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ASYMPTOTIC RUIN PROBABILITIES IN A GENERALIZED JUMP-DIFFUSION RISK MODEL WITH CONSTANT FORCE OF INTEREST

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Abstract. This paper studies the asymptotic behavior of the finite-time ruin probability in a jump-diffusion risk model with constant force of interest, upper tail asymptotically independent claims and a general counting arrival process. Particularly, if the claim inter-arrival times follow a certain dependence structure, the obtained result also covers the case of the infinite-time ruin probability.

1. Introduction

In this paper, we consider the asymptotic ruin probabilities in a generalized jump-diffusion risk model with constant force of interest, where the claim sizes $\{X_i, i \geq 1\}$ are a sequence of nonnegative, but not necessarily independent, random variables (r.v.s) with distributions F_i, $i \geq 1$, respectively, while the claim arrival process $\{N(t), t \geq 0\}$ is a general counting process, independent of $\{X_i, i \geq 1\}$. Hence, the aggregate claim amount up to time $t \geq 0$ is

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

with $S(t) = 0$ if $N(t) = 0$. Assume that the total amount of premiums accumulated up to time $t \geq 0$, denoted by $C(t)$, is a nonnegative and nondecreasing stochastic process with $C(0) = 0$ and $C(t) < \infty$ almost surely (a.s.) for every $0 \leq t < \infty$, and that the diffusion process, as a perturbed term, $\{B(t), t \geq 0\}$ is a standard Brownian motion with volatility parameter $\sigma \geq 0$ and independent of the other sources of randomness. We notice that in practice, the diffusion-perturbed term can be interpreted as an additional uncertainty of the aggregate claims or the premium income of an insurance company. Let $r \geq 0$ be the constant force of interest and $x \geq 0$ be the insurer's initial reserve. Then the total

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reserve up to time $t \geq 0$, denoted by $U_r(t)$, satisfies

(1.1)
$$U_r(t) = xe^{rt} + \int_0^t e^{r(t-s)} dC(s) - \int_0^t e^{r(t-s)} dS(s) + \sigma \int_0^t e^{r(t-s)} dB(s),\ t \geq 0.$$

Clearly, one can see that for any fixed $0 < t < \infty$,

$$0 \leq \widetilde{C}(t) = \int_0^t e^{-rs} dC(s) < \infty \quad \text{a.s.}, \tag{1.2}$$

where $\widetilde{C}(t)$ denotes the discounted value of premiums accumulated up to time $t > 0$.

As usual, the ruin probability within a finite time $T > 0$ is defined as

$$\psi_r(x, T) = P(U_r(t) < 0 \text{ for some } 0 \leq t \leq T), \tag{1.3}$$

and the infinite-time ruin probability is

$$\psi_r(x, \infty) = P(U_r(t) < 0 \text{ for some } 0 \leq t < \infty). \tag{1.4}$$

For later use, we denote the claim inter-arrival times by $\{\theta_i, i \geq 1\}$. Then $\tau_k = \sum_{i=1}^k \theta_i$, $k \geq 1$, are the arrival times of successive claims, and generate a counting process

$$N(t) = \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_k \leq t\}}, \quad t \geq 0, \tag{1.5}$$

where $\mathbf{1}_A$ is the indicator function of an event A.

To our knowledge, the asymptotic ruin probabilities with constant interest and heavy-tailed claims were investigated extensively. For example, Veraverbeke [22] and Jiang and Yan [12] considered the compound Poisson model with diffusion, while Tang [18, 19], Hao and Tang [11], etc., considered the standard renewal model with no diffusion (i.e., $\sigma = 0$). Recently, many researchers devoted themselves to a risk model with dependent claim sizes and/or dependent inter-arrival times, see Yang and Wang [26], Li and Wu [15], Liu et al. [16], Wang et al. [25], Gao and Liu [9], Gao et al. [8], etc., where there is no diffusion term. Also, Li et al. [14] and Chen and Yuen [4] allowed some dependence structures between the claim sizes and their inter-arrival times. Therein, Wang et al. [25] introduced a dependence structure below.

Definition 1.1. Say that r.v.s $\{X_i, i \geq 1\}$ are widely upper orthant dependent (WUOD), if there exists a sequence of finite positive numbers $\{g_U(n), n \geq 1\}$ such that for each $n \geq 1$ and all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P\left(\bigcap_{i=1}^n \left\{X_i > x_i\right\}\right) \leq g_U(n) \prod_{i=1}^n P(X_i > x_i).$$

If we change the above inequality into

$$P\left(\bigcap_{i=1}^{n}\left\{X_i \le x_i\right\}\right) \le g_L(n)\prod_{i=1}^{n} P(X_i \le x_i),$$

where $\{g_L(n), n \ge 1\}$ is another sequence of finite positive numbers, then $\{X_i : i \ge 1\}$ are said to be widely lower orthant dependent (WLOD).

