n - x_n\| + k_n l_n \mu_n\|u_n - x_n\| + \|(PT)^n z_n - x_n\| \end{aligned} \tag{3.17}$$

and

$$
\begin{aligned}
&\|(PI)^n x_n - x_n\| \\
&\le \|(PI)^n x_n - (PI)^n y_n\| + \|(PI)^n y_n - x_n\| \\
&\le l_n \|x_n - y_n\| + \|(PI)^n y_n - x_n\| \\
&\le l_n \{b_n \|(PT)^n z_n - x_n\| + c_n \|(PT)^n x_n - x_n\| + \nu_n \|v_n - x_n\|\} \\
&\quad + \|(PI)^n y_n - x_n\| \\
&\le b_n l_n \|(PT)^n z_n - x_n\| + c_n l_n \{a_n k_n l_n \|(PI)^n x_n - x_n\| + k_n l_n \mu_n \|u_n - x_n\| \\
&\quad + \|(PT)^n z_n - x_n\|\} + l_n \nu_n \|v_n - x_n\| + \|(PI)^n y_n - x_n\| \\
&= (b_n + c_n) l_n \|(PT)^n z_n - x_n\| + a_n c_n k_n l_n^2 \|(PI)^n x_n - x_n\| + \|(PI)^n y_n - x_n\| \\
&\quad + c_n k_n l_n^2 \mu_n \|u_n - x_n\| + l_n \nu_n \|v_n - x_n\|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
(1 - a_n c_n k_n l_n^2)\|(PI)^n x_n - x_n\| \le{}& (b_n + c_n) l_n \|(PT)^n z_n - x_n\| + \|(PI)^n y_n - x_n\| \\
&+ c_n k_n l_n^2 \mu_n \|u_n - x_n\| + l_n \nu_n \|v_n - x_n\|.
\end{aligned} \tag{3.18}
$$

By Lemma 3.1, there exists a positive integer N_1 and $\gamma \in (0,1)$ such that $a_n c_n k_n l_n^2 < \gamma$ for all $n \ge N_1$. This together with (3.18) implies that for $n \ge N_1$

$$
\begin{aligned}
(1 - \gamma)\|(PI)^n x_n - x_n\| \le{}& (b_n + c_n) l_n \|(PT)^n z_n - x_n\| + \|(PI)^n y_n - x_n\| \\
&+ c_n k_n l_n^2 \mu_n \|u_n - x_n\| + l_n \nu_n \|v_n - x_n\|.
\end{aligned} \tag{3.19}
$$

Taking limit of both sides (3.19), it follows from (3.13) and (3.14) that

$$\lim_{n\to\infty} \|(PI)^n x_n - x_n\| = 0$$

This with (3.13) and (3.17) implies that

$$\lim_{n\to\infty} \|(PT)^n x_n - x_n\| = 0.$$

Noting that

$$
\begin{aligned}
\|(PI)^n z_n - x_n\| &\le \|(PI)^n z_n - (PI)^n x_n\| + \|(PI)^n x_n - x_n\| \\
&\le l_n \|z_n - x_n\| + \|(PI)^n x_n - x_n\| \\
&\le l_n \{a_n \|(PI)^n x_n - x_n\| + \mu_n \|u_n - x_n\|\} + \|(PI)^n x_n - x_n\| \\
&\le (1 + a_n l_n)\|(PI)^n x_n - x_n\| + l_n \mu_n \|u_n - x_n\|,
\end{aligned}
$$

we have

$$\lim_{n\to\infty} \|(PI)^n z_n - x_n\| = 0.$$

This completes the proof. □

THEOREM 3.1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T : K \to E$ be a I-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $I : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{l_n\} \subset [1, \infty)$ such that $l_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. Suppose that $\{x_n\}$ is the sequence defined by (1.3) satisfying the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$.

If PT, PI are completely continuous and $F = F(T) \cap F(I) \neq \phi$, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to a common fixed point of T and I.

Proof. By Lemma 3.2 and 3.3, we have

$$\lim_{n\to\infty} \|(PT)^n z_n - x_n\| = 0, \quad \lim_{n\to\infty} \|(PI)^n y_n - x_n\| = 0,$$

$$\lim_{n\to\infty} \|(PI)^n x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|(PT)^n x_n - x_n\| = 0, \tag{3.20}$$

$$\lim_{n\to\infty} \|(PI)^n z_n - x_n\| = 0.$$

Since

$$x_{n+1} - x_n = \alpha_n((PI)^n y_n - x_n) + \beta_n((PI)^n z_n - x_n) + \lambda_n(w_n - x_n),$$

