Management Science and Financial Engineering
Vol 20, No 1, May 2014, pp.17-19
ISSN 2287-2043 | EISSN 2287-2361 |
http://dx.doi.org/10.7737/MSFE.2014.20.1.017

Note on the Inverse Metric Traveling Salesman Problem Against the Minimum Spanning Tree Algorithm

Yerim Chung*
School of Business, Yonsei University

(Received: March 14, 2014 / Revised: May 10, 2014 / Accepted: May 12, 2014)

ABSTRACT

In this paper, we consider an interesting variant of the inverse minimum traveling salesman problem. Given an instance (G, w) of the minimum traveling salesman problem defined on a metric space, we fix a specified Hamiltonian cycle HC_0. The task is then to adjust the edge cost vector w to w' so that the new cost vector w' satisfies the triangle inequality condition and HC_0 can be returned by the minimum spanning tree algorithm in the TSP-instance defined with w'. The objective is to minimize the total deviation between the original and the new cost vectors with respect to the L_1-norm. We call this problem the inverse metric traveling salesman problem against the minimum spanning tree algorithm and show that it is closely related to the inverse metric spanning tree problem.

Keywords: Combinatorial Optimization, Inverse Optimization, TSP, Minimum Spanning Tree Algorithm

* Corresponding Author, E-mail: yerimchung@yonsei.ac.kr

1. INTRODUCTION

In (Chung, 2010; Chung and Demange, 2012), some variant of the inverse optimization problem is defined with respect to a specific algorithm. For a combinatorial optimization problem, we are given an instance $I(w)$ with a weight system w, a solution x_0 and a specific (optimal or not) algorithm A. The objective is to find a new weight system w' such that (*i*) $w' \in W$ where W denotes the property that should be satisfied by the weight system, (*ii*) x_0 can be returned by the fixed algorithm A in the new instance $I(w')$ and (*iii*) the deviation $\|w - w'\|_p$ from the original weight system w becomes minimum with respect to the L_p-norm, $p \in \{1, 2, \cdots, \infty\}$. This problem is called *the inverse optimization against a specific algorithm*.

In (Chung and Demange, 2012), Chung and Demange studied the inverse traveling salesman problem with respect to two well-known approximation algorithms: a greedy algorithm *closest neighbor* and a local search algorithm *2-opt*. They proved that the inverse traveling salesman problem (against any optimal algorithm) is co-NP-complete, while the problem against *closest neighbor* and the problem against *2-opt* are both polynomially solvable by means of linear programming for any $W \in \{\mathbb{R}^m, \Delta, \mathbb{N}^m\}$, where $W = \Delta$ denotes a metric space.

In this paper, we consider the inverse traveling salesman problem in a metric space (i.e., $W = \Delta$) with respect to another approximation algorithm, called *minimum spanning tree*. To investigate the computational complexity of the problem, we point out a close link to the inverse metric spanning tree problem.

The paper is organized as follows. In Section 2, we give a definition of the inverse metric traveling salesman problem against the minimum spanning tree algorithm and point out the equivalence between our problem and the inverse metric spanning tree problem. In Section 3, we present a mathematical model for our problem, and show that it can be solved by using a linear programming technique. Section 4 concludes the work.

2. PROBLEM DEFINITION

Together with *closest neighbor* and *2-opt*, the minimum spanning tree algorithm (*MST* for short) is a well-known approximation algorithm for solving the minimum traveling salesman problem. Given a minimum spanning tree T_0, *MST* creates a graph by doubling each edge of T_0 and finds an Eulerian tour of the graph obtained. Then it converts the tour into a Hamiltonian cycle by using triangle inequality shortcuts (Garey and Johnson, 1979). Contrary to the poor approximation behavior of *closest neighbor*, *MST* performs quite well and guarantees an approximation ratio of 2 (see (Garey and Johnson, 1979)).

The inverse traveling salesman problem against *MST* in a metric space can be defined as follows. Given an instance of the traveling salesman problem (a weighted graph (G, w) with $G = (V, E)$ and an edge cost vector w) and a fixed Hamiltonian cycle HC_0, the task is to find a new edge cost vector w' such that (*i*) w' satisfies the triangle inequality condition, (*ii*) there exists in (G, w') a minimum spanning tree that can be transformed to HC_0, and (*iii*) the deviation between w and w' is minimum under the L_p-norm for $p \in \{1, \cdots, \infty\}$.

As other inverse problems, however, the above problem is ill-posted in the sense that it does not have a unique solution. In addition, it might not be easy to determine how to modify the given weight system because a Hamiltonian cycle can be obtained from a large number of different spanning trees; indeed it is shown in (Chung, 2010) that there are an exponential number of spanning trees that can induce the same Hamiltonian cycle. Hence, we define the inverse metric traveling salesman problem against *MST* by assuming a certain spanning tree as given.