Clearly, if $\{X_i, i \ge 1\}$ are WLOD, then $\{-X_i, i \ge 1\}$ are WUOD, and for each $n \ge 1$ and any $s > 0$,

$$E\exp\left\{-s\sum_{i=1}^{n} X_i\right\} \le g_L(n)\prod_{i=1}^{n} Ee^{-sX_i}. \tag{1.6}$$

Besides, Geluk and Tang [10] proposed a more general dependence structure as follows.

Definition 1.2. Say that r.v.s $\{X_i, i \ge 1\}$ are upper tail asymptotically independent (UTAI), if $P(X_i > x) > 0$ for all $x \in (-\infty, \infty)$, $i \ge 1$, and

$$\lim_{\min\{x_i, x_j\}\to\infty} P(X_i > x_i | X_j > x_j) = 0 \quad \text{for all } 1 \le i \ne j < \infty.$$

If the above relation is changed to

$$\lim_{\min\{x_i, x_j\}\to\infty} P(|X_i| > x_i | X_j > x_j) = 0 \quad \text{for all } 1 \le i \ne j < \infty,$$

then we say that $\{X_i, i \ge 1\}$ are tail asymptotically independent (TAI).

The UTAI and TAI structures were also studied by Liu at al. [16], Chen et al. [2], Gao and Liu [9], and Li [13]. Clearly, the UTAI structure properly covers the WUOD structure, see Example 3.1 of Liu et al. [16]. In addition, Chen and Yuen [3] put forward a similar dependence structure, i.e., pairwise quasi-asymptotic independence (PQAI), and obtained some results that are relevant for the current study.

Henceforth, all limit relationships are for $x \to \infty$ unless stated otherwise. For two positive functions $a(\cdot)$ and $b(\cdot)$ satisfying $C^- = \liminf a(x)/b(x) \le \limsup a(x)/b(x) = C^+$, we write $a(x) \gtrsim b(x)$ if $C^- \ge 1$, write $a(x) \lesssim b(x)$ if $C^+ \le 1$, write $a(x) \sim b(x)$ if both, write $a(x) = o(1)b(x)$ if $C^+ = 0$, and write $a(x) \asymp b(x)$ if $0 < C^- \le C^+ < \infty$. For a distribution F and any $y > 0$, we set

$$J_F^+ = -\lim_{y\to\infty} \log \overline{F}_*(y)/\log y \quad \text{and} \quad J_F^- = -\lim_{y\to\infty} \log \overline{F}^*(y)/\log y$$

with $\overline{F}_*(y) = \liminf_{x\to\infty} \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x)$.

In the paper, we assume that the claim-size distributions on $[0,\infty)$ are heavy-tailed, which can model the large claims. An important class of heavy-tailed distributions is the subexponential class, we say that a distribution F on $[0,\infty)$ is subexponential, denoted by $F \in \mathcal{S}$, if $\overline{F^{*2}}(x) \sim 2\overline{F}(x)$, where F^{*2} is the 2-fold convolution of F. Clearly, if $F \in \mathcal{S}$ then F is long-tailed, denoted by $F \in \mathcal{L}$ and characterized by $\overline{F}(x+y) \sim \overline{F}(x)$ for all $y > 0$. Another important class of heavy-tailed distributions is the dominated variation class $\mathcal{D}$, we say that a distribution F on $[0,\infty)$ belongs to the class $\mathcal{D}$, denoted by $F \in \mathcal{D}$, if $\overline{F}^*(y) < \infty$ for all $y > 0$. A slightly smaller subclass of $\mathcal{L} \cap \mathcal{D}$ is the consistent variation class $\mathcal{C}$, we say that a distribution F on $[0,\infty)$ belongs to the class $\mathcal{C}$, denoted by $F \in \mathcal{C}$, if $\lim_{y\searrow 1} \overline{F}_*(y) = 1$, or equivalently, $\lim_{y\nearrow 1} \overline{F}^*(y) = 1$. In conclusion, $\mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}$. For more details of heavy-tailed distributions and their applications, we refer the readers to Bingham et al. [1] and Embrechts et al. [7].

We know that, Jiang and Yan [12] considered the compound Poisson risk model perturbed by diffusion, and established an asymptotic formula for the finite-time ruin probability with the claim-size distribution $F \in \mathcal{S}$. Recently, for a nonstandard renewal risk model with diffusion, UTAI claim sizes and WLOD inter-arrival times, Chen et al. [2] in their Corollary 2.1 gave a uniformly asymptotic formula of the finite-time ruin probability for times in a finite interval, if $F \in \mathcal{L} \cap \mathcal{D}$, and $\{C(t), t \geq 0\}$ and $\{S(t), t \geq 0\}$ are mutually independent.

