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INEQUALITIES FOR THE RIEMANN-STIELTJES INTEGRAL OF PRODUCT INTEGRATORS WITH APPLICATIONS

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Abstract. We show amongst other that if $f, g : [a, b] \to \mathbb{C}$ are two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists, then for any continuous functions $h : [a, b] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_a^b h(t)\, d(f(t)\, g(t))$ exists and

$$\int_a^b h(t)\, d(f(t)\, g(t)) = \int_a^b h(t)\, f(t)\, d(g(t)) + \int_a^b h(t)\, g(t)\, d(f(t)).$$

Using this identity we then provide sharp upper bounds for the quantity

$$\left| \int_a^b h(t)\, d(f(t)\, g(t)) \right|$$

and apply them for trapezoid and Ostrowski type inequalities. Some applications for continuous functions of selfadjoint operators on complex Hilbert spaces are given as well.

1. Introduction

One of the most important properties of the *Riemann-Stieltjes integral*

$$\int_a^b f(t)\, dg(t)$$

is the fact that this integral exists if one of the function is of *bounded variation* while the other is *continuous*. The following sharp inequality holds

$$\left| \int_a^b f(t)\, dg(t) \right| \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (g), \tag{1.1}$$

provided that $f : [a, b] \to \mathbb{C}$ is continuous on $[a, b]$ and $g : [a, b] \to \mathbb{C}$ is of bounded variation on this interval. Here $\bigvee_a^b (g)$ denotes the *total variation* of g on $[a, b]$.

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When g is *Lipschitzian* with the constant $L > 0$, i.e.,

$$|g(t) - g(s)| \le L|t - s|$$

for any $t, s \in [a, b]$, then we have

$$\left|\int_a^b f(t)\, dg(t)\right| \le L \int_a^b |f(t)|\, dt \tag{1.2}$$

for any *Riemann integrable* function $f : [a, b] \to \mathbb{C}$.

Moreover, if the integrator g is *monotonic nondecreasing* on the interval $[a, b]$ and $f : [a, b] \to \mathbb{C}$ is continuous, then we have the *modulus inequality*

$$\left|\int_a^b f(t)\, dg(t)\right| \le \int_a^b |f(t)|\, dg(t). \tag{1.3}$$

The above inequalities have been used by many authors to derive various integral inequalities. We provide here some simple examples.

The following *generalized trapezoidal inequality* for the function of bounded variation $f : [a, b] \to \mathbb{C}$ was obtained in 1999 by the author [21, Proposition 1]

$$\begin{aligned} &\left|\int_a^b f(t)\, dt - (x-a) f(a) - (b-x) f(b)\right| \\ &\le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \end{aligned} \tag{1.4}$$

where $x \in [a, b]$. The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity. See also [19] for a different proof and other details.

The best inequality one can derive from (1.4) is the *trapezoid inequality*

$$\left|\int_a^b f(t)\, dt - \frac{f(a) + f(b)}{2}(b-a)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f). \tag{1.5}$$

Here the constant $\frac{1}{2}$ is also best possible.

For related results, see [11]-[15], [17]-[20], [24]-[25], [29]-[32], [35], [41], [34], [43]-[45] and [53]-[55].

In order to extend the classical *Ostrowski's inequality* for differentiable functions with bounded derivatives to the larger class of functions of bounded variation, the author obtained in 1999 (see [21] or the RGMIA preprint version of [23]) the following result

$$\left|\int_a^b f(t)\, dt - f(x)(b-a)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \tag{1.6}$$

for any $x \in [a, b]$ and $f : [a, b] \to \mathbb{C}$ a function of bounded variation on $[a, b]$. Here $\bigvee_a^b (f)$ denotes the *total variation* of f on $[a, b]$ and the constant $\frac{1}{2}$ is best possible in (1.6). The best inequality one can obtain from (1.6) is the *midpoint inequality*, namely

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \le \frac{1}{2}(b-a)\bigvee_a^b (f) \tag{1.7}$$

for which the constant $\frac{1}{2}$ is also sharp.

For related results, see [1]-[11], [16]-[17], [21], [23], [25]-[27], [31], [36]-[38], [42], [46]-[52] and [56]-[59].

Motivated by the above results, we establish in this paper bounds for the quantity

$$\left|\int_a^b h(t)\,d(f(t)g(t))\right|$$

in the case when the integrand h is continuous while the functions f and g are of bounded variation. Applications for the trapezoidal and midpoint inequalities are given. Some applications for continuous functions of selfadjoint operators on complex Hilbert spaces are provided as well.

2. The main results

The following identity is of interest in itself:

Lemma 1. *Let $f,g:[a,b]\to\mathbb{C}$ be two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists. If $h:[a,b]\to\mathbb{C}$ is continuous, then the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)g(t))$ exists and*

$$\int_a^b h(t)\,d(f(t)g(t)) = \int_a^b h(t)f(t)\,d(g(t)) + \int_a^b h(t)g(t)\,d(f(t)). \tag{2.1}$$

Proof. Since $f,g:[a,b]\to\mathbb{C}$ are of bounded variation, then fg is of bounded variation and since $h:[a,b]\to\mathbb{C}$ is continuous, it follows that the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)g(t))$ exists.

Observe that, since the integral $\int_a^b f(t)\,dg(t)$ exists, then for any $s\in[a,b]$ the integral $\ell(s):=\int_a^s f(t)\,dg(t)$ exists and the function ℓ is of bounded variation on $[a,b]$.

Indeed, let

$$a=s_0<s_1<\cdots<s_{n-1}<s_n=b$$

a division of the interval $[a,b]$. Then we have

$$\begin{aligned}\sum_{i=0}^{n-1}|\ell(s_{i+1})-\ell(s_i)| &= \sum_{i=0}^{n-1}\left|\int_a^{s_{i+1}} f(t)\,dg(t) - \int_a^{s_i} f(t)\,dg(t)\right| \\ &= \sum_{i=0}^{n-1}\left|\int_{s_i}^{s_{i+1}} f(t)\,dg(t)\right| \le \sum_{i=0}^{n-1}\left(\sup_{t\in[s_i,s_{i+1}]}|f(t)|\bigvee_{s_i}^{s_{i+1}}(g)\right)\end{aligned}$$

$$\le \sup_{t\in[a,b]} |f(t)| \sum_{i=0}^{n-1} \left(\bigvee_{s_i}^{s_{i+1}} (g) \right) = \sup_{t\in[a,b]} |f(t)| \bigvee_a^b (g) < \infty,$$

which shows that ℓ is of bounded variation on $[a,b]$ and

$$\bigvee_a^b (\ell) \le \|f\|_\infty \bigvee_a^b (g),$$

where $\|f\|_\infty := \sup_{t\in[a,b]} |f(t)|$.

