J. Korean Math. Soc. **51** (2014), No. 4, pp. 773–789
http://dx.doi.org/10.4134/JKMS.2014.51.4.773

LINEAR OPERATORS THAT PRESERVE SETS OF PRIMITIVE MATRICES

LeRoy B. Beasley, Kyung-Tae Kang, and Seok-Zun Song

Abstract. We consider linear operators on square matrices over antinegative semirings. Let $\mathcal{E}_k$ denote the set of all primitive matrices of exponent k. We characterize those linear operators which preserve the set $\mathcal{E}_1$ and the set $\mathcal{E}_2$, and those that preserve the set $\mathcal{E}_{n^2-2n+2}$ and the set $\mathcal{E}_{n^2-2n+1}$. We also characterize those linear operators that strongly preserve $\mathcal{E}_2$, $\mathcal{E}_{n^2-2n+2}$ or $\mathcal{E}_{n^2-2n+1}$.

1. Introduction

The characterization of linear operators on vector space of matrices which leave functions, sets or relations invariant began over a century ago when in 1897 Fröbenius [9] characterized the linear operators that leave the determinant function invariant. Since then, several researchers have investigated the preservers of nearly every function, set and relation on matrices over fields. See [12, 14] for an excellent survey of preserver problems through 2001.

In the 1980's research began on linear preserver problems over semirings, in particular linear operators on spaces of $(0, 1)$-matrices. (See for example [3].) Many functions, sets and relations concerning matrices do not depend upon the magnitude or nature of the individual entries of a matrix, but rather only on whether the entry is zero or nonzero. These combinatorially significant matrices have become increasingly important in recent years. Primitive matrices are an example of this type of combinatorially significant set of matrices.

To begin the investigation concerning the present article, we define an antinegative semiring.

Let $\mathbb{S}$ be a commutative semiring, that is: $\mathbb{S}$ is a set with two binary operations, addition $(+)$ and multiplication $(\cdot)$; there is a zero element (for addition) and an identity element (for multiplication) in $\mathbb{S}$; $(\mathbb{S}, +)$ is closed, commutative

Received November 20, 2013; Revised March 10, 2014.
2010 *Mathematics Subject Classification.* Primary 15A04, 15A86, 15B34.
Key words and phrases. Linear operator, primitive matrix, line matrix, double star matrix.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(No. 2012R1A1A2042193).

and associative, but may not have additive inverses, except for the zero; $(\mathbb{S}, \cdot)$ is closed, associative and commutative, but may not have multiplicative inverses, except for the identity; and the distributive laws hold. We only consider semirings which have no zero divisors, that is nonzero elements, s, for which there is some nonzero element, t in $\mathbb{S}$ such that $st = 0$. Of particular interest in this article, is the binary Boolean algebra, $\mathbb{B} = \{0, 1\}$ with the usual addition and multiplication, except that $1 + 1 = 1$. A semiring is *antinegative* if the only element with an additive inverse is the zero element.

In this article we only consider commutative antinegative semirings with no zero divisors, the reason is that primitivity is not well defined for other semirings. Antinegative semirings of special interest to us include $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ (the nonegative integers, rationals and reals, resp.), the fuzzy semiring $\mathbb{F} = [0, 1]$ with maximum for addition and minimum for multiplication, and chain semirings, including the binary Boolean algebra $\mathbb{B}$.

Let $\mathcal{M}_n(\mathbb{S})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{S}$. Let $\mathcal{M}_n^{(0)}(\mathbb{S})$ denote those members of $\mathcal{M}_n(\mathbb{S})$ with all zeros on the main diagonal. Let $\boldsymbol{I}_n$ denote the identity matrix, $\boldsymbol{J}_n$ denote the matrix of all ones and $\boldsymbol{O}_n$ denote the zero matrix. Further, let $\boldsymbol{K}_n = \boldsymbol{J}_n \setminus \boldsymbol{I}_n$, the $n \times n$ matrix with all diagonal entries zero and all off diagonal entries one. The subscripts are usually omitted as the order is usually obvious from the context, and we write $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{O}$ and $\boldsymbol{K}$. Let $E_{i,j}$ denote the matrix in $\mathcal{M}_n(\mathbb{S})$ that has only one nonzero entry, that being a "1" in the (i, j) location. Such a matrix is called a *cell.*

A matrix $L \in \mathcal{M}_n(\mathbb{S})$ is called a *line matrix* if $L = \sum_{l=1}^{n} E_{i,l}$ or $L = \sum_{s=1}^{n} E_{s,j}$ for some $i \in \{1, \ldots, n\}$ or for some $j \in \{1, \ldots, n\}$; $R_i = \sum_{l=1}^{n} E_{i,l}$ is the ith *row matrix* and $C_j = \sum_{s=1}^{n} E_{s,j}$ is the jth *column matrix.* A matrix in $\mathcal{M}_n(\mathbb{S})$ is a *double star* if it is a sum of a row matrix and a column matrix which share a diagonal entry. That is, $D_k = R_k + C_k$ is a double star for all $k = 1, \ldots, n$.

Suppose that $A, B \in \mathcal{M}_n(\mathbb{S})$. We say that A *dominates* B (written $A \sqsupseteq B$ or $B \sqsubseteq A$) if $a_{i,j} = 0$ implies that $b_{i,j} = 0$ for all i and j. If $A \sqsupseteq B$ we write $A \setminus B = C$ to denote the matrix with $c_{i,j} = a_{i,j}$ if $b_{i,j} = 0$ and $c_{i,j} = 0$ if $b_{i,j} \neq 0$. Let $A \circ B$ denote the *Hadamard* product of A and B, that is $A \circ B = [a_{i,j} b_{i,j}]$. Note that if $X \in \mathcal{M}_n(\mathbb{S})$ then $X \circ \boldsymbol{K} \in \mathcal{M}_n^{(0)}(\mathbb{S})$.

The matrix $A \in \mathcal{M}_n(\mathbb{S})$ is said to be *primitive* if A^l has all nonzero entries for some positive integer l. A primitive matrix A is said to have *exponent* k if A^k has all nonzero entries and A^s has a zero entry if $s < k$. That is, k is the minimum exponent of A that produces a strictly nonzero matrix, denoted $\mathbf{exp}(A)$. For notational convenience, we say that the exponent of a non-primitive matrix is zero. Let $\mathcal{E}_k = \{A \in \mathcal{M}_n(\mathbb{S}) : \mathbf{exp}(A) = k\}$. So $\mathcal{E}_0$ is the set of non-primitive matrices, and further, $\mathcal{E}_1 = \{A \in \mathcal{M}_n(\mathbb{S}) : \boldsymbol{J} \sqsubseteq A\}$.

Primitive matrices appear in many physical models. The study of the exponents of primitive matrices began in 1950 when Wielandt published the first upper bound on the exponent of a primitive matrix [15]. The study of exponents of primitive matrices has been an active area of research by several authors, including Dulmadge and Mendelsohn [8], Lewin and Vitek [11], Holladay and R. S. Varga [10], and more recently by Liu [13], to name only a few.

Let $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be an operator. Then, T is said to be *linear* if $T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$ for all $A, B \in \mathcal{M}_n(\mathbb{S})$ and $\alpha, \beta \in \mathbb{S}$. For permutation matrices, P and Q in $\mathcal{M}_n(\mathbb{S})$, T is said to be a (P, Q)-*operator* if $T(X) = PXQ$ for all X, or $T(X) = PX^tQ$ for all X where X^t denotes the transpose of X. Further, T is said to be a (P, P^t)-*operator* if $T(X) = PXP^t$ for all X, or $T(X) = PX^tP^t$ for all X. A linear operator T is *singular* if there is some nonzero X such that $T(X) = \boldsymbol{O}$. Otherwise, T is *nonsingular*. Note that over antinegative semirings, a linear operator being nonsingular is not equivalent to being invertible, for example if $A \in \mathcal{M}_n(\mathbb{B})$, $A \neq \boldsymbol{O}$, and $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ is a linear operator such that $T(X) = A$ for all nonzero X, then T is nonsingular, but obviously not invertible.

Let Φ be a subset of $\mathcal{M}_n(\mathbb{S})$. We say that T *preserves* Φ if $X \in \Phi$ implies that $T(X) \in \Phi$. Further, T *strongly preserves* Φ if, $X \in \Phi$ if and only if $T(X) \in \Phi$.