Inspired by the references above, in this paper we aim to investigate the finite-time and infinite-time ruin probabilities $\psi_r(x,T)$, $0 < T \leq \infty$, in the generalized jump-diffusion risk model (1.1), where two cases are considered, one is that the premium process $\{C(t), t \geq 0\}$ is independent of the other sources of randomness, and the other is that $\{C(t), t \geq 0\}$ is not necessarily so. The following are the main results, among which the first one is concerned with the finite-time ruin probability with UTAI, non-identically distributed claim sizes and a general claim-arrival process.

Theorem 1.1. *Consider the risk model* (1.1) *with* $r \geq 0$*, in which the claim sizes* $\{X_i, i \geq 1\}$ *are UTAI r.v.s with distributions* $F_i, i \geq 1$*, respectively, and for any fixed* $0 < T < \infty$ *such that* $EN(T) > 0$*, the general claim-arrival process* $\{N(t), t \geq 0\}$ *satisfies* $E(N(T))^{p+1} < \infty$ *for some* $p > J_F^+$*. Assume that there are a sequence of positive numbers* $\{l_i, i \geq 1\}$ *and a distribution* $F \in \mathcal{L} \cap \mathcal{D}$ *such that* $\overline{F}_i(x) \sim l_i \overline{F}(x)$ *holds for each* $i \geq 1$ *and*

$$0 < \underline{l} = \inf_{n\geq 1} \frac{1}{n}\sum_{i=1}^{n} l_i \leq \sup_{n\geq 1} \frac{1}{n}\sum_{i=1}^{n} l_i = \overline{l} < \infty. \tag{1.7}$$

Then for the fixed $0 < T < \infty$,

$$\underline{l}\int_0^T \overline{F}(xe^{rt})dEN(t) \lesssim \psi_r(x,T) \lesssim \overline{l}\int_0^T \overline{F}(xe^{rt})dEN(t), \tag{1.8}$$

if the premium process $\{C(t), t \geq 0\}$ *is independent of the other sources of randomness. Further, if* $F_i \equiv F, i \geq 1$*, then for the fixed* $0 < T < \infty$,

$$\psi_r(x,T) \sim \int_0^T \overline{F}(xe^{rt})dEN(t). \tag{1.9}$$

Obviously, from relation $\overline{F_i}(x) \sim l_i\overline{F}(x)$, $i \geq 1$, and $F \in \mathcal{L} \cap \mathcal{D}$, it follows that $F_i \in \mathcal{L} \cap \mathcal{D}$ and $J^{\pm}_{F_i} = J^{\pm}_F$, $i \geq 1$. Compared to Theorem 1.1, the second main result discusses the case that $\{C(t), t \geq 0\}$ is not necessarily independent of the other sources of randomness.

Theorem 1.2. *Let $F \in \mathcal{C}$ and the other conditions of Theorem 1.1 be true. Then relation* (1.8) *still holds for any fixed $0 < T < \infty$, if the discounted value of premiums accumulated up to time T, define in* (1.2), *satisfies*

$$P(\widetilde{C}(T) > x) = o(1)\overline{F}(x). \tag{1.10}$$

Further, if $F_i \equiv F$, $i \geq 1$, then (1.9) *holds for the fixed $0 < T < \infty$.*

Applying Theorems 1.1 and 1.2, we now present a corollary for a special case when $r = 0$.

Corollary 1.1. *For the risk model* (1.1) *with $r = 0$, if the conditions of Theorem 1.1* (*or Theorem 1.2*) *are true, then for any fixed $0 < T < \infty$ and any $\alpha > 0$,*

$$\underline{l}\ \overline{F}(x)EN(T) \lesssim \psi_0(x, T) \lesssim \overline{l}\ \overline{F}(x)EN(T),$$

and

$$\alpha^{-1}\underline{l}\int_x^{x+\alpha EN(T)} \overline{F}(y)dy \lesssim \psi_0(x, T) \lesssim \alpha^{-1}\overline{l}\int_x^{x+\alpha EN(T)} \overline{F}(y)dy.$$

If $F_i \equiv F$, $i \geq 1$, then

$$\psi_0(x, T) \sim \overline{F}(x)EN(T) \sim \alpha^{-1}\int_x^{x+\alpha EN(T)} \overline{F}(y)dy.$$

In the third main result, we extend the set for T from $(0, \infty)$ to an infinite set $(0, \infty]$.

Theorem 1.3. *Under the conditions of Theorem 1.2 with $r > 0$, we further assume that the claim sizes $\{X_i, i \geq 1\}$ are identically distributed by F with $J^-_F > 0$, and the claim inter-arrival times $\{\theta_i, i \geq 1\}$ are WLOD such that for every $\epsilon > 0$,*

$$\lim_{n\to\infty} g_L(n)e^{-\epsilon n} = 0, \tag{1.11}$$

and the total discounted amount of premiums is finite, namely,

$$0 \leq \widetilde{C} = \int_0^\infty e^{-rs}dC(s) < \infty \quad a.s..$$

Then relation (1.9) *holds for all $0 < T \leq \infty$, if one of the following conditions is true*:

1. *the premium process $\{C(t), t \geq 0\}$ is independent of the other sources of randomness*;
2. *the total discounted amount of premiums satisfies*

$$P(\widetilde{C} > x) = o(1)\overline{F}(x). \tag{1.12}$$

Remark 1.1. The main results above show that the dependence structures of the claim sizes and their inter-arrival times, and the perturbed term generated by a diffusion process $\{B(t), t \geq 0\}$ do not influence the asymptotic behaviors of the finite-time and infinite-time ruin probabilities.