Now, by the integration by parts theorem, since $\int_a^s f(t)\,dg(t)$ exists for any $s \in [a,b]$, then $\int_a^s g(t)\,df(t)$ also exists and we have the equality

$$f(s)\,g(s) = f(a)\,g(a) + \int_a^s f(t)\,dg(t) + \int_a^s g(t)\,df(t) \tag{2.2}$$

for any $s \in [a,b]$.

Since the functions $\int_a^\cdot f(t)\,dg(t)$ and $\int_a^\cdot g(t)\,df(t)$ are of bounded variation, then the Riemann-Stieltjes integrals

$$\int_a^b h(s)\,d\left(\int_a^s f(t)\,dg(t)\right) \text{ and } \int_a^b h(s)\,d\left(\int_a^s g(t)\,df(t)\right)$$

exist and

$$\int_a^b h(s)\,d\left(\int_a^s f(t)\,dg(t)\right) = \int_a^b h(s)\,f(s)\,dg(s), \tag{2.3}$$

and

$$\int_a^b h(s)\,d\left(\int_a^s g(t)\,df(t)\right) = \int_a^b h(s)\,g(s)\,df(s). \tag{2.4}$$

Now, on utilizing (2.2), (2.3) and (2.4) we have

$$\begin{aligned}
\int_a^b h(s)\,d(f(s)\,g(s)) &= \int_a^b h(s)\,d(f(a)\,g(a)) \\
&\quad + \int_a^b h(s)\,d\left(\int_a^s f(t)\,dg(t)\right) \\
&\quad + \int_a^b h(s)\left(\int_a^s g(t)\,df(t)\right) \\
&= \int_a^b h(s)\,f(s)\,dg(s) + \int_a^b h(s)\,g(s)\,df(s)
\end{aligned}$$

and the equality (2.1) is proved. □

Remark 1. The dual case also holds, namely, when the functions $f, g : [a,b] \to \mathbb{C}$ are continuous and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists, then for any function $h : [a,b] \to \mathbb{C}$ of bounded variation the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)\,g(t))$ exists and the equality (2.1) is satisfied.

Corollary 1. *Let $f : [a, b] \to \mathbb{C}$ be a functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\, df(t)$ exists. If $h : [a, b] \to \mathbb{C}$ is continuous, then the Riemann-Stieltjes integral $\int_a^b h(t)\, d\left(f^2(t)\right)$ exists and*

$$\int_a^b h(t)\, d\left(f^2(t)\right) = 2\int_a^b h(t) f(t)\, d(f(t)). \tag{2.5}$$

If $\int_a^b f(t)\, d\overline{f(t)}$ exists, then for any continuous function $h : [a, b] \to \mathbb{C}$, the Riemann-Stieltjes integral $\int_a^b h(t)\, d\left(|f(t)|^2\right)$ exists and

$$\int_a^b h(t)\, d\left(|f(t)|^2\right) = \int_a^b h(t) f(t)\, d\left(\overline{f(t)}\right) + \int_a^b h(t)\, \overline{f(t)} d(f(t)). \tag{2.6}$$

In particular, if $h : [a, b] \to \mathbb{R}$, then

$$\int_a^b h(t)\, d\left(|f(t)|^2\right) = 2\mathrm{Re}\left(\int_a^b h(t)\, \overline{f(t)} d(f(t))\right). \tag{2.7}$$

The first bound for the Riemann-Stieltjes integral of product integrators is as follows:

Theorem 1. *Let $f, g : [a, b] \to \mathbb{C}$ be two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists. If $h : [a, b] \to \mathbb{C}$ is continuous, then*

$$\left|\int_a^b h(t)\, d(f(t) g(t))\right| \le \|hf\|_\infty \bigvee_a^b (g) + \|hg\|_\infty \bigvee_a^b (f) \tag{2.8}$$
$$\le \|h\|_\infty \left[\|f\|_\infty \bigvee_a^b (g) + \|g\|_\infty \bigvee_a^b (f)\right].$$

Both inequalities in (2.8) *are sharp.*

Proof. We know that if $p : [a, b] \to \mathbb{C}$ is bounded, $v : [a, b] \to \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(s)\, dv(s)$ exists, then we have the inequality

$$\left|\int_a^b p(s)\, dv(s)\right| \le \|p\|_\infty \bigvee_a^b (v), \tag{2.9}$$

where $\|p\|_\infty = \sup_{t\in[a,b]} |p(t)| < \infty$.

Taking the modulus in (2.1) and using the property (2.9) we have

$$\left|\int_a^b h(t)\, d(f(t) g(t))\right| \le \left|\int_a^b h(t) f(t)\, d(g(t))\right| + \left|\int_a^b h(t) g(t)\, d(f(t))\right|$$
$$\le \|hf\|_\infty \bigvee_a^b (g) + \|hg\|_\infty \bigvee_a^b (f)$$

$$\leq \|h\|_{\infty} \|f\|_{\infty} \bigvee_{a}^{b} (g) + \|h\|_{\infty} \|g\|_{\infty} \bigvee_{a}^{b} (f)$$
$$= \|h\|_{\infty} \left[\|f\|_{\infty} \bigvee_{a}^{b} (g) + \|g\|_{\infty} \bigvee_{a}^{b} (f) \right]$$

and the inequality (2.8) is proved.

Now, to prove the sharpness of the inequalities (2.8) we consider the functions $f, g : [a, b] \to \mathbb{R}$ given by

$$f(t) := \begin{cases} 0 & \text{if} \quad t = a \\ 1 & \text{if} \quad t \in (a, b], \end{cases} \quad g(t) := \begin{cases} 1 & \text{if} \quad t \in [a, b) \\ 0 & \text{if} \quad t = b. \end{cases}$$

The functions f and g are of bounded variation, $\bigvee_{a}^{b} (f) = \bigvee_{a}^{b} (g) = 1$ and $\|f\|_{\infty} = \|g\|_{\infty} = 1$. We have

$$f(t) g(t) := \begin{cases} 0 & \text{if } t = a \\ 1 & \text{if } t \in (a, b) \\ 0 & \text{if } t = b. \end{cases}$$

The function fg is of bounded variation and for a continuous function $h : [a, b] \to \mathbb{C}$ the Riemann-Stieltjes integral $\int_a^b h(t) \, d(f(t) g(t))$ exists and integrating by parts we have

$$\int_a^b h(t) \, d(f(t) g(t)) \tag{2.10}$$
$$= f(b) g(b) h(b) - f(a) g(a) h(a) - \int_a^b f(t) g(t) \, d(h(t))$$
$$= -\int_a^b f(t) g(t) \, d(h(t)).$$