In 1989 Beasley and Pullman [5] characterized the linear operators that strongly preserve the set of primitive matrices. They also characterized the linear operators that preserve the index of imprimitivity [6]. Recently Beasley and Guterman have characterized the linear operators that preserve sets of tuples that generalize the concept of primitivity, [1, 2]. In this article we investigate preservers of certain subsets of primitive matrices defined by their exponent. In particular, we shall characterize linear operators that preserve $\mathcal{E}_1$ and $\mathcal{E}_2$, and those that preserve $\mathcal{E}_{n^2-2n+2}$ and $\mathcal{E}_{n^2-2n+1}$. We also characterize those linear operators that strongly preserve $\mathcal{E}_2$, $\mathcal{E}_{n^2-2n+2}$, or $\mathcal{E}_{n^2-2n+1}$.

Let $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{S})$. The *support* of A, $\overline{A} = [\overline{a}_{i,j}]$, is the element of $\mathcal{M}_n(\mathbb{B})$ whose entries equal to 1 are in precisely the locations that A has nonzero entries. That is $\overline{a}_{i,j} = 1$ if and only if $a_{i,j} \neq 0$ for all i and j. Since $\mathbb{S}$ is antinegative, a matrix being primitive does not depend on the nature of the nonzero entries of A. Thus, A is primitive in $\mathcal{M}_n(\mathbb{S})$ if and only if $\overline{A}$ is primitive in $\mathcal{M}_n(\mathbb{B})$. For a linear operator $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ we define the corresponding operator $\overline{T} : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ by $\overline{T}(\overline{X}) = \overline{T(X)}$ for all $X \in \mathcal{M}_n(\mathbb{S})$. Let $\overline{\mathcal{E}_k}$ be the subset of $\mathcal{M}_n(\mathbb{B})$ of primitive matrices of exponent k. Then, T (strongly) preserves $\mathcal{E}_k$ if and only if $\overline{T}$ (strongly) preserves $\overline{\mathcal{E}_k}$. For that reason, we restrict our attention in the next section to linear operators on $\mathcal{M}_n(\mathbb{B})$.

2. Exponent preservers on $\boldsymbol{\mathcal{M}_n}(\mathbb{B})$

In this section, we consider only linear operators on $\mathcal{M}_n(\mathbb{B})$ or $\mathcal{M}_n^{(0)}(\mathbb{B})$, and will not use the notation $\overline{A}$ since for A in $\mathcal{M}_n(\mathbb{B})$ or $\mathcal{M}_n^{(0)}(\mathbb{B})$, $A = \overline{A}$.

Further, we assume that $n \geq 3$ since the main theorem is false when $n = 2$ as the following example shows.

Example 2.1. Let $T : \mathcal{M}_2(\mathbb{B}) \to \mathcal{M}_2(\mathbb{B})$ be defined by $T(E_{1,1}) = E_{1,1} + E_{1,2}, T(E_{1,2}) = O, T(E_{2,1}) = E_{2,1}$ and $T(E_{2,2}) = E_{2,2} + E_{1,2}$. Then T preserves $\mathcal{E}_1$ and $\mathcal{E}_2$, but clearly T is singular, and hence, not a (P, P^t)-operator.

Note that the lemma below may be found in previous literature (see for example [4, Theorem 5.3]) in different formats. The proofs are included here for completeness.

Lemma 2.2. *Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a bijective linear operator. Then T maps line matrices to line matrices if and only if T is a (P, Q)-operator.*

Proof. Since T is bijective, $T(\boldsymbol{J}_n) = \boldsymbol{J}_n$. Let T map line matrices to line matrices. Now we claim that either

(1) T maps $\{R_1, \ldots, R_n\}$ onto $\{R_1, \ldots, R_n\}$ and maps $\{C_1, \ldots, C_n\}$ onto $\{C_1, \ldots, C_n\}$, or
(2) T maps $\{R_1, \ldots, R_n\}$ onto $\{C_1, \ldots, C_n\}$ and maps $\{C_1, \ldots, C_n\}$ onto $\{R_1, \ldots, R_n\}$.

Suppose that the claim is not true. Then there are two distinct row matrices R_i and R_j (or column matrices C_i and C_j) such that $T(R_i)$ is a row matrix and $T(R_j)$ is a column matrix. But then $T(\boldsymbol{J}_n) = T(R_1) + \cdots + T(R_i) + \cdots + T(R_j) + \cdots + T(R_n)$ cannot dominate $\boldsymbol{J}_n$. This contradicts $T(\boldsymbol{J}_n) = \boldsymbol{J}_n$. Hence the claim is true.

Case (1): We note that $T(R_i) = R_{\alpha(i)}$ for all i and $T(C_j) = C_{\beta(j)}$ for all j, where α and β are permutations of $\{1, \ldots, n\}$. Then for any cell $E_{i,j}$, we have $T(E_{i,j}) = E_{\alpha(i),\beta(j)}$. Let P and Q be the permutation matrices corresponding to α and β, respectively. Then for any matrix $X = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} E_{i,j} \in \mathcal{M}_n(\mathbb{B})$, we have

$$T(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} E_{\alpha(i),\beta(j)} = PXQ.$$

Hence T is a (P, Q)-operator.

Case (2): $T(R_i) = C_{\alpha(i)}$ for all i and $T(C_j) = R_{\beta(j)}$ for all j, where α and β are permutations of $\{1, \ldots, n\}$. By a parallel argument similar to Case (1), we obtain that $T(X)$ is of the form $T(X) = PX^tQ$, and thus T is a (P, Q)-operator.

The converse is obvious. □

Lemma 2.3. *Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a bijective linear operator. Then, T maps line matrices to line matrices and double stars to double stars if and only if T is a (P, P^t)-operator.*

Proof. Suppose that T maps line matrices to line matrices and double stars to double stars. By Lemma 2.2, T is a (P, Q)-operator. The permutations α and β defined in the proof of Lemma 2.2 must be equal since T maps double stars to double stars. Thus $Q = P^t$, and hence T is a (P, P^t)-operator.

The converse is obvious. $\square$

For a line matrix L, we say that $L^{(0)} = L \setminus E$ is an *off diagonal line matrix*, where E is a diagonal cell with $E \sqsubseteq L$. Further, $D_k^{(0)} = D_k \setminus E_{k,k}$ is an *off diagonal double star* for $k = 1, \ldots, n$.

Noting that a (P, P^t)-operator maps $\mathcal{M}_n^{(0)}(\mathbb{B})$ to $\mathcal{M}_n^{(0)}(\mathbb{B})$, the off diagonal version of Lemma 2.2 is the following:

Lemma 2.4. *Let $T : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a bijective linear operator. Then, T maps off diagonal line matrices to off diagonal line matrices if and only if T is a (P, P^t)-operator.*

Proof. Suppose that T maps off diagonal line matrices to off diagonal line matrices. By Lemma 2.2, T is a (P, Q)-operator. If T does not map an off diagonal double star $D_k^{(0)}$ to an off diagonal double star, then there are distinct indices i and j such that $T(D_k^{(0)}) = R_i^{(0)} + C_j^{(0)}$. This contradicts the fact that T is bijective since $D_k^{(0)}$ is the sum of $2n - 2$ cells and $R_i^{(0)} + C_j^{(0)}$ is the sum of $2n - 3$ cells. Thus, T maps off diagonal double stars to off diagonal double stars.

Now, by defining $T' : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ by $T'(E_{i,i}) = E_{k,k}$ if $T(D_i^{(0)}) = D_k^{(0)}$, we have that T' preserves line matrices and double stars and hence, by Lemma 2.3, T' is a (P, P^t)-operator. By considering the mapping $X \to P(X \circ \boldsymbol{K})P^t$, we can see that T is a (P, P^t)-operator.

The converse is obvious. $\square$

Theorem 2.5. *Let $n \geq 3$ and $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. Then, T preserves $\mathcal{E}_1$ and $\mathcal{E}_2$ if and only if T is a (P, P^t)-operator.*

Proof. If T is a (P, P^t)-operator, clearly T preserves $\mathcal{E}_1$ and $\mathcal{E}_2$. Conversely assume that T preserves $\mathcal{E}_1$ and $\mathcal{E}_2$. Then $T(\boldsymbol{J}) = \boldsymbol{J}$. If $T(X) = \boldsymbol{O}$ for some nonzero X then there is a cell $E_{i,j}$ such that $T(E_{i,j}) = \boldsymbol{O}$. But then, $T(\boldsymbol{J}) = T(\boldsymbol{J} \setminus E_{i,j})$, a contradiction since for $n \geq 3$, $\boldsymbol{J} \setminus E_{i,j} \in \mathcal{E}_2$ while $\boldsymbol{J} \in \mathcal{E}_1$. Thus, T is nonsingular.