The remaining part of this paper is divided into two parts: Section 2 states some lemmas and Section 3 proves the main results.

2. Some lemmas

In this section, we present some lemmas that are helpful to prove the main results. The first lemma is a direct consequence of Proposition 2.2.1 of Bingham et al. [1] and Lemma 3.5 of Tang and Tsitsiashvili [20].

Lemma 2.1. *If a distribution* $F \in \mathcal{D}$ *with* $J_F^- > 0$, *then*

(1) *for any* $0 < \hat{p} < J_F^- \leq J_F^+ < p < \infty$, *there exist positive constants* $C > 1$ *and* $D > 0$ *such that*

$$C^{-1}(x/y)^{\hat{p}} \leq \frac{\overline{F}(y)}{\overline{F}(x)} \leq C(x/y)^p \quad \text{for all } x \geq y \geq D; \tag{2.1}$$

(2) *for any* $p > J_F^+$, *it holds that* $x^{-p} = o(1)\overline{F}(x)$.

The second lemma is a combination of Theorem 3.3(iv) of Cline and Samorodnitsky [6] and Lemma 2.5 of Wang et al. [24].

Lemma 2.2. *Let* X *be a r.v. with distribution* F, *and* Y *be a nonnegative r.v. independent of* X *and such that* $EY^p < \infty$ *for some* $p > J_F^+$.

(1) *If* $F \in \mathcal{D}$, *then* $P(XY > x) \asymp \overline{F}(x)$.

(2) *If* $F \in \mathcal{C}$, *then the distribution of* XY *still belongs to the class* $\mathcal{C}$.

The third lemma is a restatement of Lemma 3.3 of Gao and Liu [9]. Also, see Lemma 3.1(i) of Chen et al. [2] or Theorem 2.1 of Li [13]. It should be mentioned that the asymptotic formula in the lemma was first developed by Tang and Tsitsiashvili [21].

Lemma 2.3. *Let* $\{X_i, 1 \leq i \leq n\}$ *be* n *TAI and real-valued r.v.s with distributions* $F_i \in \mathcal{L} \cap \mathcal{D}$, $1 \leq i \leq n$, *respectively. Then for any fixed* $0 < a \leq b < \infty$,

$$P\left(\sum_{i=1}^n c_i X_i > x\right) \sim \sum_{i=1}^n P(c_i X_i > x)$$

holds uniformly for all $(c_1, c_2, \ldots, c_n) \in [a, b]^n$.

The lemma below comes from and can extend Lemma 3.5 of Wang [23].

Lemma 2.4. *In the risk model* (1.1) *with a general claim-arrival process satisfying* $EN(T) > 0$ *for any fixed* $0 < T < \infty$, *if the claim sizes* $\{X_i, i \geq 1\}$ *are* *non-identically distributed by F_i, $i \geq 1$, respectively, such that $\overline{F_i}(x) \sim l_i \overline{F}(x)$, $i \geq 1$, and (1.7) hold, then*

$$\begin{aligned} \underline{l} \int_0^T \overline{F}(xe^{rt}) dEN(t) &\lesssim \sum_{i=1}^{\infty} P(X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \leq T\}} > x) \\ &\lesssim \overline{l} \int_0^T \overline{F}(xe^{rt}) dEN(t). \end{aligned} \tag{2.2}$$

Further, if $\{X_i, i \geq 1\}$ are identically distributed by F, then

$$\sum_{i=1}^{\infty} P(X_i e^{-r\tau_i} \mathbf{1}_{\{\tau_i \leq T\}} > x) = \int_0^T \overline{F}(xe^{rt}) dEN(t). \tag{2.3}$$

Proof. Clearly, relation (2.3) is from Lemma 3.5 of Wang [23]. As for (2.2), it can be given by copying the proof of Lemma 3.5 of Wang [23] with some obvious modifications. □

The following lemma is due to Lemma 3.3 of Gao et al. [8].

Lemma 2.5. *Consider the counting process $\{N(t), t \geq 0\}$ defined by (1.5) with WLOD inter-arrival times $\{\theta_i, i \geq 1\}$ such that (1.11) holds for every $\epsilon > 0$. Then for any fixed $T > 0$ and any $p > 0$,*

$$E(N(T))^p < \infty.$$

Finally, we present Lemma 3.5 of Jiang and Yan [12], which is due to Lemma 4.5 of Tang [17].