Consider the following sequence of divisions and intermediate points

$$a = x_0^{(n)} < \xi_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < \xi_{n-1}^{(n)} < x_n^{(n)} = b$$

such that the norm of the division $\Delta_n := \max_{i \in \{0, \dots, n-1\}} \left(x_{i+1}^{(n)} - x_i^{(n)} \right) \to 0$ as $n \to \infty$. By the definition of the Riemann-Stieltjes integral we have

$$\int_a^b f(t) g(t) \, d(h(t)) = \lim_{n \to \infty} \sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right) g\left(\xi_i^{(n)}\right) \left(h\left(x_{i+1}^{(n)}\right) - h\left(x_i^{(n)}\right) \right)$$
$$= \lim_{n \to \infty} \sum_{i=0}^{n-1} \left(h\left(x_{i+1}^{(n)}\right) - h\left(x_i^{(n)}\right) \right) = h(b) - h(a),$$

and then, by (2.10) we have

$$\int_a^b h(t) \, d(f(t) g(t)) = -h(b) + h(a).$$

We also have

$$h(t)\, f(t) := \begin{cases} 0 & \text{if } t = a \\ h(t) & \text{if } t \in (a,b], \end{cases} \quad h(t)\, g(t) := \begin{cases} h(t) & \text{if } t \in [a,b) \\ 0 & \text{if } t = b, \end{cases}$$

which implies that

$$\|hf\|_\infty = \|hg\|_\infty = \|h\|_\infty .$$

Therefore the inequality (2.8) reduces to

$$|h(b) - h(a)| \le 2\|h\|_\infty \le 2\|h\|_\infty . \tag{2.11}$$

We observe that, this inequality is sharp since for continuous functions $h : [a,b] \to \mathbb{R}$ for which

$$0 < h(b) = -h(a) = \sup_{t\in[a,b]} h(t),$$

we get equality in (2.11).

For instance, if we take

$$h(t) = t - \frac{a+b}{2},\ t \in [a,b],$$

then

$$|h(b) - h(a)| = b - a,\ \|h\|_\infty = \frac{b-a}{2}$$

and the equality in (2.11) is realized. □

Remark 2. We observe that if one of the functions f or g is constant, then (2.8) reduces to (1.1).

We say that the function $f : [a,b] \to \mathbb{C}$ is *Lipschitzian* with the constant $L > 0$ if

$$|f(t) - f(s)| \le L|t - s|$$

for any $t, s \in [a,b]$.

Theorem 2. *Assume that the function $f : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, $g : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$ and $h : [a,b] \to \mathbb{C}$ a continuous function on $[a,b]$. Then we have the inequality*

$$\left|\int_a^b h(t)\, d(f(t)\, g(t))\right| \le K\int_a^b |h(t)\, f(t)|\, dt + L\int_a^b |h(t)\, g(t)|\, dt \tag{2.12}$$
$$\le \max\{K, L\}\int_a^b |h(t)|\,(|f(t)| + |g(t)|)\, dt.$$

The inequalities in (2.12) *are sharp.*

Proof. It is known that, if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b p(s)\,dv(s)$ exists and we have the inequality

$$\left|\int_a^b p(s)\,dv(s)\right|\le L\int_a^b |p(s)|\,ds. \tag{2.13}$$

Taking the modulus in (2.1) and using the property (2.9) we have

$$\begin{aligned}\left|\int_a^b h(t)\,d(f(t)\,g(t))\right| &\le \left|\int_a^b h(t)\,f(t)\,d(g(t))\right|+\left|\int_a^b h(t)\,g(t)\,d(f(t))\right|\\ &\le K\int_a^b |h(t)\,f(t)|\,dt+L\int_a^b |h(t)\,g(t)|\,dt\\ &\le \max\{K,L\}\int_a^b |h(t)|\,(|f(t)|+|g(t)|)\,dt,\end{aligned}$$

and the inequality (2.12) is proved.

Consider now the functions $f,g:[a,b]\to\mathbb{R}$, $f(t)=g(t)=\left|t-\frac{a+b}{2}\right|$. We observe that f and g are Lipschitzian with the constant $L=1$.

Indeed, for any $t,s\in[a,b]$ we have

$$\begin{aligned}|f(t)-f(s)| &= \left|\left|t-\frac{a+b}{2}\right|-\left|s-\frac{a+b}{2}\right|\right|\\ &\le |t-s|,\end{aligned}$$

which shows that the function f is Lipschitzian with the constant $L=1$.

Now

$$\begin{aligned}\left|\int_a^b h(t)\,d(f(t)\,g(t))\right| &= \int_a^b h(t)\,d\left(\left(t-\frac{a+b}{2}\right)^2\right)\\ &= 2\left|\int_a^b h(t)\left(t-\frac{a+b}{2}\right)dt\right|\end{aligned}$$

and

$$K\int_a^b |h(t)\,f(t)|\,dt+L\int_a^b |h(t)\,g(t)|\,dt=2\int_a^b |h(t)|\left|t-\frac{a+b}{2}\right|dt$$

and the first inequality in (2.12) becomes

$$\left|\int_a^b h(t)\left(t-\frac{a+b}{2}\right)dt\right|\le\int_a^b |h(t)|\left|t-\frac{a+b}{2}\right|dt.$$

We observe that the equality case holds if we take $h:[a,b]\to\mathbb{R}$, $h(t)=t-\frac{a+b}{2}$. □

Remark 3. We remark that in the dual case, namely when the functions $f, g : [a, b] \to \mathbb{C}$ are continuous and $h : [a, b] \to \mathbb{C}$ is Lipschitzian with a constant $M > 0$, then the Riemann-Stieltjes integral $\int_a^b h(t)\, d(f(t)\, g(t))$ exists and integrating by parts we have

$$\int_a^b h(t)\, d(f(t)\, g(t)) = f(b)\, g(b)\, h(b) - f(a)\, g(a)\, h(a) - \int_a^b f(t)\, g(t)\, d(h(t))$$

or, equivalently

$$f(b)\, g(b)\, h(b) - f(a)\, g(a)\, h(a) - \int_a^b h(t)\, d(f(t)\, g(t)) = \int_a^b f(t)\, g(t)\, d(h(t)). \tag{2.14}$$

Taking the modulus in (2.14) and using the property (2.13) we get the inequality

$$\left| f(b)\, g(b)\, h(b) - f(a)\, g(a)\, h(a) - \int_a^b h(t)\, d(f(t)\, g(t)) \right| \le M \sup_{t\in[a,b]} |f(t)\, g(t)|.$$