Suppose that the image of some cell, E, is not a cell. Then, there are at most $n^2 - 2$ cells, $E_1, \ldots, E_{n^2-2}$, such that $T(E) + T(E_1) + \cdots + T(E_{n^2-2}) = \boldsymbol{J}$, but for $n \geq 3$, $E + E_1 + \cdots + E_{n^2-2} \in \mathcal{E}_2$ while $\boldsymbol{J} \in \mathcal{E}_1$, a contradiction. Thus, the image of any cell is a cell. Further, since $T(\boldsymbol{J}) = \boldsymbol{J}$, it follows that T is bijective on the set of cells.

It is easily shown that any matrix with exponent 2 with the minimum number of nonzero entries is a double star, $D_k = [d_{i,j}]$, where $d_{i,j} = 1$ if and only if $i = k$ or $j = k$. Thus, T maps double stars to double stars. Suppose that T

does not preserve line matrices. Then without loss of generality, we may assume that the image of R_1 is not a line matrix. But, by permuting, $T(D_1) = D_1$. That is, $T(R_1 + C_1) = R_1 + C_1$, so that the preimage of R_1, is a sum of some cells dominated by R_1 and some cells dominated by C_1, other than $E_{1,1}$. By permuting, we may assume that the preimage of $U = \begin{bmatrix} 1 & \mathbf{0}_{n-1}^t \\ \boldsymbol{j}_{n-1} & \boldsymbol{J}_{n-1} \end{bmatrix}$ is

$$V = \begin{bmatrix} 1 & \boldsymbol{j}_u^t & \mathbf{0}_{n-u-1}^t \\ \mathbf{0}_u & \boldsymbol{J}_u & \boldsymbol{J}_{u,n-u-1} \\ \boldsymbol{j}_{n-u-1} & \boldsymbol{J}_{n-u-1,u} & \boldsymbol{J}_{n-u-1,n-u-1} \end{bmatrix},$$

where $\boldsymbol{J}_{\alpha,\beta}$ is the $\alpha \times \beta$ matrix of all ones, and $\boldsymbol{j}_\gamma$ is the column vector of all ones of size γ. But $V \in \mathcal{E}_2$ while $U \in \mathcal{E}_0$, a contradiction. Thus, line matrices are mapped into line matrices and double stars into double stars. By Lemma 2.3, T is a (P, P^t)-operator. □

It is known that if $A \in \mathcal{M}_n(\mathbb{B})$ is primitive, then $\mathbf{exp}(A) \le n^2 - 2n + 2$, and if $\mathbf{exp}(A) = n^2 - 2n + 2$, then A is permutationally equivalent to $W_n = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} + E_{n,1} + E_{n-1,1}$, the Wielandt matrix. See [15] or [7, Pages 82 & 83]. From [15], we also know that for $n \ge 4$ the only matrices whose exponents are $n^2 - 2n + 1$ are permutationally equivalent to $W_n' = E_{1,2} + E_{2,3} + \cdots + E_{n-1,n} + E_{n,1} + E_{n-1,1} + E_{n,2}$ so that $W_n' = W_n + E_{n,2}$. Thus, for $n \ge 3$, if a matrix A in $\mathcal{M}_n(\mathbb{B})$ dominates a diagonal cell, then $A \notin \mathcal{E}_{n^2-2n+2}$. If $n \ge 4$ and a matrix A in $\mathcal{M}_n(\mathbb{B})$ dominates a diagonal cell, then $A \notin \mathcal{E}_{n^2-2n+1}$. This shows the following:

Lemma 2.6. *Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator and $n \ge 3$. If T preserves $\mathcal{E}_{n^2-2n+2}$, or if $n \ge 4$ and T preserves $\mathcal{E}_{n^2-2n+1}$, then $T(\mathcal{M}_n^{(0)}(\mathbb{B})) \subseteq \mathcal{M}_n^{(0)}(\mathbb{B})$.*

Lemma 2.7. *Let $n \ge 3$ and $T : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a linear operator. Then T preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$ if and only if T is a (P, P^t)-operator.*

Proof. Assume that T preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$. Suppose that there is some off diagonal cell, E, such that $T(E) = \boldsymbol{O}$. By permuting we may assume that $E = E_{n,2}$. Then, $W_n + E = W_n'$. But then, $T(W_n') = T(W_n + E) = T(W_n)$, a contradiction since $W_n \in \mathcal{E}_{n^2-2n+2}$ while $W_n' \in \mathcal{E}_{n^2-2n+1}$. Hence T is nonsingular. Now suppose that the image of an off diagonal cell, E, is not an off diagonal cell. Then, $T(E)$ must dominate at least two off diagonal cells.

Let $L = T^d$ where d is chosen so that L is idempotent. Since $T(E)$ dominates two cells, so does $L(E)$. Then, we may assume that $L(E) = F + X$ where F is a cell with $F \ne E$. If E and F are collinear, we may permute and/or transpose if necessary, so that $E = E_{n,1}$ and $F = E_{n,2}$. If E and F are not collinear, we may permute and/or transpose if necessary, so that $E = E_{n-1,1}$ and $F = E_{n,2}$. Then, in either case, $W_n \sqsupseteq E$ and $W_n + F = W_n'$. But then $L(W_n) = L(W_n + E) = L(W_n) + L(E) = L(W_n) + F + X$. Since L is idempotent, we have that

$$\begin{aligned} L(W_n) &= L^2(W_n) = L(L(W_n) + F + X) \\ &= L(L(W_n) + F + X + F) \\ &= L(L(W_n) + F) = L(W_n) + L(F) \\ &= L(W_n + F) = L(W_n'), \end{aligned}$$

a contradiction since T, and hence L, preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$. Thus, T maps off diagonal cells to off diagonal cells.

If T is not bijective on the off diagonal cells then, say $T(E) = T(F)$ for some off diagonal cells E and F. But, by permuting, we may assume that $W_n' \sqsupseteq E + F$, so that, as above, $W_n \sqsupseteq E$ and $W_n' = W_n + F$. But then, $T(W_n') = T(W_n + F) = T(W_n) + T(F) = T(W_n) + T(E) = T(W_n + E) = T(W_n)$, a contradiction. Thus, T is bijective on the set of off diagonal cells.

Now, suppose that T does not preserve off diagonal line matrices. Then, since T is bijective on the off diagonal cells, there are two noncollinear cells, E and F, whose images are collinear. By permuting, we may assume that $W_n \setminus E_{n-1,1} \sqsupseteq E+F$, $T(W_n) = W_n$, and $T(W_n \setminus E_{n-1,1}) \sqsupseteq E_{n-1,1}$. But there are at least n off diagonal cells G such that $W_n \setminus E_{n-1,1} + G$ has exponent n^2-2n+2 while there is only one off diagonal cell such that $T(W_n \setminus E_{n-1,1}) + G$ has exponent $n^2 - 2n + 2$, a contradiction since T is bijective on the set of off diagonal cells. Thus, T maps off diagonal line matrices to off diagonal line matrices. By Lemma 2.4, T is a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$.

The converse is obvious. $\square$

Definition 2.8. Let Δ_n denote the subset of $\mathcal{M}_n(\mathbb{S})$ consisting of the diagonal matrices. That is, $\Delta_n = \{A \in \mathcal{M}_n(\mathbb{S}) : A \sqsubseteq \boldsymbol{I}_n\}$. A *diagonal transformation* is any linear mapping $R : \Delta_n \to \mathcal{M}_n(\mathbb{S})$.

Lemma 2.9. *Let $n \geq 3$, $T_o : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a linear operator and $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ be any diagonal transformation. Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator defined by $T(X) = T_o(X \circ \boldsymbol{K}) + R(X \circ \boldsymbol{I})$ for all $X \in \mathcal{M}_n(\mathbb{B})$. Then, if T_o preserves $\mathcal{E}_{n^2-2n+2}$ (resp., $\mathcal{E}_{n^2-2n+1}$, $n \geq 4$), then T does.*

Proof. Notice that any matrix in $\mathcal{E}_{n^2-2n+2}$ cannot dominate a diagonal cell. Now, suppose that T_o preserves $\mathcal{E}_{n^2-2n+2}$, and let $A \in \mathcal{M}_n(\mathbb{B})$ be any matrix in $\mathcal{E}_{n^2-2n+2}$. Then $A \circ \boldsymbol{I} = \boldsymbol{O}$ and so $A \in \mathcal{M}_n^{(0)}(\mathbb{B})$. Thus $T_o(A) \in \mathcal{E}_{n^2-2n+2}$. It follows from $R(A \circ \boldsymbol{I}) = R(\boldsymbol{O}) = \boldsymbol{O}$ that $T(A) = T_o(A \circ \boldsymbol{K}) + R(A \circ \boldsymbol{I}) = T_o(A) + \boldsymbol{O} = T_o(A)$. Thus, $T(A) \in \mathcal{E}_{n^2-2n+2}$. That is, T preserves $\mathcal{E}_{n^2-2n+2}$.