Lemma 2.6. *Let X_1 and X_2 be two independent r.v.s with distributions F_1 and F_2, respectively. If $F_1 \in \mathcal{S}$ and $\overline{F}_2(x) = o(1)\overline{F}_1(x)$, then $P(X_1 + X_2 > x) \sim \overline{F}_1(x)$.*

3. Proofs of main results

Proof of Theorem 1.1. From (1.1) and (1.3), the finite-time ruin probability satisfies

$$\psi_r(x, T) = P(S_r(t) - \sigma I_t > x + \widetilde{C}(t) \text{ for some } 0 < t \leq T), \tag{3.1}$$

where $S_r(t) = \sum_{i=1}^{N(t)} X_i e^{-r\tau_i}$, $I_t = \int_0^t e^{-rs} dB(s)$, and $\widetilde{C}(t)$ is that in (1.2). Set $Y_T = \sigma \sup_{t \in [0,T]} |I_t|$, $0 < T \leq \infty$. It is well-known that the stochastic integral I_t, $0 < t \leq \infty$, follows a normal distribution with mean 0 and variance $\int_0^t e^{-2rs} ds$. So by many classic martingale inequalities, Y_T, $0 < T \leq \infty$, has finite moments of arbitrary orders, and then

$$P(Y_T > x) = o(1)\overline{F}(x), \quad 0 < T \leq \infty. \tag{3.2}$$

Hence from (3.1), it follows that for any fixed $0 < T < \infty$,

$$P(S_r(T) - Y_T > x + \widetilde{C}(T)) \leq \psi_r(x, T) \leq P(S_r(T) + Y_T > x). \tag{3.3}$$

Note that for any fixed $0 < T < \infty$ satisfying $EN(T) > 0$, it holds that $E(N(T))^{p+1} < \infty$ for some $p > J_F^+$, thus for any given $\varepsilon > 0$, there exists a positive integer $m_0 = m_0(\varepsilon, T) > 1$ such that

$$E(N(T))^{p+1}\mathbf{1}_{\{N(T)>m_0\}} \le \varepsilon. \tag{3.4}$$

Firstly, we deal with $P(S_r(T) > x)$. Let m_0 be fixed as above, we get

$$\begin{aligned} P(S_r(T) > x) &= \left(\sum_{n=1}^{m_0} + \sum_{n=m_0+1}^{\infty}\right) P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, N(T) = n\right) \\ &= H_1 + H_2. \end{aligned} \tag{3.5}$$

For H_1, by Lemma 2.3 and the independence between $\{X_i, i \ge 1\}$ and $\{N(t), t \ge 0\}$, we have

$$\begin{aligned} H_1 &= \sum_{n=1}^{m_0} \int_{\{0<t_1\le t_2\le\cdots\le t_n\le T,\ t_{n+1}>T\}} P\left(\sum_{i=1}^{n} X_i e^{-rt_i} > x\right) dG(t_1, t_2, \ldots, t_{n+1}) \\ &\sim \sum_{n=1}^{m_0}\sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x, N(T) = n) \\ &\le \sum_{i=1}^{\infty} P(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i\le T\}} > x), \end{aligned} \tag{3.6}$$

where $G(t_1, t_2, \ldots, t_{n+1})$ is the joint distribution of $(\tau_1, \tau_2, \ldots, \tau_{n+1})$, $1 \le n \le m_0$. For H_2, it holds

$$\begin{aligned} H_2 &\le \left(\sum_{m_0<n<x/D} + \sum_{n\ge x/D}\right) P\left(\sum_{i=1}^{n} X_i > x\right) P(N(T) = n) \\ &= H_{21} + H_{22}, \end{aligned} \tag{3.7}$$

where D is the constant in (2.1) such that $m_0 < x/D$. Then, we combine (2.1), (1.7) and (3.4) to obtain that

$$\begin{aligned} H_{21} &\le \sum_{m_0<n<x/D}\sum_{i=1}^{n} \overline{F}_i\left(\frac{x}{n}\right) P(N(T) = n) \\ &\lesssim C\overline{F}(x) \sum_{m_0<n<x/D} \left(\frac{1}{n}\sum_{i=1}^{n} l_i\right) n^{p+1} P(N(T) = n) \\ &\le C\bar{l}\ \overline{F}(x) E(N(T))^{p+1}\mathbf{1}_{\{N(T)>m_0\}} \le C\bar{l}\varepsilon\overline{F}(x). \end{aligned} \tag{3.8}$$