Theorem 3. *Assume that $f, g : [a, b] \to \mathbb{R}$ are monotonic nondecreasing on $[a, b]$ and such that the Riemann-Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists, and $h : [a, b] \to \mathbb{C}$ is continuous on $[a, b]$. Then we have the inequality*

$$\left| \int_a^b h(t)\, d(f(t)\, g(t)) \right| \le \int_a^b |f(t)\, h(t)|\, dg(t) + \int_a^b |g(t)\, h(t)|\, df(t). \tag{2.15}$$

The inequality (2.15) *is sharp.*

Proof. It is well known that if $p : [a, b] \to \mathbb{C}$ is continuous and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b p(t)\, dv(t)$ exists and

$$\left| \int_a^b p(t)\, dv(t) \right| \le \int_a^b |p(t)|\, dv(t). \tag{2.16}$$

Taking the modulus in (2.1) we have

$$\begin{aligned} \left| \int_a^b h(t)\, d(f(t)\, g(t)) \right| &\le \left| \int_a^b f(t)\, h(t)\, dg(t) \right| + \left| \int_a^b g(t)\, h(t)\, df(t) \right| \\ &\le \int_a^b |f(t)\, h(t)|\, dg(t) + \int_a^b |g(t)\, h(t)|\, df(t). \end{aligned}$$

Consider the functions $f, g : [a, b] \to \mathbb{R}$ defined by

$$f(t) := \begin{cases} 0 & \text{if } t = a \\ 1 & \text{if } t \in (a, b], \end{cases} \quad g(t) := \begin{cases} -1 & \text{if } t \in [a, b) \\ 0 & \text{if } t = b. \end{cases}$$

The functions f and g are monotonic nondecreasing. We will show that the Riemann-Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists.

Take the sequence of divisions and intermediate points

$$d_n : a = x_0^{(n)} < \xi_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < \xi_{n-1}^{(n)} < x_n^{(n)} = b$$

such that $\Delta(d_n) := \max_{i\in\{0,\dots,n-1\}} \left\{ x_{i+1}^{(n)} - x_i^{(n)} \right\} \to 0$ as $n \to \infty$.

By the definition of the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ we have

$$\begin{aligned}\int_a^b f(t)\,dg(t) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right] \\ &= \lim_{n\to\infty} \sum_{i=0}^{n-2} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right] \\ &\quad + \lim_{n\to\infty} f\left(\xi_{n-1}^{(n)}\right)\left[g(b) - g\left(x_{n-1}^{(n)}\right)\right] \\ &= 0 + 1 = 1.\end{aligned}$$

Now, define the function $\ell : [a,b] \to \mathbb{R}$ by

$$\ell(t) := f(t)\,g(t) = \begin{cases} 0 & \text{if } t = a \\ -1 & \text{if } t \in (a,b) \\ 0 & \text{if } t = a. \end{cases}$$

For a continuous function $h : [a,b] \to \mathbb{R}$, since ℓ is of bounded variation, then the integral $\int_a^b h(t)\,d\ell(t)$ exists.

Take the sequence of divisions and intermediate points

$$d_n : a = x_0^{(n)} < \xi_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < \xi_{n-1}^{(n)} < x_n^{(n)} = b$$

such that $\Delta(d_n) := \max_{i\in\{0,\dots,n-1\}} \left\{ x_{i+1}^{(n)} - x_i^{(n)} \right\} \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned}\int_a^b h(t)\,d\ell(t) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} h\left(\xi_i^{(n)}\right)\left[\ell\left(x_{i+1}^{(n)}\right) - \ell\left(x_i^{(n)}\right)\right] \\ &= \lim_{n\to\infty} h\left(\xi_0^{(n)}\right)\left[\ell\left(x_1^{(n)}\right) - \ell(a)\right] \\ &\quad + \lim_{n\to\infty} \sum_{i=1}^{n-2} h\left(\xi_i^{(n)}\right)\left[\ell\left(x_{i+1}^{(n)}\right) - \ell\left(x_i^{(n)}\right)\right] \\ &\quad + \lim_{n\to\infty} h\left(\xi_{n-1}^{(n)}\right)\left[\ell(b) - \ell\left(x_{n-1}^{(n)}\right)\right] \\ &= \lim_{n\to\infty} h\left(\xi_0^{(n)}\right)(-1-0) + 0 + \lim_{n\to\infty} h\left(\xi_{n-1}^{(n)}\right)[0-(-1)] \\ &= h(b) - h(a).\end{aligned}$$

Consider the functions $u, v : [a,b] \to \mathbb{R}$ given by

$$u(t) := |f(t)\,h(t)| = \begin{cases} 0 & \text{if } t = a \\ |h(t)| & \text{if } t \in (a,b], \end{cases}$$

and

$$v(t) := |g(t)h(t)| = \begin{cases} |h(t)| & \text{if } t \in [a,b) \\ 0 & \text{if } t = b. \end{cases}$$

Take the sequence of divisions and intermediate points

$$d_n : a = x_0^{(n)} < \xi_0^{(n)} < x_1^{(n)} < \cdots < x_{n-1}^{(n)} < \xi_{n-1}^{(n)} < x_n^{(n)} = b$$

such that $\Delta(d_n) := \max_{i\in\{0,\dots,n-1\}} \left\{x_{i+1}^{(n)} - x_i^{(n)}\right\} \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned}\int_a^b u(t)\,dg(t) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} u\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right) - g\left(x_i^{(n)}\right)\right] \\ &= \lim_{n\to\infty} u\left(\xi_{n-1}^{(n)}\right)\left[g(b) - g\left(x_{n-1}^{(n)}\right)\right] \\ &= \lim_{n\to\infty} \left|h\left(\xi_{n-1}^{(n)}\right)\right| [0-(-1)] = |h(b)|\end{aligned}$$

and

$$\begin{aligned}\int_a^b v(t)\,df(t) &= \lim_{n\to\infty} \sum_{i=0}^{n-1} v\left(\xi_i^{(n)}\right)\left[f\left(x_{i+1}^{(n)}\right) - f\left(x_i^{(n)}\right)\right] \\ &= \lim_{n\to\infty} v\left(\xi_0^{(n)}\right)\left[f\left(x_1^{(n)}\right) - f(a)\right] \\ &= \lim_{n\to\infty} \left|h\left(\xi_0^{(n)}\right)\right| (1-0) = |h(a)|.\end{aligned}$$

Replacing these values in (2.15) we have

$$|h(b) - h(a)| \le |h(b)| + |h(a)|. \tag{2.17}$$

This inequality reduces to an equality if we choose a continuous function $h : [a,b] \to \mathbb{R}$ such that $h(b) = -h(a)$. For instance, for $h : [a,b] \to \mathbb{R}$, $h(t) = t - \frac{a+b}{2}$, we get in both sides of (2.17) the same quantity $b - a$. □