The case of $\mathcal{E}_{n^2-2n+1}$ with $n \geq 4$ is parallel. $\square$

Theorem 2.10. *Let $n \geq 3$ and $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. Then, T preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$ if and only if T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a diagonal transformation $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$. That is, $T(X) = P(X \circ \boldsymbol{K})P^t + R(X \circ \boldsymbol{I})$ for all X, or $T(X) = P(X \circ \boldsymbol{K})^t P^t + R(X \circ \boldsymbol{I})$ for all X.*

Proof. Suppose that T preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$. By Lemma 2.6, we can define a linear operator $T_o : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ by $T_o(Y) = T(Y)$ for all $Y \in \mathcal{M}_n^{(0)}(\mathbb{B})$. Clearly T_o preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$. Hence T_o is a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ by Lemma 2.7. Define a diagonal transformation $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ by $R(Z) = T(Z)$ for all $Z \in \Delta_n$. Then we note that for any $X \in \mathcal{M}_n(\mathbb{B})$, letting $Y = X \circ \boldsymbol{K} \in \mathcal{M}_n^{(0)}(\mathbb{B})$ and $Z = X \circ \boldsymbol{I} \in \Delta_n$, we have that $X = Y + Z$ and $T(X) = T(Y) + T(Z)$. Then, for all $X \in \mathcal{M}_n(\mathbb{B})$,

$$T(X) = T(X \circ \boldsymbol{K}) + T(X \circ \boldsymbol{I}) = T_o(X \circ \boldsymbol{K}) + R(X \circ \boldsymbol{I}).$$

Hence T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a diagonal transformation R.

The converse is obvious by Lemmas 2.7 and 2.9. □

Remark 2.11. Suppose that $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ is a linear operator that preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$. Then T may not preserve $\mathcal{E}_2$. For example, let $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ be a diagonal transformation defined by $R(A) = \boldsymbol{I}$ for all nonzero $A \in \Delta_n$. Define a linear operator $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ by $T(X) = (X \circ \boldsymbol{K}) + R(X \circ \boldsymbol{I})$ for all X. By Theorem 2.10, T preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$. But

$$T(\boldsymbol{J} \setminus E_{1,1}) = (\boldsymbol{K}) + \boldsymbol{I} = \boldsymbol{J},$$

and so T does not preserve $\mathcal{E}_2$ since $\mathbf{exp}(\boldsymbol{J} \setminus E_{1,1}) = 2$ while $\mathbf{exp}(\boldsymbol{J}) = 1$.

3. Strong preservers of exponents on $\mathcal{M}_n(\mathbb{B})$

Theorem 3.1. *Let $n \geq 2$ and $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. Then T strongly preserves $\mathcal{E}_2$ if and only if T is a (P, P^t)-operator.*

Proof. Let $n = 2$. Note that if $A \in \mathcal{E}_2$, then A is the sum of 3 cells and any sum of 2 cells is not primitive.

Suppose T strongly preserves $\mathcal{E}_2$. If $T(X) = \boldsymbol{O}$ for some nonzero X, then there is a cell, E, such that $T(E) = \boldsymbol{O}$. In this case there are two cells, F and G such that $E + F + G \in \mathcal{E}_2$, but then $T(F + G) = T(E + F + G) \in \mathcal{E}_2$, a contradiction. Thus, T is nonsingular.

Now suppose that the image of a cell is not a cell. Then there is a cell E such that $T(E)$ dominates at least two cells. Then there are cells F and G such that $E + F + G \in \mathcal{E}_2$. But then, either $T(E + F)$ or $T(E + G)$ dominates 3 cells, and hence equals $T(E + F + G)$ and so is in $\mathcal{E}_2$, a contradiction.

Suppose that $T(E) = T(F)$ for some distinct cells E and F. Let G and H be cells such that $E + F + G + H = J_2$. Now, at least one of $E + G + H$, $F + G + H$, $E + F + G$, or $E + F + H$ is in $\mathcal{E}_2$. If $E + G + H \in \mathcal{E}_2$, then $T(J_2) \in \mathcal{E}_2$, a contradiction; if $F+G+H \in \mathcal{E}_2$, then $T(J_2) \in \mathcal{E}_2$, a contradiction; if $E + F + G \in \mathcal{E}_2$, then $T(E + G) = T(E + F + G) \in \mathcal{E}_2$, a contradiction; and if $E + F + H \in \mathcal{E}_2$, then $T(E + H) = T(E + F + H) \in \mathcal{E}_2$, a contradiction. In any case we have a contradiction and hence, T is bijective.

Since T is bijective, and T strongly preserves $\mathcal{E}_2$, T preserves $\mathcal{E}_1$. Since the two double stars are the only members of $\mathcal{E}_2$, T preserves double stars. Also, it is easy to show that T preserves lines. By Lemma 2.3, T is a (P, P^t)-operator.

Now, let $n \geq 3$ and suppose T strongly preserves $\mathcal{E}_2$. If $T(X) = \boldsymbol{O}$ for some nonzero X, then there is a cell, E, such that $T(E) = \boldsymbol{O}$. In this case $T(\boldsymbol{J} \setminus E) = T(\boldsymbol{J})$. But $\boldsymbol{J} \notin \mathcal{E}_2$ while $\boldsymbol{J} \setminus E \in \mathcal{E}_2$, contradicting that T strongly preserves $\mathcal{E}_2$. Thus, T is nonsingular.

Now suppose that the image of a cell is not a cell. Then there is a cell E such that $T(E)$ dominates at least two cells. Then, there are at most $n^2 - 1$ cells whose image dominates $T(\boldsymbol{J})$. That is, there is a cell F such that $T(\boldsymbol{J} \setminus F) = T(\boldsymbol{J})$, again a contradiction. That is, the image of a cell is a cell.

Let E and F be distinct cells. Suppose that $T(E) = T(F)$, Then $T(\boldsymbol{J}) = T(\boldsymbol{J} \setminus (E + F)) + T(E) + T(F)$. So, $T(\boldsymbol{J} \setminus E) = T(\boldsymbol{J})$, again a contradiction. Thus, T is bijective on the set of cells, and hence, $T(\boldsymbol{J}) = \boldsymbol{J}$. That is, T preserves $\mathcal{E}_1$. By Theorem 2.5, T is a (P, P^t)-operator.

The converse is obvious. □

Definition 3.2. We define $\mathcal{W}_n$ to be the set of all matrices dominated by a matrix of exponent $n^2 - 2n + 2$, that is, $\mathcal{W}_n = \{A \in \mathcal{M}_n(\mathbb{B}) : A \sqsubseteq B \text{ for some } B \in \mathcal{E}_{n^2-2n+2}\}$. Similarly we define $\mathcal{W}'_n$ to be the set of all matrices dominated by a matrix of exponent $n^2 - 2n + 1$, that is, $\mathcal{W}'_n = \{A \in \mathcal{M}_n(\mathbb{B}) : A \sqsubseteq B \text{ for some } B \in \mathcal{E}_{n^2-2n+1}\}$.

Remark 3.3. If $\mathcal{X}$ is a subset of $\mathcal{M}_n(\mathbb{B})$, let $\mathcal{X}^c$ denote the complement of $\mathcal{X}$, $\mathcal{X}^c = \mathcal{M}_n(\mathbb{B}) \setminus \mathcal{X}$. Notice that $\Delta_n \subseteq (\mathcal{W}_n)^c$ and $\Delta_n \subseteq (\mathcal{W}'_n)^c$.

Lemma 3.4. *Let $n \geq 3$, $T_o : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a linear operator and $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ be any diagonal transformation such that $R(\Delta_n) \subseteq (\mathcal{W}_n)^c$ (resp., $(\mathcal{W}'_n)^c$). Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator defined by $T(X) = T_o(X \circ \boldsymbol{K}) + R(X \circ \boldsymbol{I})$ for all $X \in \mathcal{M}_n(\mathbb{B})$. Then, if T_o strongly preserves $\mathcal{E}_{n^2-2n+2}$ (resp., $\mathcal{E}_{n^2-2n+1}$), then T does.*

Proof. Suppose that T_o strongly preserves $\mathcal{E}_{n^2-2n+2}$ and R is a diagonal transformation with $R(\Delta_n) \subseteq (\mathcal{W}_n)^c$. Then T preserves $\mathcal{E}_{n^2-2n+2}$ by Lemma 2.9. Suppose that $T(X) \in \mathcal{E}_{n^2-2n+2}$ for some $X \in \mathcal{M}_n(\mathbb{B})$. Then $T(X) = T_o(X \circ \boldsymbol{K}) + R(X \circ \boldsymbol{I}) \in \mathcal{E}_{n^2-2n+2}$ and hence $R(X \circ \boldsymbol{I}) \in \mathcal{W}_n$. But it follows from $R(\Delta_n) \subseteq (\mathcal{W}_n)^c$ that $R(X \circ \boldsymbol{I})$ is a member of the empty set. Hence X cannot dominate a diagonal cell. But then $T(X) = T_o(X \circ \boldsymbol{K}) = T_o(X)$. Since T_o strongly preserves $\mathcal{E}_{n^2-2n+2}$, we have $X \in \mathcal{E}_{n^2-2n+2}$. Hence T strongly preserves $\mathcal{E}_{n^2-2n+2}$.