By Markov's inequality, Lemma 2.1(2) and (3.4), there exists an $x_1 = x_1(\varepsilon)$ such that for all $x \ge x_1$,

$$\begin{aligned} H_{22} &\le P(N(T) \ge x/D) \le (x/D)^{-(p+1)} E(N(T))^{p+1}\mathbf{1}_{\{N(T)>x/D\}} \\ &\le \varepsilon\overline{F}(x) E(N(T))^{p+1}\mathbf{1}_{\{N(T)>m_0\}} \le \varepsilon^2\overline{F}(x). \end{aligned} \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7) and considering the arbitrariness of $\varepsilon > 0$ can imply that for any fixed $0 < T < \infty$,

$$H_2 = o(1)\overline{F}(x) = o(1)P(X_1 e^{-r\tau_1}\mathbf{1}_{\{\tau_1 \le T\}} > x), \tag{3.10}$$

where the second step is due to $\overline{F}_1(x) \sim l_1\overline{F}(x)$ and Lemma 2.2(1). So from (3.5), (3.6) and (3.10), we arrive at

$$P(S_r(T) > x) \lesssim \sum_{i=1}^{\infty} P(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le T\}} > x). \tag{3.11}$$

On the other hand, we derive by the derivation of H_1 that

$$\begin{aligned} P(S_r(T) > x) \ge H_1 &\sim \left(\sum_{n=1}^{\infty} - \sum_{n=m_0+1}^{\infty}\right)\sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x, N(T) = n) \\ &= \sum_{i=1}^{\infty} P(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le T\}} > x) - H_3, \end{aligned} \tag{3.12}$$

where m_0 is the same as that in (3.4). For H_3, similarly to (3.8), it follows that

$$H_3 \lesssim \overline{F}(x) \sum_{n=m_0+1}^{\infty} \left(\frac{1}{n}\sum_{i=1}^{n} l_i\right) nP(N(T) = n) \le \bar{l}\varepsilon\overline{F}(x),$$

Thus by the similar derivation of (3.10), we also get that for any fixed $0 < T < \infty$,

$$H_3 = o(1)\overline{F}(x) = o(1)P(X_1 e^{-r\tau_1}\mathbf{1}_{\{\tau_1 \le T\}} > x),$$

which, along with (3.12), yields that

$$P(S_r(T) > x) \gtrsim \sum_{i=1}^{\infty} P(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le T\}} > x). \tag{3.13}$$

Consequently, from (3.11), (3.13) and Lemma 2.4, we show that

$$\begin{aligned} \underline{l}\int_0^T \overline{F}(xe^{rt})dEN(t) &\lesssim P(S_r(T) > x) \\ &\sim \sum_{i=1}^{\infty} P(X_i e^{-r\tau_i}\mathbf{1}_{\{\tau_i \le T\}} > x) \lesssim \bar{l}\int_0^T \overline{F}(xe^{rt})dEN(t). \end{aligned} \tag{3.14}$$

Now we turn to estimate $\psi_r(x,T)$. Clearly, a combination of the right-hand side inequality in (3.3), (3.2), (3.14), Lemma 2.6 and the independence between Y_T and $S_r(T)$, can prove that

$$\psi_r(x,T) \lesssim \bar{l}\int_0^T \overline{F}(xe^{rt})\, dEN(t). \tag{3.15}$$

By (3.14), we find that the distribution of $S_r(T)$ is long-tailed. Then by the dominated convergence theorem and the independence between $\{C(t), t \ge 0\}$

and the other sources of randomness, we know that

$$\lim_{x\to\infty}\frac{P(S_r(T)-\widetilde{Y}_T>x)}{P(S_r(T)>x)}=\int_0^\infty \lim_{x\to\infty}\frac{P(S_r(T)>x+y)}{P(S_r(T)>x)}P(\widetilde{Y}_T\in dy)=1, \tag{3.16}$$

where $\widetilde{Y}_T=Y_T+\widetilde{C}(T)$. By the left-hand side inequality in (3.3), (3.14) and (3.16), it follows that

$$\psi_r(x,T)\gtrsim \underline{l}\int_0^T \overline{F}(xe^{rt})dEN(t). \tag{3.17}$$

So by (3.15) and (3.17), relation (1.8) holds for the fixed $0<T<\infty$.

If $F_i\equiv F$, $i\geq 1$, then $\underline{l}=\overline{l}=1$, and relation (1.9) follows from (1.8) immediately. □

Proof of Theorem 1.2. According to the proof of Theorem 1.1, we only need to estimate the asymptotic lower-bound of $\psi_r(x,T)$. The condition $F\in\mathcal{C}$ ensures that for any given $\varepsilon>0$, there exist a $u_0>0$ and an $x_2=x_2(\varepsilon)$ such that for all $x\geq x_2$,

$$\overline{F}((1+u_0)x)\geq(1-\varepsilon)\overline{F}(x). \tag{3.18}$$

By the left-hand side inequality in (3.3), we see that for $u_0>0$ as above,

$$\psi_r(x,T)\geq P(S_r(T)-Y_T>(1+u_0)x)-P(\widetilde{C}(T)>u_0x)=H_4-H_5. \tag{3.19}$$