Remark 4. We remark that in the dual case, namely when the functions $f, g : [a,b] \to \mathbb{C}$ are continuous and $h : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b h(t)\,d(f(t)g(t))$ exists and integrating by parts we have the equality (2.14). Taking the modulus in this equality and using the property (2.16) we get

$$\begin{aligned}&\left|f(b)g(b)h(b) - f(a)g(a)h(a) - \int_a^b h(t)\,d(f(t)g(t))\right| \\ &\quad\le \int_a^b |f(t)g(t)|\,d(h(t)).\end{aligned} \tag{2.18}$$

We observe that in our Theorems 1, 2 and 3 we assumed that the factors of the integrator, namely f and g have the same properties. The reader can also consider the other cases when one has a property and the other has a different property, for example f is of bounded variation and g is Lipschitzian with a constant L. In this case, if h is continuous, then we have

$$\begin{aligned} &\left|\int_a^b h(t)\, d(f(t)\, g(t))\right| \\ &\leq \left|\int_a^b h(t)\, f(t)\, d(g(t))\right| + \left|\int_a^b h(t)\, g(t)\, d(f(t))\right| \\ &\leq L\left|\int_a^b h(t)\, f(t)\, dt\right| + \sup_{t\in[a,b]} |h(t)\, g(t)| \bigvee_a^b (f). \end{aligned} \tag{2.19}$$

The details in the other cases are omitted.

3. Applications for trapezoid type inequalities

The following result holds:

Proposition 1. *Let $f, g : [a,b] \to \mathbb{C}$ be two functions of bounded variation and such that the Riemann-Stieltjes integral $\int_a^b f(t)\, dg(t)$ exists. Then for any $x \in [a,b]$ we have*

$$\begin{aligned} &\left|f(b)\, g(b)\,(b-x) + f(a)\, g(a)\,(x-a) - \int_a^b f(t)\, g(t)\, dt\right| \\ &\leq \sup_{t\in[a,b]} |(t-x)\, g(t)| \bigvee_a^b (f) + \sup_{t\in[a,b]} |(t-x)\, f(t)| \bigvee_a^b (g) \\ &\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[\|g\|_\infty \bigvee_a^b (f) + \|f\|_\infty \bigvee_a^b (g)\right]. \end{aligned} \tag{3.1}$$

In particular, we have

$$\begin{aligned} &\left|\frac{f(b)\, g(b) + f(a)\, g(a)}{2}(b-a) - \int_a^b f(t)\, g(t)\, dt\right| \\ &\leq \sup_{t\in[a,b]} \left|\left(t - \frac{a+b}{2}\right) g(t)\right| \bigvee_a^b (f) + \sup_{t\in[a,b]} \left|\left(t - \frac{a+b}{2}\right) f(t)\right| \bigvee_a^b (g) \\ &\leq \frac{1}{2}(b-a)\left[\|g\|_\infty \bigvee_a^b (f) + \|f\|_\infty \bigvee_a^b (g)\right]. \end{aligned} \tag{3.2}$$

The inequalities (3.2) *are sharp.*

Proof. We use the following identity

$$F(b)\,(b-x) + F(a)\,(x-a) - \int_a^b F(t)\, dt = \int_a^b (t-x)\, dF(t) \tag{3.3}$$

that holds for any function of bounded variation $F:[a,b]\to\mathbb{C}$ and any $x\in[a,b]$.

If we write the equality (3.3) for $F=fg$ we get

$$f(b)\,g(b)\,(b-x)+f(a)\,g(a)\,(x-a)-\int_a^b f(t)\,g(t)\,dt = \int_a^b (t-x)\,d\,(f(t)\,g(t)) \tag{3.4}$$

for any $x\in[a,b]$.

If we use Theorem 1 for the function $h(t)=t-x,\ t\in[a,b]$, then we have the inequality

$$\begin{aligned}\left|\int_a^b (t-x)\,d\,(f(t)\,g(t))\right| &\le \sup_{t\in[a,b]}|(t-x)\,g(t)|\bigvee_a^b(f)+\sup_{t\in[a,b]}|(t-x)\,g(f)|\bigvee_a^b(g)\\ &\le \sup_{t\in[a,b]}|t-x|\left[\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)\right]\\ &= \max\{x-a,b-x\}\left[\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)\right]\\ &= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)\right]\end{aligned} \tag{3.5}$$

and the desired result (3.1) is obtained.

The inequality (3.2) follows from (3.1) for $x=\frac{a+b}{2}$.

Let us prove the sharpness of the inequalities (3.2).

Consider the functions $f,g:[a,b]\to\mathbb{R}$ defined by

$$f(t):=\begin{cases}0 & \text{if } t=a\\ 1 & \text{if } t\in(a,b],\end{cases}\qquad g(t):=\begin{cases}1 & \text{if } t\in[a,b)\\ 0 & \text{if } t=b.\end{cases}$$

We observe that f and g are of bounded variation and

$$\bigvee_a^b(f)=\bigvee_a^b(g)=1.$$

Take the sequence of divisions and intermediate points

$$d_n : a=x_0^{(n)}<\xi_0^{(n)}<x_1^{(n)}<\cdots<x_{n-1}^{(n)}<\xi_{n-1}^{(n)}<x_n^{(n)}=b$$

such that $\Delta(d_n):=\max_{i\in\{0,\dots,n-1\}}\left\{x_{i+1}^{(n)}-x_i^{(n)}\right\}\to 0$ as $n\to\infty$.

By the definition of the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ we have

$$\begin{aligned}\int_a^b f(t)\,dg(t) &= \lim_{n\to\infty}\sum_{i=0}^{n-1} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right)-g\left(x_i^{(n)}\right)\right]\\ &= \lim_{n\to\infty}\sum_{i=0}^{n-2} f\left(\xi_i^{(n)}\right)\left[g\left(x_{i+1}^{(n)}\right)-g\left(x_i^{(n)}\right)\right]\\ &\quad+\lim_{n\to\infty} f\left(\xi_{n-1}^{(n)}\right)\left[g(b)-g\left(x_{n-1}^{(n)}\right)\right]\\ &= 0-1=-1,\end{aligned}$$

which shows that this integral exists.