The case for $\mathcal{E}_{n^2-2n+1}$ and $R(\Delta_n) \subseteq (\mathcal{W}'_n)^c$ is parallel. □

Lemma 3.5. *Let $n \geq 3$ and $T : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ be a linear operator. If T strongly preserves $\mathcal{E}_{n^2-2n+2}$ or $\mathcal{E}_{n^2-2n+1}$, then T is bijective.*

Proof. Assume that T strongly preserves $\mathcal{E}_{n^2-2n+2}$. If $T(X) = \boldsymbol{O}$ for some nonzero X, then $T(E) = \boldsymbol{O}$ for some off diagonal cell E. Now choose a member

U of $\mathcal{E}_{n^2-2n+2}$ that dominates E. But then $T(U \setminus E) = T(U)$, while $U \setminus E \notin \mathcal{E}_{n^2-2n+2}$, a contradiction since T strongly preserves $\mathcal{E}_{n^2-2n+2}$. Hence T is nonsingular.

Suppose that the image of an off diagonal cell, E, is not an off diagonal cell, say $T(E) \sqsupseteq F + G$ for some off diagonal cells F and G. Since every member of $\mathcal{E}_{n^2-2n+2}$ has exactly $n+1$ nonzero entries, there is some member U of $\mathcal{E}_{n^2-2n+2}$ that dominates E. But then $T(U \setminus H) = T(U)$, for some cell H dominated by U. But $U \setminus H \notin \mathcal{E}_{n^2-2n+2}$, a contradiction since T strongly preserves $\mathcal{E}_{n^2-2n+2}$. Thus, T maps off diagonal cells to off diagonal cells.

Suppose that $T(E) = T(F)$ for distinct off diagonal cells E and F. Extend $E + F$ to a member U of $\mathcal{E}_{n^2-2n+2}$. But then $T(U \setminus F) = T(U)$, again a contradiction. Hence T is bijective.

By similar argument as above, if T strongly preserves $\mathcal{E}_{n^2-2n+1}$, then T is bijective. □

Example 3.6. Let $A \in \mathcal{M}_3(\mathbb{B})$ be a matrix of exponent $n^2 - 2n + 2 = 5$. Then A is permutationally equivalent to $W_3 = \begin{bmatrix} 0&1&0 \\ 1&0&1 \\ 1&0&0 \end{bmatrix}$. Notice that there are only six matrices of exponent 5 in $\mathcal{M}_3(\mathbb{B})$ and they are the following: $\begin{bmatrix} 0&1&0 \\ 1&0&1 \\ 1&0&0 \end{bmatrix}$, $\begin{bmatrix} 0&0&1 \\ 1&0&0 \\ 1&1&0 \end{bmatrix}$, $\begin{bmatrix} 0&1&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix}$, $\begin{bmatrix} 0&1&1 \\ 0&0&1 \\ 1&0&0 \end{bmatrix}$, $\begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 1&1&0 \end{bmatrix}$, and $\begin{bmatrix} 0&0&1 \\ 1&0&1 \\ 0&1&0 \end{bmatrix}$.

Lemma 3.7. *If $T : \mathcal{M}_3^{(0)}(\mathbb{B}) \to \mathcal{M}_3^{(0)}(\mathbb{B})$ is a linear operator which strongly preserves $\mathcal{E}_5$, then T maps off diagonal line matrices to off diagonal line matrices.*

Proof. By Lemma 3.5, T is bijective. Suppose that T does not map off diagonal line matrices to off diagonal line matrices. Then, there is a pair of collinear off diagonal cells whose images are not collinear. By permuting and/or transposing, we may assume that $T(E_{1,2}) = E_{1,2}$ and, $T(E_{1,3}) = E_{2,1}$ or $T(E_{1,3}) = E_{2,3}$.

Notice that there are exactly two choices of 2 cells whose sum with $E_{1,2}+E_{1,3}$ (resp., $E_{1,2}+E_{2,1}$) is in $\mathcal{E}_5$. But there are exactly three choices of 2 cells whose sum with $E_{1,2} + E_{2,3}$ is in $\mathcal{E}_5$. Since T is bijective and strongly preserves $\mathcal{E}_5$, we must have $T(E_{1,2}) = E_{1,2}$ and $T(E_{1,3}) = E_{2,1}$. It follows from $\begin{bmatrix} 0&1&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix} \in \mathcal{E}_5$ that $T(E_{2,1} + E_{3,2}) = E_{2,3} + E_{3,1}$ or $T(E_{2,1} + E_{3,2}) = E_{1,3} + E_{3,2}$, say that $T(E_{2,1} + E_{3,2}) = E_{2,3} + E_{3,1}$. But then we must have $T(E_{2,3} + E_{3,1}) = E_{1,3} + E_{3,2}$. Further

$$T(W_3) = T(E_{1,2} + E_{2,1} + E_{2,3} + E_{3,1}) = E_{1,2} + T(E_{2,1}) + E_{1,3} + E_{3,2}$$

has exponent 5 so that $T(E_{2,1}) = E_{2,1}$, a contradiction since $T(E_{2,1} + E_{3,2}) = E_{2,3} + E_{3,1}$. Hence T maps off diagonal line matrices to off diagonal line matrices. □

Example 3.8. Let $A \in \mathcal{M}_4(\mathbb{B})$ be a matrix of exponent $n^2 - 2n + 2 = 10$. Then A is s permutationally equivalent to $W_4 = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&1 \\ 1&0&0&0 \end{bmatrix}$. Notice that there are only 24 matrices of exponent 10 in $\mathcal{M}_4(\mathbb{B})$. Furthermore, $\left[\begin{smallmatrix} 0&1&1&0\\ 0&0&0&1\\ 0&1&0&0\\ 1&0&0&0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0&1&1&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{smallmatrix}\right]$ are the only two matrices in $\mathcal{E}_{10}$ that dominate $E_{1,2}+E_{1,3}$. Similarly there are only four matrices in $\mathcal{E}_9$ that dominate $E_{1,2}+E_{1,3}$.

Lemma 3.9. *Let $T : \mathcal{M}_4^{(0)}(\mathbb{B}) \to \mathcal{M}_4^{(0)}(\mathbb{B})$ be a linear operator. If T strongly preserves $\mathcal{E}_{10}$ or $\mathcal{E}_9$, then T maps off diagonal line matrices to off diagonal line matrices.*

Proof. By Lemma 3.5, T is bijective. Suppose that T does not map off diagonal line matrices to off diagonal line matrices. Then, there is a pair of collinear off diagonal cells whose images are not collinear. By permuting and/or transposing, we may assume that there are three cases to consider:

(1) $T(E_{1,2}) = E_{1,2}$ and $T(E_{1,3}) = E_{2,1}$;
(2) $T(E_{1,2}) = E_{1,2}$ and $T(E_{1,3}) = E_{2,3}$; or
(3) $T(E_{1,2}) = E_{1,2}$ and $T(E_{1,3}) = E_{3,4}$.

First, suppose that T strongly preserves $\mathcal{E}_{10}$. Notice that there are exactly two choices of 3 cells whose sum together with $E_{1,2}+E_{1,3}$ is in $\mathcal{E}_{10}$. In case (1), there are not 3 cells whose sum with $E_{1,2}+E_{2,1}$ is in $\mathcal{E}_{10}$, a contradiction since T strongly preserves $\mathcal{E}_{10}$. In case (2), there are exactly six choices of 3 cells whose sum with $E_{1,2}+E_{2,3}$ is in $\mathcal{E}_{10}$, a contradiction since T is bijective and strongly preserves $\mathcal{E}_{10}$. In case (3), there are exactly four choices of 3 cells whose sum with $E_{1,2}+E_{3,4}$ is in $\mathcal{E}_{10}$, a contradiction since T is bijective and strongly preserves $\mathcal{E}_{10}$. Thus, T maps off diagonal line matrices to off diagonal line matrices.