For H_4, by (3.14) and the similar derivation to (3.16), we have that for all large $x\geq x_2$,

$$H_4\gtrsim\ \underline{l}\int_0^T \overline{F}((1+u_0)xe^{rt})dEN(t)\geq(1-\varepsilon)\ \underline{l}\int_0^T \overline{F}(xe^{rt})dEN(t), \tag{3.20}$$

where the second step is due to (3.18). For H_5, by (1.10) and $F\in\mathcal{L}\cap\mathcal{D}\subset\mathcal{D}$, we get

$$H_5=o(1)\overline{F}(u_0x)=o(1)\overline{F}(x).$$

This, along with (2.1), yields that there exists an $x_3=x_3(\varepsilon)$ such that for all $x\geq\max\{x_3,D\}$,

$$H_5\leq\varepsilon\overline{F}(x)\leq\frac{C_0\varepsilon}{\underline{l}}\cdot\underline{l}\int_0^T\overline{F}(xe^{rt})dEN(t), \tag{3.21}$$

where $C_0=\frac{Ce^{rTp}}{EN(T)}$. Hence, substituting (3.20) and (3.21) into (3.19) and using the arbitrariness of $\varepsilon>0$ can prove that relation (3.17) still holds under the conditions of Theorem 1.2. □

Proof of Theorem 1.3. For the case when $0<T<\infty$, we know from Lemma 2.5 that Theorem 1.3 is a special case of Theorems 1.1 and 1.2. Hence, it suffices to deal with the case of $T=\infty$. By (1.1) and (1.4), we have

$$\psi_r(x,\infty)=P\left(\sum_{i=1}^\infty X_ie^{-r\tau_i}\mathbf{1}_{\{\tau_i\leq t\}}-\sigma I_t>x+\widetilde{C}(t)\text{ for some }0<t<\infty\right),$$

where $\widetilde{C}(t)$ and I_t are the same as those in (1.2) and (3.1), respectively. Hence,

$$P\left(\sum_{i=1}^{\infty} X_i e^{-r\tau_i} - Y_\infty > x + \widetilde{C}\right) \le \psi_r(x,\infty) \le P\left(\sum_{i=1}^{\infty} X_i e^{-r\tau_i} + Y_\infty > x\right), \tag{3.22}$$

where $\widetilde{C}$ and Y_∞ are those in (1.12) and (3.2) with $T = \infty$.

Firstly, we estimate the asymptotic upper-bound of $\psi_r(x,\infty)$. Following the proof of Lemma 3.5 of Gao and Liu [9], there exists a positive integer n_0 such that for any $0 < v < 1$,

$$P\left(\sum_{i=n_0+1}^{\infty} X_i e^{-r\tau_i} > \frac{vx}{2}\right) = o(1)P(X_1 e^{-r\tau_1} > x). \tag{3.23}$$

Note that $F \in \mathcal{C}$, then by Lemma 2.2(2), the distributions of $X_i e^{-r\tau_i}$, $i \ge 1$, all belong to the class $\mathcal{C}$. So for any given $\varepsilon > 0$, there exist a v_0, $0 < v_0 < 1$, and an $x_4 = x_4(\varepsilon) > 0$ such that for all $x \ge x_4$,

$$\sum_{i=1}^{n_0} P\left(X_i e^{-r\tau_i} > (1-v_0)x\right) \le (1+\varepsilon) \sum_{i=1}^{n_0} P\left(X_i e^{-r\tau_i} > x\right). \tag{3.24}$$

Let n_0 and v_0 be fixed as above. By the right-hand side inequality in (3.22), it holds that

$$\begin{aligned}\psi_r(x,\infty) &\le P\left(\sum_{i=1}^{n_0} X_i e^{-r\tau_i} > (1-v_0)x\right) + P\left(\sum_{i=n_0+1}^{\infty} X_i e^{-r\tau_i} > \frac{v_0 x}{2}\right) \\ &\quad + P\left(Y_\infty > \frac{v_0 x}{2}\right) \\ &= H_6 + H_7 + H_8. \end{aligned} \tag{3.25}$$

For H_6, by Theorem 1 of Chen et al. [5] and (3.24), we derive that for all large $x \ge x_4$,

$$H_6 \sim \sum_{i=1}^{n_0} P\left(X_i e^{-r\tau_i} > (1-v_0)x\right) \le (1+\varepsilon) \sum_{i=1}^{n_0} P\left(X_i e^{-r\tau_i} > x\right).$$

For H_7, by (3.23) with v replaced by v_0, we get

$$H_7 = o(1)P(X_1 e^{-r\tau_1} > x).$$

For H_8, by (3.2) with $T = \infty$, $F \in \mathcal{L} \cap \mathcal{D} \subset \mathcal{D}$ and Lemma 2.2(1), we obtain

$$H_8 = o(1)P(X_1 e^{-r\tau_1} > x).$$

Therefore, substituting the derivations of $H_i, i = 6, 7, 8$, into (3.25) and considering the arbitrariness of $\varepsilon > 0$, it follows that

$$\psi_r(x,\infty) \lesssim \sum_{i=1}^{\infty} P\left(X_i e^{-r\tau_i} > x\right) = \int_0^\infty \overline{F}(xe^{rt})dEN(t). \tag{3.26}$$