Observe that

$$\left(t-\frac{a+b}{2}\right)f(t) := \begin{cases} 0 & \text{if } t=a \\ t-\frac{a+b}{2} & \text{if } t\in(a,b], \end{cases}$$

$$\left(t-\frac{a+b}{2}\right)g(t) := \begin{cases} t-\frac{a+b}{2} & \text{if } t\in[a,b) \\ 0 & \text{if } t=b. \end{cases}$$

Then

$$\sup_{t\in[a,b]}\left|\left(t-\frac{a+b}{2}\right)g(t)\right| = \frac{b-a}{2}$$

and

$$\sup_{t\in[a,b]}\left|\left(t-\frac{a+b}{2}\right)f(t)\right| = \frac{b-a}{2}.$$

We also have

$$\frac{f(b)\,g(b)+f(a)\,g(a)}{2}(b-a)-\int_a^b f(t)\,g(t)\,dt = -(b-a).$$

Now, if we replace these values in (3.2) then we get in all terms the same quantity $b-a$. □

In a similar way we can prove the following results as well:

Proposition 2. *Let $f:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $L>0$ and $g:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $K>0$. Then for any $x\in[a,b]$ we have*

$$\begin{aligned}(3.6)\qquad &\left|f(b)\,g(b)(b-x)+f(a)\,g(a)(x-a)-\int_a^b f(t)\,g(t)\,dt\right|\\ &\le L\int_a^b |t-x|\,|g(t)|\,dt + K\int_a^b |t-x|\,|f(t)|\,dt\\ &\le \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left(L\|g\|_\infty + K\|f\|_\infty\right).\end{aligned}$$

In particular, we have

$$\left|\frac{f(b)g(b)+f(a)g(a)}{2}(b-a)-\int_a^b f(t)g(t)\,dt\right| \tag{3.7}$$
$$\leq L\int_a^b \left|t-\frac{a+b}{2}\right| |g(t)|\,dt + K\int_a^b \left|t-\frac{a+b}{2}\right| |f(t)|\,dt$$
$$\leq \frac{1}{4}(b-a)^2\left(L\|g\|_\infty + K\|f\|_\infty\right).$$

We also have:

Proposition 3. *Assume that $f,g:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on $[a,b]$ and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,dg(t)$ exists. Then for any $x\in[a,b]$ we have*

$$\left|f(b)g(b)(b-x)+f(a)g(a)(x-a)-\int_a^b f(t)g(t)\,dt\right| \tag{3.8}$$
$$\leq \int_a^b |t-x|\,|g(t)|\,df(t)+\int_a^b |t-x|\,|f(t)|\,dg(t)$$
$$\leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left(\int_a^b |g(t)|\,df(t)+\int_a^b |f(t)|\,dg(t)\right).$$

In particular, we have

$$\left|\frac{f(b)g(b)+f(a)g(a)}{2}(b-a)-\int_a^b f(t)g(t)\,dt\right| \tag{3.9}$$
$$\leq \int_a^b \left|t-\frac{a+b}{2}\right| |g(t)|\,df(t)+\int_a^b \left|t-\frac{a+b}{2}\right| |f(t)|\,dg(t)$$
$$\leq \frac{1}{2}(b-a)\left(\int_a^b |g(t)|\,df(t)+\int_a^b |f(t)|\,dg(t)\right).$$

4. Applications for Ostrowski type inequalities

The following result holds:

Proposition 4. *Let $f,g:[a,b]\to\mathbb{C}$ be two functions of bounded variation and such that for $x\in(a,b)$ the Riemann-Stieltjes integrals $\int_a^x f(t)\,dg(t)$ and $\int_x^b f(t)\,dg(t)$ exist. Then we have*

$$\left|f(x)g(x)(b-a)-\int_a^b f(t)g(t)\,dt\right| \tag{4.1}$$
$$\leq (x-a)\sup_{t\in[a,x]}\{|f(t)|\}\bigvee_a^x(g)+(x-a)\sup_{t\in[a,x]}\{|g(t)|\}\bigvee_a^x(f)$$

$$+ (b-x) \sup_{t\in[x,b]} \{|f(t)|\} \bigvee_x^b (g) + (b-x) \sup_{t\in[x,b]} \{|g(t)|\} \bigvee_x^b (f)$$

$$\leq \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \left[\|g\|_\infty \bigvee_a^b (f) + \|f\|_\infty \bigvee_a^b (g)\right].$$

In particular, if the Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} f(t)dg(t)$ *and* $\int_{\frac{a+b}{2}}^b f(t)dg(t)$ *exist, then we have*

$$\left| f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)\, g(t)\, dt \right| \tag{4.2}$$

$$\leq \frac{b-a}{2} \left[\sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|f(t)|\} \bigvee_a^{\frac{a+b}{2}} (g) + \sup_{t\in\left[a,\frac{a+b}{2}\right]} \{|g(t)|\} \bigvee_a^{\frac{a+b}{2}} (f) \right.$$

$$\left. \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|f(t)|\} \bigvee_{\frac{a+b}{2}}^b (g) + \sup_{t\in\left[\frac{a+b}{2},b\right]} \{|g(t)|\} \bigvee_{\frac{a+b}{2}}^b (f) \right]$$

$$\leq \frac{1}{2}(b-a) \left[\|g\|_\infty \bigvee_a^b (f) + \|f\|_\infty \bigvee_a^b (g)\right].$$

The inequalities in (4.2) *are sharp.*

Proof. We use the following identity (see for instance [23])

$$F(x)(b-a) - \int_a^b F(t)\, dt = \int_a^x (t-a)\, dF(t) + \int_x^b (t-b)\, dF(t) \tag{4.3}$$

that holds for any function of bounded variation $F : [a,b] \to \mathbb{C}$ and any $x \in [a,b]$.

If we write the equality (4.3) for $F = fg$ we get

$$f(x)\, g(x)\, (b-a) - \int_a^b f(t)\, g(t)\, dt \tag{4.4}$$

$$= \int_a^x (t-a)\, d(f(t)\, g(t)) + \int_x^b (t-b)\, d(f(t)\, g(t))$$

for any functions $f, g : [a,b] \to \mathbb{C}$ of bounded variation and any $x \in [a,b]$.

Taking the modulus on (4.4) and utilizing Theorem 1 on the intervals $[a,x]$ and $[x,b]$ we have successively that

$$\left| f(x)\, g(x)\, (b-a) - \int_a^b f(t)\, g(t)\, dt \right| \tag{4.5}$$

$$\leq \left| \int_a^x (t-a)\, d(f(t)\, g(t)) \right| + \left| \int_x^b (t-b)\, d(f(t)\, g(t)) \right|$$

$$
\begin{aligned}
&\leq \sup_{t\in[a,x]}\{(t-a)|f(t)|\}\bigvee_a^x(g)+\sup_{t\in[a,x]}\{(t-a)|g(t)|\}\bigvee_a^x(f)\\
&\quad+\sup_{t\in[x,b]}\{(b-t)|f(t)|\}\bigvee_x^b(g)+\sup_{t\in[x,b]}\{(b-t)|g(t)|\}\bigvee_x^b(f)\\
&\leq (x-a)\sup_{t\in[a,x]}\{|f(t)|\}\bigvee_a^x(g)+(x-a)\sup_{t\in[a,x]}\{|g(t)|\}\bigvee_a^x(f)\\
&\quad+(b-x)\sup_{t\in[x,b]}\{|f(t)|\}\bigvee_x^b(g)+(b-x)\sup_{t\in[x,b]}\{|g(t)|\}\bigvee_x^b(f)\\
&\leq \max\{x-a,b-x\}\sup_{t\in[a,b]}\{|f(t)|\}\bigvee_a^b(g)\\
&\quad+\max\{x-a,b-x\}\sup_{t\in[a,b]}\{|g(t)|\}\bigvee_a^b(f)\\
&= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)\right],
\end{aligned}
$$

which proves the desired result (4.1).