Next, suppose that T strongly preserves $\mathcal{E}_9$. Notice that there are exactly four choices of 4 cells whose sum together with $E_{1,2}+E_{1,3}$ is in $\mathcal{E}_9$. In case (1), there are not 4 cells whose sum with $E_{1,2}+E_{2,1}$ is in $\mathcal{E}_9$, a contradiction since T strongly preserves $\mathcal{E}_9$. In case (2), there are exactly eight choices of 4 cells whose sum with $E_{1,2}+E_{2,3}$ is in $\mathcal{E}_9$, a contradiction since T is bijective and strongly preserves $\mathcal{E}_9$. In case (3), there are exactly six choices of 4 cells whose sum with $E_{1,2}+E_{3,4}$ is in $\mathcal{E}_9$, a contradiction since T is bijective and strongly preserves $\mathcal{E}_9$. Thus, T maps off diagonal line matrices to off diagonal line matrices. $\square$

Theorem 3.10. *Let $n \geq 3$ and $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator. Then T strongly preserves $\mathcal{E}_{n^2-2n+2}$ if and only if T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a diagonal transformation $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ such that $R(\Delta_n) \subseteq (\mathcal{W}_n)^c$. That is, $T(X) = P(X \circ \boldsymbol{K})P^t + R(X \circ \boldsymbol{I})$ for all X, or $T(X) = P(X \circ \boldsymbol{K})^t P^t + R(X \circ \boldsymbol{I})$ for all X.*

Proof. Suppose that T strongly preserves $\mathcal{E}_{n^2-2n+2}$. By Lemma 2.6, we can define a linear operator $T_o : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ by $T_o(Y) = T(Y)$ for all

$Y \in \mathcal{M}_n^{(0)}(\mathbb{B})$. Clearly T_o strongly preserves $\mathcal{E}_{n^2-2n+2}$ and hence T_o is bijective by Lemma 3.5.

Now we claim that T_o maps off diagonal line matrices to off diagonal line matrices. But, by Lemmas 3.7 and 3.9, we only consider the case of $n \geq 5$. If the claim is not true, then there are two collinear off diagonal cells whose images are not collinear. By permuting and/or transposing, we may assume that there are three cases to consider:

(1) $T_o(E_{1,2}) = E_{1,2}$ and $T_o(E_{1,3}) = E_{2,1}$;
(2) $T_o(E_{1,2}) = E_{1,2}$ and $T_o(E_{1,3}) = E_{2,3}$; or
(3) $T_o(E_{1,2}) = E_{1,2}$ and $T_o(E_{1,3}) = E_{3,4}$.

Notice that there are exactly $2\cdot(n-3)!$ choices of $n-1$ cells whose sum together with $E_{1,2} + E_{1,3}$ is in $\mathcal{E}_{n^2-2n+2}$. In case (1), there are not $n-1$ cells whose sum with $E_{1,2}+E_{2,1}$ is in $\mathcal{E}_{n^2-2n+2}$, a contradiction since T_o strongly preserves $\mathcal{E}_{n^2-2n+2}$. In case (2), there are $n\cdot(n-3)!$ choices of $n-1$ cells whose sum with $E_{1,2} + E_{2,3}$ is in $\mathcal{E}_{n^2-2n+2}$. This contradicts that T_o is bijective and strongly preserves $\mathcal{E}_{n^2-2n+2}$. In case (3), there are $(n-3)!$ choices of $n-1$ cells whose sum with $E_{1,2} + E_{3,4}$ is in $\mathcal{E}_{n^2-2n+2}$. This contradicts that T_o is bijective and strongly preserves $\mathcal{E}_{n^2-2n+2}$.

Thus, T_o maps off diagonal line matrices to off diagonal line matrices. By Lemma 2.4, T_o is a (P, P^t)-operator so that $T_o(Y) = PYP^t$ or $T_o(Y) = PY^tP^t$ for all $Y \in \mathcal{M}_n^{(0)}(\mathbb{B})$. Now, we define a diagonal transformation $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ by $R(D) = T(D)$ for all $D \in \Delta_n$. Then we have $T(X) = T_o(X \circ \boldsymbol{K}) + R(X \circ \boldsymbol{I})$ for all $X \in \mathcal{M}_n(\mathbb{B})$. Hence T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a diagonal transformation R.

Suppose that $R(D) \in \mathcal{W}_n$ for some nonzero $D \in \Delta_n$. Then there is a member U of $\mathcal{E}_{n^2-2n+2}$ such that $R(D) \sqsubseteq U$. For the case of $T_o(Y) = PYP^t$, we have that $T(P^tUP + D) = T_o(P^tUP) + R(D\circ \boldsymbol{I}) = P(P^tUP)P^t + R(D) = U + R(D) = U \in \mathcal{E}_{n^2-2n+2}$, a contradiction since $P^tUP + D \notin \mathcal{E}_{n^2-2n+2}$. Considering the matrix $P^tU^tP + D$, we also get a contradiction for the case of $T_o(Y) = PY^tP^t$. That is, R is a diagonal transformation such that $R(\Delta_n) \subseteq (\mathcal{W}_n)^c$.

The converse is obvious by Lemma 3.4. $\square$

Theorem 3.11. *Let $T : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ be a linear operator and $n \geq 4$. Then T strongly preserves $\mathcal{E}_{n^2-2n+1}$ if and only if T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a diagonal transformation $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ such that $R(\Delta_n) \subseteq (\mathcal{W}_n')^c$. That is, $T(X) = P(X\circ K)P^t + R(X \circ \boldsymbol{I})$ for all X, or $T(X) = P(X \circ \boldsymbol{K})^tP^t + R(X \circ \boldsymbol{I})$ for all X.*

Proof. Suppose that T strongly preserves $\mathcal{E}_{n^2-2n+1}$. By Lemma 2.6, we can define a linear operator $T_o : \mathcal{M}_n^{(0)}(\mathbb{B}) \to \mathcal{M}_n^{(0)}(\mathbb{B})$ by $T_o(Y) = T(Y)$ for all $Y \in \mathcal{M}_n^{(0)}(\mathbb{B})$. Clearly T_o strongly preserves $\mathcal{E}_{n^2-2n+1}$ and hence T_o is bijective by Lemma 3.5.

Now we claim that T_o maps off diagonal line matrices to off diagonal line matrices. But, by Lemma 3.9, we only consider the case of $n \geq 5$. If the claim is not true, then there are two collinear off diagonal cells whose images are not collinear. By permuting and/or transposing, we may assume that there are three cases to consider:

(1) $T_o(E_{1,2}) = E_{1,2}$ and $T_o(E_{1,3}) = E_{2,1}$;
(2) $T_o(E_{1,2}) = E_{1,2}$ and $T_o(E_{1,3}) = E_{2,3}$; or
(3) $T_o(E_{1,2}) = E_{1,2}$ and $T_o(E_{1,3}) = E_{3,4}$.

Notice that there are exactly $4 \cdot (n-3)!$ choices of n cells whose sum together with $E_{1,2} + E_{1,3}$ is in $\mathcal{E}_{n^2-2n+1}$. In case (1), there are not n cells whose sum with $E_{1,2} + E_{2,1}$ is in $\mathcal{E}_{n^2-2n+1}$, a contradiction since T_o strongly preserves $\mathcal{E}_{n^2-2n+1}$. In case (2), there are $n \cdot (n-3)!$ choices of n cells whose sum with $E_{1,2} + E_{2,3}$ is in $\mathcal{E}_{n^2-2n+1}$. Since $n \geq 5$, this contradicts that T_o is bijective and strongly preserves $\mathcal{E}_{n^2-2n+1}$. In case (3), there are $(n-3)!$ choices of n cells whose sum with $E_{1,2} + E_{3,4}$ is in $\mathcal{E}_{n^2-2n+1}$. This contradicts that T_o is bijective and strongly preserves $\mathcal{E}_{n^2-2n+1}$.

Thus, T_o maps off diagonal line matrices to off diagonal line matrices. By Lemma 2.4, T_o is a (P, P^t)-operator so that $T_o(Y) = PYP^t$ or $T_o(Y) = PY^tP^t$ for all $Y \in \mathcal{M}_n^{(0)}(\mathbb{B})$. Now, we define a diagonal transformation $R : \Delta_n \to \mathcal{M}_n(\mathbb{B})$ by $R(D) = T(D)$ for all $D \in \Delta_n$. Then we have $T(X) = T_o(X \circ \boldsymbol{K}) + R(X \circ \boldsymbol{I})$ for all $X \in \mathcal{M}_n(\mathbb{B})$. Hence T is the sum of a (P, P^t)-operator on $\mathcal{M}_n^{(0)}(\mathbb{B})$ and a diagonal transformation R.