Subsequently, we estimate the asymptotic lower-bound of $\psi_r(x,\infty)$. By (2.1), we see that for all $x \geq D$ and any $0 < T < \infty$,

$$\frac{\int_T^\infty \overline{F}(xe^{rt})dEN(t)}{\int_0^\infty \overline{F}(xe^{rt})dEN(t)} = \frac{\int_T^\infty \overline{F}(xe^{rt})/\overline{F}(x)dEN(t)}{\int_0^\infty \overline{F}(xe^{rt})/\overline{F}(x)dEN(t)}$$
$$\leq C^2 \frac{\int_T^\infty e^{-r\hat{p}t}dEN(t)}{\int_0^\infty e^{-rpt}dEN(t)}. \tag{3.27}$$

Clearly, by (1.6), it holds that

$$\int_0^\infty e^{-rpt}dEN(t) = \sum_{n=1}^\infty \int_0^\infty e^{-rpt}dP(\tau_n \leq t)$$
$$= \sum_{n=1}^\infty E(e^{-rp\tau_n}) \leq \sum_{n=1}^\infty g_L(n)(Ee^{-rp\theta_1})^n.$$

For (1.11), take $\epsilon = -\log(Ee^{-rp\tau_1}) - c$ for some $c > 0$, then there exists a positive integer n_1 such that for all $n \geq n_1$,

$$g_L(n) \leq e^{-cn}\exp\{-n\log(Ee^{-rp\theta_1})\}.$$

Thus, we have

$$\int_0^\infty e^{-rpt}dEN(t) \leq \sum_{n=1}^{n_1-1} g_L(n)\left(Ee^{-rp\theta_1}\right)^n + \sum_{n=n_1}^\infty e^{-cn} < \infty.$$

Similarly, we also have

$$\int_0^\infty e^{-r\hat{p}t}dEN(t) < \infty.$$

Hence, the third item of (3.27) tends to 0 as $T \to \infty$, which yields that for the given $\varepsilon > 0$, there exists some T_0, $0 < T_0 < \infty$, such that for all $x \geq D$,

$$\int_{T_0}^\infty \overline{F}(xe^{rt})dEN(t) \leq \varepsilon \int_0^\infty \overline{F}(xe^{rt})dEN(t). \tag{3.28}$$

Under condition 1 of Theorem 1.3, by the left-hand side inequality in (3.22), the similar argument of (3.16), and (3.14) with T replaced by T_0, we show that for all $x \geq D$,

$$\psi_r(x,\infty) \geq P(S_r(T_0) - Y_\infty > x + \widetilde{C}) \gtrsim \int_0^{T_0} \overline{F}(xe^{rt})dEN(t)$$
$$= \left(\int_0^\infty - \int_{T_0}^\infty\right)\overline{F}(xe^{rt})dEN(t) \geq (1-\varepsilon)\int_0^\infty \overline{F}(xe^{rt})dEN(t), \tag{3.29}$$

where the last step is due to (3.28). Therefore, by (3.26), (3.29) and the arbitrariness of $\varepsilon > 0$, we obtain that relation (1.9) holds for $T = \infty$ under condition 1 of this theorem. Under condition 2 of Theorem 1.3, again by the left-hand side inequality in (3.22), one has

$$\psi_r(x,\infty) \geq P\left(S_r(T_0) - Y_\infty > (1+u_0)x\right) - P(\widetilde{C} > u_0x) = H_9 - H_{10}, \tag{3.30}$$

where $u_0 > 0$ and $0 < T_0 < \infty$ are those in (3.18) and (3.28), respectively. For H_9, by the similar derivation of (3.20), we prove that for all $x \geq \max\{x_2, D\}$,

$$\begin{aligned} H_9 &\geq (1-\varepsilon)\int_0^{T_0} \overline{F}(xe^{rt})dEN(t) \\ &= (1-\varepsilon)\left(\int_0^{\infty} - \int_{T_0}^{\infty}\right)\overline{F}(xe^{rt})dEN(t) \\ &\geq (1-\varepsilon)^2\int_0^{\infty} \overline{F}(xe^{rt})dEN(t), \end{aligned} \tag{3.31}$$

where the last step is due to (3.28). For H_{10}, by (1.12) and the similar derivation of (3.21), there exists an $x_5 = x_5(\varepsilon)$ such that for all $x \geq \max\{x_5, D\}$,

$$H_{10} \leq \varepsilon\overline{F}(x) \leq C_0\varepsilon\int_0^{\infty} \overline{F}(xe^{rt})dEN(t), \tag{3.32}$$

where C_0 is the same as that in (3.21). Consequently, by (3.26), (3.30)-(3.32) and the arbitrariness of $\varepsilon > 0$, we arrive at relation (1.9) for $T = \infty$ under condition 2 of the theorem, and hence the proof is completed. □

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