we have

$$\begin{aligned}
&\|x_{n+1} - (PI)^n x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|x_n - (PI)^n x_n\| + \|(PI)^n x_n - (PI)^n x_{n+1}\| \\
&\le (1 + l_n)\|x_{n+1} - x_n\| + \|x_n - (PI)^n x_n\| \\
&\le (1 + l_n)\alpha_n\|(PI)^n y_n - x_n\| + (1 + l_n)\beta_n\|(PI)^n z_n - x_n\| \\
&\quad + (1 + l_n)\lambda_n\|w_n - x_n\| + \|x_n - (PI)^n x_n\|
\end{aligned}$$

and

$$\begin{aligned}
&\|x_{n+1} - (PT)^n x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|x_n - (PT)^n x_n\| + \|(PT)^n x_n - (PT)^n x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \|x_n - (PT)^n x_n\| + k_n\|(PI)^n x_n - (PI)^n x_{n+1}\| \\
&\le (1 + k_n l_n)\|x_{n+1} - x_n\| + \|x_n - (PT)^n x_n\| \\
&\le (1 + k_n l_n)\alpha_n\|(PI)^n y_n - x_n\| + (1 + k_n l_n)\beta_n\|(PI)^n z_n - x_n\| \\
&\quad + (1 + k_n l_n)\lambda_n\|w_n - x_n\| + \|x_n - (PT)^n x_n\|.
\end{aligned}$$

It follows from (3.20) that

$$\lim_{n\to\infty} \|x_{n+1} - (PI)^n x_{n+1}\| = 0$$

and

$$\lim_{n\to\infty} \|x_{n+1} - (PT)^n x_{n+1}\| = 0.$$

Thus we obtain

$$\begin{aligned}
\|x_{n+1} - (PI)x_{n+1}\| &\le \|x_{n+1} - (PI)^{n+1}x_{n+1}\| + \|(PI)^{n+1}x_{n+1} - (PI)x_{n+1}\| \\
&\le \|x_{n+1} - (PI)^{n+1}x_{n+1}\| + l_1\|(PI)^n x_{n+1} - x_{n+1}\| \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1} - (PT)x_{n+1}\| &\le \|x_{n+1} - (PT)^{n+1}x_{n+1}\| + \|(PT)^{n+1}x_{n+1} - (PT)x_{n+1}\| \\
&\le \|x_{n+1} - (PT)^{n+1}x_{n+1}| + k_1 l_1\|(PT)^n x_{n+1} - x_{n+1}\| \\
&\to 0 \quad \text{as} \quad n \to \infty. \qquad (3.21)
\end{aligned}$$

Since PT, PI are completely continuous and $\{x_n\} \subseteq K$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{(PT)x_{n_k}\}$, $\{(PI)x_{n_k}\}$ converge strongly to q. From (3.21) we have $x_{n_k} \to q$ as $k \to \infty$. By the continuities of P, T and I, we have $q = (PI)q = (PT)q$ and $q \in F(T) \cap F(I)$ by Lemma 2.3. By Lemma 3.2, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists. Therefore $\{x_n\}$ converges strongly to q as $n \to \infty$. Since

$$\begin{aligned}
\|y_n - x_n\| &\le b_n\|(PT)^n z_n - x_n\| + c_n\|(PT)^n x_n - x_n\| + \nu_n\|v_n - x_n\| \\
&\to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - x_n\| &\le a_n\|(PI)^n x_n - x_n\| + \mu_n\|u_n - x_n\| \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

it follows that $\lim_{n\to\infty} y_n = q$ and $\lim_{n\to\infty} z_n = q$. This completes the proof. □

THEOREM 3.2. *Let E be a real smooth and uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T : K \to E$ be a weakly inward and I-asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $k_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $I : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with a sequence $\{l_n\} \subset [1,\infty)$ such that $l_n \to 1$ as $n \to \infty$ and $\sum_{n=1}^{\infty}(l_n - 1) < \infty$. Suppose that $\{x_n\}$ is the sequence defined by (1.3) satisfying the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \nu_n) < 1$.

If $F = F(T) \cap F(I) \ne \phi$, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge weakly to a common fixed point of T and I.

Proof. Let $q \in F(T) \cap F(I)$. Then, by Lemma 3.2, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists. We now prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T) \cap F(I)$.

We assume that q_1 and q_2 are weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By (3.21), $\lim_{k\to\infty} \|x_{n_k} - (PT)x_{n_k}\| = 0$ and $\lim_{k\to\infty} \|x_{n_k} - (PI)x_{n_k}\| = 0$. By Chang et al. [3,Theorem 1], we conclude that

$$q_1 = (PT)q_1 \quad \text{and} \quad q_1 = (PI)q_1.$$

Since $F(PT) = F(T)$ and $F(PI) = F(I)$ by Lemma 2.3, we have $Tq_1 = q_1$ and $Iq_1 = q_1$. In the same way, $Tq_2 = q_2$ and $Iq_2 = q_2$. Therefore we have $q_1, q_2 \in F(T) \cap F(I)$. From Lemma 2.4 we have $q_1 = q_2$. This completes the proof. □

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