Suppose that $R(D) \in \mathcal{W}'_n$ for some nonzero $D \in \Delta_n$. Then there is a member U of $\mathcal{E}_{n^2-2n+1}$ such that $R(D) \sqsubseteq U$. For the case of $T_o(Y) = PYP^t$, we have that $T(P^tUP + D) = T_o(P^tUP) + R(D \circ \boldsymbol{I}) = P(P^tUP)P^t + R(D) = U + R(D) = U \in \mathcal{E}_{n^2-2n+1}$, a contradiction since $P^tUP + D \notin \mathcal{E}_{n^2-2n+1}$. Considering the matrix $P^tU^tP + D$, we also get a contradiction for the case of $T_o(Y) = PY^tP^t$. That is, R is a diagonal transformation such that $R(\Delta_n) \subseteq (\mathcal{W}'_n)^c$.

The converse is obvious by Lemma 3.4. $\square$

Remark 3.12. If $n = 3$, Theorem 3.11 may be not true. See Theorem 3.15.

In $\mathcal{M}_3(\mathbb{B})$, let $C_3 = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{bmatrix}$ and $A \in \mathcal{M}_3(\mathbb{B})$ be a matrix of exponent $n^2 - 2n + 1 = 4$. Then A is permutationally equivalent to $W'_3 = \begin{bmatrix} 0&1&0 \\ 1&0&1 \\ 1&1&0 \end{bmatrix}$, $E_{i,i} + C_3$, or $E_{i,i} + C_3^t$ for some i. Further, we can easily check that every matrix in $\mathcal{M}_3(\mathbb{B})$ that is the sum of five off diagonal cells is of exponent 4, equivalently, $\boldsymbol{K} \setminus E \in \mathcal{E}_4$ for any off diagonal cell E.

Lemma 3.13. *If $T : \mathcal{M}_3(\mathbb{B}) \to \mathcal{M}_3(\mathbb{B})$ strongly preserves $\mathcal{E}_4$, then T is a bijective linear operator such that $T(\boldsymbol{I}) = \boldsymbol{I}$.*

Proof. Suppose that $T(X) = \boldsymbol{O}$ for some nonzero X. Then $T(E) = \boldsymbol{O}$ for some cell E. If $E = E_{i,i}$ then $T(E_{i,i} + C_3) = T(C_3)$, a contradiction since

$E_{i,i}+C_3 \in \mathcal{E}_4$ and $C_3 \notin \mathcal{E}_4$. If $E=E_{i,j}$ with $i \neq j$, then $T(\boldsymbol{K} \setminus E_{i,j}) = T(\boldsymbol{K})$, a contradiction since $\boldsymbol{K} \setminus E_{i,j} \in \mathcal{E}_4$ and $\boldsymbol{K} \notin \mathcal{E}_4$. Thus T is nonsingular.

Suppose that the image of an off diagonal cell dominates a diagonal cell, so that $T(E) \sqsupseteq E_{i,i}$ for some off diagonal cell E and for some i. Notice that $E_{i,i}+C_3$ and $E_{i,i}+C_3^t$ are only matrices in $\mathcal{E}_4$ dominate $E_{i,i}$. Let F, G and H be distinct off diagonal cells different from E. Then $T(\boldsymbol{K} \setminus F)$, $T(\boldsymbol{K} \setminus G)$ and $T(\boldsymbol{K} \setminus H)$ are matrices in $\mathcal{E}_4$ that dominate $E_{i,i}$. Hence by the above, we assume that $T(\boldsymbol{K} \setminus F) = T(\boldsymbol{K} \setminus G)$. But then $T(\boldsymbol{K}) = T((\boldsymbol{K} \setminus F)+(\boldsymbol{K} \setminus G)) = T(\boldsymbol{K} \setminus F)$, a contradiction since $\boldsymbol{K} \notin \mathcal{E}_4$ and $\boldsymbol{K} \setminus F \in \mathcal{E}_4$. Hence we have established that $T(\mathcal{M}_3^{(0)}(\mathbb{B})) \subseteq \mathcal{M}_3^{(0)}(\mathbb{B})$.

Suppose that the image of an off diagonal cell dominates two off diagonal cells. Then, $T(\boldsymbol{K} \setminus E) = T(\boldsymbol{K})$ for some off diagonal cell E, a contradiction. Thus, the image of an off diagonal cell is an off diagonal cell.

Suppose that $T(E) = T(F)$ for some distinct off diagonal cells E and F. Then, $T(\boldsymbol{K} \setminus E) = T((\boldsymbol{K} \setminus E)+F) = T(\boldsymbol{K} \setminus E)+T(F) = T(\boldsymbol{K} \setminus E)+T(E) = T(\boldsymbol{K})$, a contradiction. Thus, T is injective on $\mathcal{M}_3^{(0)}(\mathbb{B})$, and hence bijective on $\mathcal{M}_3^{(0)}(\mathbb{B})$ since $\mathcal{M}_3^{(0)}(\mathbb{B})$ is finite.

Suppose now that $T(E_{i,i}) \circ \boldsymbol{I} = \boldsymbol{O}$ for some i. Then $\boldsymbol{K} \sqsupseteq T(E_{i,i})$ so that there is a matrix $X \in \mathcal{M}_3^{(0)}(\mathbb{B})$ such that $T(E_{i,i}) = X$. If $X = \boldsymbol{K}$ then $T(E_{i,i}+C_3) = \boldsymbol{K}$, a contradiction. Thus $X \neq \boldsymbol{K}$ so that there are off diagonal cells E and F such that $T(E) = F \not\sqsubseteq X$. But then $T((\boldsymbol{K} \setminus E)+E_{i,i}) = (\boldsymbol{K} \setminus F)+X = \boldsymbol{K} \setminus F$, a contradiction since $\boldsymbol{K} \setminus F \in \mathcal{E}_4$ and $(\boldsymbol{K} \setminus E)+E_{i,i} \notin \mathcal{E}_4$. Thus we have that $T(E_{i,i}) \circ \boldsymbol{I} \neq \boldsymbol{O}$ for all i.

If $T(\boldsymbol{I}) \not\sqsubseteq \boldsymbol{I}$, then there is a diagonal cell $E_{i,i}$ such that $T(E_{i,i}) \sqsupseteq E_{u,v}$ for some off diagonal cell $E_{u,v}$. Since T is bijective on $\mathcal{M}_3^{(0)}(\mathbb{B})$, we assume that $T(C_3) \not\sqsupseteq E_{u,v}$. But then $T(E_{i,i}+C_3)$ must dominate a diagonal cell and at least four off diagonal cells, a contradiction since there is no such matrix in $\mathcal{E}_4$. Hence $T(\boldsymbol{I}) \sqsubseteq \boldsymbol{I}$.

For some i, suppose that $T(E_{i,i})$ dominates two diagonal cells. Then $T(E_{i,i}+C_3)$ dominates two diagonal cells, a contradiction. Thus $T(E_{i,i})$ is a diagonal cell for all i. If $T(E_{i,i}) = T(E_{j,j})$ and $i \neq j$ then $T(E_{j,j}+E_{i,i}+C_3) = T(E_{i,i}+C_3)$, a contradiction. Hence T is a bijection of Δ_3, and consequently, T is a bijection on $\mathcal{M}_3(\mathbb{B})$ and $T(\boldsymbol{I}) = \boldsymbol{I}$. $\square$

Definition 3.14. Let $\Gamma_1, \Gamma_2 : \mathcal{M}_3^{(0)}(\mathbb{B}) \to \mathcal{M}_3^{(0)}(\mathbb{B})$ denote the linear operators defined by

$$\Gamma_1\left(\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & c \\ b & 0 & d \\ e & f & 0 \end{bmatrix}$$

and

$$\Gamma_2\left(\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & f \\ b & 0 & e \\ d & c & 0 \end{bmatrix}.$$

Theorem 3.15. *Let $T : \mathcal{M}_3(\mathbb{B}) \to \mathcal{M}_3(\mathbb{B})$ be a linear operator. Then T strongly preserves $\mathcal{E}_4$ if and only if T is the sum of a permutation on Δ_3, and the composition of a (P, P^t)-operator on $\mathcal{M}_3^{(0)}(\mathbb{B})$ and possibly one or more of the operators Γ_1 and Γ_2.*

Proof. Suppose that T strongly preserves $\mathcal{E}_4$. By Lemma 3.13 T is bijective and $T(\boldsymbol{I}) = \boldsymbol{I}$.