The inequality (4.2) is obvious from (4.1).

Consider now the functions $f,g:[a,b]\to\mathbb{R}$ defined by

$$
f(t):=\begin{cases}0 & \text{if } t\in\left[a,\frac{a+b}{2}\right)\\ 1 & \text{if } t\in\left[\frac{a+b}{2},b\right]\end{cases}\qquad g(t):=\begin{cases}1 & \text{if } t\in\left[a,\frac{a+b}{2}\right]\\ 0 & \text{if } t\in\left(\frac{a+b}{2},b\right].\end{cases}
$$

We observe that f and g are of bounded variation and

$$
\bigvee_a^b(f)=\bigvee_a^b(g)=1.
$$

The Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}}f(t)\,dg(t)$ and $\int_{\frac{a+b}{2}}^b f(t)\,dg(t)$ exist since one function is continuous while the other is of bounded variation on those intervals.

We observe that for these functions we have

$$
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)\,g(t)\,dt=b-a,
$$

$$
\sup_{t\in\left[a,\frac{a+b}{2}\right]}\{|f(t)|\}\bigvee_a^{\frac{a+b}{2}}(g)+\sup_{t\in\left[a,\frac{a+b}{2}\right]}\{|g(t)|\}\bigvee_a^{\frac{a+b}{2}}(f)
$$

$$+\sup_{t\in\left[\frac{a+b}{2},b\right]}\{|f(t)|\}\bigvee_{\frac{a+b}{2}}^{b}(g)+\sup_{t\in\left[\frac{a+b}{2},b\right]}\{|g(t)|\}\bigvee_{\frac{a+b}{2}}^{b}(f)=2$$

and

$$\|g\|_\infty\bigvee_a^b(f)+\|f\|_\infty\bigvee_a^b(g)=2.$$

Replacing these values in (4.2) we obtain in all terms the same quantity $b-a$, which proves the sharpness of the inequalities. □

In a similar way we can prove the following results as well:

Proposition 5. *Let $f:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $L>0$ and $g:[a,b]\to\mathbb{C}$ be Lipschitzian with the constant $K>0$. Then for any $x\in[a,b]$ we have*

$$\left|f(x)g(x)(b-a)-\int_a^b f(t)g(t)\,dt\right| \tag{4.6}$$
$$\leq L\left(\int_a^x (t-a)|g(t)|\,dt+\int_x^b (b-t)|g(t)|\,dt\right)$$
$$+K\left(\int_a^x (t-a)|f(t)|\,dt+\int_x^b (b-t)|f(t)|\,dt\right)$$
$$\leq\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right](L\|g\|_\infty+K\|f\|_\infty).$$

In particular, we have

$$\left|f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)g(t)\,dt\right| \tag{4.7}$$
$$\leq L\left(\int_a^{\frac{a+b}{2}}(t-a)|g(t)|\,dt+\int_{\frac{a+b}{2}}^b(b-t)|g(t)|\,dt\right)$$
$$+K\left(\int_a^{\frac{a+b}{2}}(t-a)|f(t)|\,dt+\int_{\frac{a+b}{2}}^b(b-t)|f(t)|\,dt\right)$$
$$\leq\frac{1}{4}(b-a)^2(L\|g\|_\infty+K\|f\|_\infty).$$

We also have:

Proposition 6. *Assume that $f,g:[a,b]\to\mathbb{R}$ are monotonic nondecreasing on $[a,b]$ and such that for $x\in(a,b)$ the Riemann-Stieltjes integrals $\int_a^x f(t)\,dg(t)$*

and $\int_x^b f(t)\,dg(t)$ *exist. Then we have*

$$\begin{aligned}(4.8)\qquad & \left| f(x)\,g(x)\,(b-a) - \int_a^b f(t)\,g(t)\,dt \right| \\ & \le \int_a^x (t-a)\,|g(t)|\,df(t) + \int_x^b (b-t)\,|g(t)|\,df(t) \\ & \quad + \int_a^x (t-a)\,|f(t)|\,dg(t) + \int_x^b (b-t)\,|f(t)|\,dg(t) \\ & \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right|\right] \left(\int_a^b |g(t)|\,df(t) + \int_a^b |f(t)|\,dg(t) \right).\end{aligned}$$

In particular, if the Riemann-Stieltjes integrals $\int_a^{\frac{a+b}{2}} f(t)dg(t)$ *and* $\int_{\frac{a+b}{2}}^b f(t)dg(t)$ *exist, then we have*

$$\begin{aligned}(4.9)\qquad & \left| f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)\,g(t)\,dt \right| \\ & \le \int_a^{\frac{a+b}{2}} (t-a)\,|g(t)|\,df(t) + \int_{\frac{a+b}{2}}^b (b-t)\,|g(t)|\,df(t) \\ & \quad + \int_a^{\frac{a+b}{2}} (t-a)\,|f(t)|\,dg(t) + \int_{\frac{a+b}{2}}^b (b-t)\,|f(t)|\,dg(t) \\ & \le \frac{1}{2}(b-a) \left(\int_a^b |g(t)|\,df(t) + \int_a^b |f(t)|\,dg(t) \right).\end{aligned}$$

5. Applications for selfadjoint operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \qquad (5.1)$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [33, p. 256]:

Theorem 4 (Spectral Representation Theorem). *Let A be a bounded selfadjoint operator on the Hilbert space H and let* $m = \min\{\lambda \mid \lambda \in Sp(A)\} =:$

$\min Sp(A)$ *and* $M = \max\{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. *Then there exists a family of projections* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, *called the spectral family of* A, *with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{m-0}^{M} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ *defined on* $\mathbb{R}$ *and for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{5.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 2. *With the assumptions of Theorem 4 for* A, E_λ *and* φ *we have the representations*

$$\varphi(A)x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{5.3}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