Let $T_o : \mathcal{M}_3^{(0)}(\mathbb{B}) \to \mathcal{M}_3^{(0)}(\mathbb{B})$ be the restriction of T to $\mathcal{M}_3^{(0)}(\mathbb{B})$. Suppose that T_o maps two collinear off diagonal cells to two noncollinear off diagonal cells. If the image of these two cells is dominated by C_3 or C_3^t, then there is one other cell together with a diagonal cell that must be in $\mathcal{E}_4$ since their image is, a contradiction. Thus, say, without loss of generality, that $T_o(E_{1,2} + E_{1,3}) = E_{1,2} + E_{2,1}$. In this case, it follows that T_o is permutationally equivalent to Γ_1 or Γ_2

The converse is established by an easy check that each of the operators strongly preserves $\mathcal{E}_4$. □

4. Exponent preservers on $\mathcal{M}_n(\mathbb{S})$

The fact that the primitivity of a matrix and its exponent do not depend on the nature of the nonzero entries, only on the fact that they are nonzero, gives that any linear operator $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ preserves some property of primitive matrices if and only if $\overline{T} : \mathcal{M}_n(\mathbb{B}) \to \mathcal{M}_n(\mathbb{B})$ preserves that property of primitive matrices. Thus we state without proof the following theorems that will be a summary of the results of this paper:

Theorem 4.1. *Let $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be a linear operator and $n \geq 3$. Then the following are equivalent*:

- *T preserves the exponent of primitive matrices.*
- *T preserves $\mathcal{E}_1$ and $\mathcal{E}_2$.*
- *T strongly preserves $\mathcal{E}_2$.*
- *There are a permutation matrix $P \in \mathcal{M}_n(\mathbb{S})$ and a matrix $B \in \mathcal{M}_n(\mathbb{S})$ with $\overline{B} = \boldsymbol{J}$ such that $T(X) = P(X \circ B)P^t$ for all X, or $T(X) = P(X \circ B)^t P^t$ for all X.*

Theorem 4.2. *Let $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be a linear operator and $n \geq 3$. Then the following are equivalent*:

- *T preserves $\mathcal{E}_{n^2-2n+1}$ and $\mathcal{E}_{n^2-2n+2}$.*
- *There are a permutation matrix $P \in \mathcal{M}_n(\mathbb{S})$ and a matrix $B \in \mathcal{M}_n(\mathbb{S})$ with $\overline{B} = \boldsymbol{K}$ such that $T(X) = P(X \circ B)P^t + R(X \circ \boldsymbol{I})$ for all X, or $T(X) = P(X \circ B)^t P^t + R(X \circ \boldsymbol{I})$ for all X, where $R : \Delta_n \to \mathcal{M}_n(\mathbb{S})$ is any linear transformation.*

Remark 4.3. By replacing $\mathbb{B}$ with $\mathbb{S}$ in Definition 3.2 we have that $\mathcal{W}_n$ is the set of all matrices in $\mathcal{M}_n(\mathbb{S})$ which are dominated by a matrix of exponent

$n^2 - 2n + 2$. Similarly, $\mathcal{W}'_n$ is the set of all matrices in $\mathcal{M}_n(\mathbb{S})$ which are dominated by a matrix of exponent $n^2 - 2n + 1$.

Theorem 4.4. *Let $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be a linear operator and $n \geq 3$. Then the following are equivalent*:

- *T strongly preserves $\mathcal{E}_{n^2-2n+2}$.*
- *There are a permutation matrix $P \in \mathcal{M}_n(\mathbb{S})$ and a matrix $B \in \mathcal{M}_n(\mathbb{S})$ with $\overline{B} = \boldsymbol{K}$ such that $T(X) = P(X \circ B)P^t + R(X \circ \boldsymbol{I})$ for all X, or $T(X) = P(X \circ B)^t P^t + R(X \circ \boldsymbol{I})$ for all X, where $R : \Delta_n \to \mathcal{M}_n(\mathbb{S})$ is any linear transformation such that $R(\Delta_n) \subseteq (\mathcal{W}_n)^c$.*

Theorem 4.5. *Let $T : \mathcal{M}_n(\mathbb{S}) \to \mathcal{M}_n(\mathbb{S})$ be a linear operator and $n \geq 4$. Then the following are equivalent*:

- *T strongly preserves $\mathcal{E}_{n^2-2n+1}$.*
- *There are a permutation matrix $P \in \mathcal{M}_n(\mathbb{S})$ and a matrix $B \in \mathcal{M}_n(\mathbb{S})$ with $\overline{B} = \boldsymbol{K}$ such that $T(X) = P(X \circ B)P^t + R(X \circ \boldsymbol{I})$ for all X, or $T(X) = P(X \circ B)^t P^t + R(X \circ \boldsymbol{I})$ for all X, where $R : \Delta_n \to \mathcal{M}_n(\mathbb{S})$ is any linear transformation such that $R(\Delta_n) \subseteq (\mathcal{W}'_n)^c$.*

Theorem 4.6. *Let $T : \mathcal{M}_3(\mathbb{S}) \to \mathcal{M}_3(\mathbb{S})$ be a linear operator. Then T strongly preserves $\mathcal{E}_4$ if and only if $T = L_B + D_B$ where $L_B(X) = \overline{L}(\overline{X}) \circ B$ where $\overline{L} : \mathcal{M}_3^{(0)}(\mathbb{B}) \to \mathcal{M}_3^{(0)}(\mathbb{B})$ is the composition of a (P, P^t)-operator and possibly one or more of the operators Γ_1, Γ_2, and $D_B : \Delta_3 \to \Delta_3$ where $D_B(X) = \overline{D}(\overline{X}) \circ B$ where $\overline{D}$ is a permutation, for some $B \in \mathcal{M}_3(\mathbb{S})$ with no zero entries.*

References

[1] L. B. Beasley and A. E. Guterman, *Operators preserving primitivity for matrix pairs*, Matrix methods: theory, algorithms and applications, World Sci. Publ., Hackensack, NJ, 2–19, 2010.

[2] ———, *The characterization of operators preserving primitivity for matrix k-tuples*, Linear Algebra Appl. **430** (2009), no. 7, 1762–1777.

[3] L. B. Beasley and N. J. Pullman, *Boolean rank preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl. **59** (1984), 55–77.

[4] ———, *Term-rank, permanent, and rook-polynomial preservers*, Linear Algebra Appl. **90** (1987), 33–46.

[5] ———, *Linear operators that strongly preserve primitivity*, Linear and Multilinear Algebra **25** (1989), no. 3, 205–213.

[6] ———, *Linear operators that strongly preserve the index of imprimitivity*, Linear and Multilinear Algebra **31** (1992), no. 1-4, 267–283.

[7] R. Brualdi and H. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991.

[8] A. L. Dulmadge and N. S. Mendelsohn, *The exponents of incident matrices*, Duke Math. J. **31** (1964), 575–584.

[9] G. Fröbenius, *Uber die Darstellung der entlichen Gruppen durch Linear Substitutionen*, S. B. Deutsch. Akad. Wiss. Berlin (1897), 994–1015.

[10] J. C. Holladay and R. S. Varga, *On powers of non-negative matrices*, Proc. Amer. Math. Soc. **9** (1985), 631–634.

[11] M. Lewin and Y. Vitek, *A system of gaps in the exponent set of primitive matrices*, Illinois J. Math. **25** (1981), no. 1, 87–98.
[12] C.-K. Li and S. J. Pierce, *Linear preserver problems*, Amer. Math. Monthly **108** (2001), no. 7, 591–605.
[13] B. Liu, *On the bounds of exponents of primitive* $(0,1)$ *matrices*, Southeast Asia Bull. Math. **22** (1998), no. 1, 57–65.
[14] S. J. Pierce, et al., *A survey of linear preserver problems*, Linear and Multilinear Algebra **33** (1992), no. 1-2, 1–129.
[15] H. Wielandt, *Unzerlegbare, nicht negative Matrizen*, Math. Z. **52** (1950), 642–648.

LeRoy B. Beasley
Department of Mathematics and Statistics
Utah State University
Logan, Utah 84322-3900, USA
E-mail address: leroy.b.beasley@aggiemail.usu.edu

Kyung-Tae Kang
Department of Mathematics
Jeju National University
Jeju 690-756, Korea
E-mail address: kangkt@jejunu.ac.kr

Seok-Zun Song
Department of Mathematics
and Research Institute for Basic Sciences
Jeju National University
Jeju 690-756, Korea
E-mail address: szsong@jejunu.ac.kr