Proposition 7. *Let* A *be a bounded selfadjoint operator on the Hilbert space* H *and let* $m = \min\{\lambda \mid \lambda \in Sp(A)\} =: \min Sp(A)$ *and* $M = \max\{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. *Assume that* $f, g : \mathbb{R} \to \mathbb{C}$ *are continuous.*

(i) *If* f,g *are of bounded variation on* $[m,M]$, *then*

$$|\langle [f(M)g(M)I - f(A)g(A)]x,y\rangle| \tag{5.4}$$
$$\le \sup_{t\in[m,M]}|\langle E_t x,y\rangle|\left[\sup_{t\in[m,M]}|f(t)|\bigvee_m^M(g)+\sup_{t\in[m,M]}|f(g)|\bigvee_m^M(f)\right]$$
$$\le \|x\|\,\|y\|\left[\sup_{t\in[m,M]}|f(t)|\bigvee_m^M(g)+\sup_{t\in[m,M]}|f(g)|\bigvee_m^M(f)\right]$$

for any $x,y\in H$.

(ii) *If* $f,g:\mathbb{R}\to\mathbb{C}$ *are Lipschitzian with the constants* L, *respectively* K *on* $[m,M]$, *then*

$$|\langle [f(M)g(M)I - f(A)g(A)]x,y\rangle| \tag{5.5}$$
$$\le K\int_{m-0}^{M}|\langle E_t x,y\rangle f(t)|\,dt + L\int_{m-0}^{M}|\langle E_t x,y\rangle g(t)|\,dt$$
$$\le \sup_{t\in[m,M]}|\langle E_t x,y\rangle|\left[K\int_m^M|f(t)|\,dt+L\int_m^M|g(t)|\,dt\right]$$
$$\le \|x\|\,\|y\|\left[K\int_m^M|f(t)|\,dt+L\int_m^M|g(t)|\,dt\right]$$

for any $x,y\in H$.

(iii) *If* $f,g:\mathbb{R}\to\mathbb{C}$ *are monotonic nondecreasing on* $[m,M]$, *then*

$$|\langle [f(M)g(M)I - f(A)g(A)]x,y\rangle| \tag{5.6}$$
$$\le \int_{m-0}^{M}|\langle E_t x,y\rangle f(t)|\,dg(t) + \int_{m-0}^{M}|\langle E_t x,y\rangle g(t)|\,df(t)$$
$$\le \sup_{t\in[m,M]}|\langle E_t x,y\rangle|\left[\int_m^M|f(t)|\,dg(t)+\int_m^M|g(t)|\,df(t)\right]$$
$$\le \|x\|\,\|y\|\left[\int_m^M|f(t)|\,dg(t)+\int_m^M|g(t)|\,df(t)\right]$$

for any $x,y\in H$.

Proof. Let $\varepsilon>0$. We use the fact that (see Remark 1)

$$\int_{m-\varepsilon}^{M}\langle E_t x,y\rangle\,d(f(t)g(t)) \tag{5.7}$$
$$=\int_{m-\varepsilon}^{M}\langle E_t x,y\rangle f(t)\,d(g(t))+\int_{m-\varepsilon}^{M}\langle E_t x,y\rangle g(t)\,d(f(t))$$

for any $x,y\in H$, since $\langle E.x,y\rangle$ is of bounded variation on $[m-\varepsilon,M]$ while f, g are continuous on $[m-\varepsilon,M]$.

Integrating by parts in the Riemann-Stieltjes integral, we also have

$$\begin{aligned}(5.8)\qquad &\int_{m-\varepsilon}^{M}\langle E_t x,y\rangle\, d\left(f\left(t\right)g\left(t\right)\right)\\ &= \left.\langle E_t x,y\rangle\left(f\left(t\right)g\left(t\right)\right)\right|_{m-\varepsilon}^{M}-\int_{m-\varepsilon}^{M}f\left(t\right)g\left(t\right)d\left(\langle E_t x,y\rangle\right)\\ &= f\left(M\right)g\left(M\right)\langle x,y\rangle-\int_{m-\varepsilon}^{M}f\left(t\right)g\left(t\right)d\left(\langle E_t x,y\rangle\right)\end{aligned}$$

for any $x, y \in H$.

From (5.7) and (5.8) we then have

$$\begin{aligned}(5.9)\qquad &f\left(M\right)g\left(M\right)\langle x,y\rangle-\int_{m-\varepsilon}^{M}f\left(t\right)g\left(t\right)d\left(\langle E_t x,y\rangle\right)\\ &= \int_{m-\varepsilon}^{M}\langle E_t x,y\rangle f\left(t\right)d\left(g\left(t\right)\right)+\int_{m-\varepsilon}^{M}\langle E_t x,y\rangle g\left(t\right)d\left(f\left(t\right)\right)\end{aligned}$$

for any $x, y \in H$ and $\varepsilon > 0$.

(i). If f and g are of bounded variation, then by (5.9) we have

$$\begin{aligned}(5.10)\qquad &\left|f\left(M\right)g\left(M\right)\langle x,y\rangle-\int_{m-\varepsilon}^{M}f\left(t\right)g\left(t\right)d\left(\langle E_t x,y\rangle\right)\right|\\ &\le \left|\int_{m-\varepsilon}^{M}\langle E_t x,y\rangle f\left(t\right)d\left(g\left(t\right)\right)\right|+\left|\int_{m-\varepsilon}^{M}\langle E_t x,y\rangle g\left(t\right)d\left(f\left(t\right)\right)\right|\\ &\le \sup_{t\in[m-\varepsilon,M]}\left|\langle E_t x,y\rangle f\left(t\right)\right|\bigvee_{m-\varepsilon}^{M}\left(g\right)+\sup_{t\in[m-\varepsilon,M]}\left|\langle E_t x,y\rangle g\left(t\right)\right|\bigvee_{m-\varepsilon}^{M}\left(f\right)\\ &\le \sup_{t\in[m-\varepsilon,M]}\left|\langle E_t x,y\rangle\right|\sup_{t\in[m-\varepsilon,M]}\left|f\left(t\right)\right|\bigvee_{m-\varepsilon}^{M}\left(g\right)\\ &\quad+\sup_{t\in[m-\varepsilon,M]}\left|\langle E_t x,y\rangle\right|\sup_{t\in[m-\varepsilon,M]}\left|g\left(t\right)\right|\bigvee_{m-\varepsilon}^{M}\left(f\right)\end{aligned}$$

for any $x, y \in H$ and $\varepsilon > 0$.

Taking the limit over $\varepsilon \to 0+$ in (5.10) and utilizing the representation (5.3) we deduce (5.4).

The statements (ii) and (iii) can be proved in a similar way, however the details are omitted. □

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