Definition 1: Inverse metric traveling salesman problem against *MST:* *Given a graph $G = (V, E)$, an edge cost vector w and a fixed spanning tree T_0 associated with HC_0, the task is to perturb the edge cost vector w to w' so that (i) the perturbed cost vector w' satisfies the triangle inequality condition, (ii) the fixed solution T_0 is a minimum spanning tree under w' and (iii) the deviation is minimum with respect to the L_p-norm for $p \in \{1, \cdots, \infty\}$.*

The mathematical model for the inverse metric traveling salesman problem against *MST* is as follows.

$$\begin{aligned} \min\ & \textstyle\sum_{l=1}^{m} \| w_l - w_l' \|_p \\ s.t.\ & w' \in \Delta, \\ & T_0 \text{ is a minimum spanning tree} \end{aligned} \tag{1}$$

The problem (1) can be seen as a variant of the inverse minimum spanning tree problem, defined with triangle inequality condition. We call it the inverse metric spanning tree problem. In the literature, the inverse minimum spanning tree problem has been extensively studied by many authors (Ahuja and Orlin, 2000; Sokkalingam *et al*., 1999; Zhang *et al*., 1996, 1997) under the L_1-norm or/and L_∞-norm. In (Zhang *et al*., 1997), Zhang, Xu and Ma use the minimum cover problem on a bipartite graph as a sub-problem and devise an $O(|E|^4)$ algorithm. Sokkalingam *et al*. (1999) formulate the inverse minimum spanning tree problem as the dual of an unbalanced assignment problem on a bipartite graph and provide an $O(|V|^3)$ algorithm.

To our knowledge, the inverse minimum spanning tree problem with triangle inequality condition has not been studied before and no one has pointed out the link between the inverse traveling salesman problem against *MST* and the inverse metric spanning tree problem. In the following section, we show that this problem can be formulated as a linear programming problem.

3. MATHEMATICAL FORMULATION

Let $(G = (V, E), w)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges. We assume that $V = \{1, 2, \cdots, n\}$ and $E = \{e_1, e_2 \cdots, e_m\}$. The cost vector is denoted by $w = (w_1, w_2, \cdots, w_m)$, each w_i denoting the cost of edge e_i. We are given a spanning tree $T_0 = \langle e_1, e_2, \cdots, e_{n-1} \rangle$; the edges in T_0 are referred to as tree edges and the edges not belonging to T_0 as non-tree edges (see (Sokkalingam *et al*., 1999)).

To formulate the inverse metric spanning tree problem as a mathematical program, we use *the path optimality condition for the minimum spanning tree,* described in (Ahuja *et al*., 1993), that is, T_0 is optimal with respect to the cost vector w' if and only if $w'_i \le w'_j$ for any $i \in P_j$, $j \in \{n, \cdots, m\}$, where P_j is the set of indices of the tree edges consisting of the unique path between two endpoints of a non-tree edge $e_j \notin T_0$. For triangle inequality constraints, we consider a triangle set $\Delta = \{(i, j, k) \mid e_i, e_j \text{ and } e_k \text{ form a triangle}\}$. The cost vector w' satisfies the triangle inequality condition if and only if for any $(i, j, k) \in \Delta$,

$$\begin{aligned} w'_i + w'_j &\ge w'_k \\ w'_j + w'_k &\ge w'_i \\ w'_k + w'_i &\ge w'_j . \end{aligned}$$

Hence, we obtain the following mathematical formulation for the inverse metric traveling salesman problem against *MST* under the L_1-norm.

$$\begin{aligned} \min\ & \textstyle\sum_{l=1}^{m} | w_l - w_l' | \\ \text{s.t.}\ & w'_i \le w'_j \ \text{ for any } i \in P_j,\ j \in \{n, \cdots, m\} \\ & w'_i + w'_j \ge w'_k \ \text{ for any } (i, j, k) \in \Delta \\ & w'_j + w'_k \ge w'_i \ \text{ for any } (i, j, k) \in \Delta \\ & w'_k + w'_i \ge w'_j \ \text{ for any } (i, j, k) \in \Delta \\ & w_l \in \mathbb{R},\ l \in \{1, \cdots, m\} \end{aligned} \tag{2}$$

By introducing the perturbation $\alpha - \beta$ such that $\alpha - \beta = w - w'$ where α_l and β_l are all nonnegative for any $l \in \{1, \cdots, m\}$, we obtain the following formulation which is equivalent to (2):

$$\begin{aligned}
\min\ & \textstyle\sum_{l=1}^{m}\alpha_l + \sum_{l=1}^{m}\beta_l \\
\text{s.t. } & w_i + \alpha_i - \beta_i \le w_j + \alpha_j - \beta_j,\ \forall i \in P_j,\ \forall j \in \{n, \cdots, m\} \\
& w_i + \alpha_i - \beta_i + w_j + \alpha_j - \beta_j \ge w_k + \alpha_k - \beta_k,\ \forall (i, j, k) \in \Delta \\
& w_j + \alpha_j - \beta_j + w_k + \alpha_k - \beta_k \ge w_i + \alpha_i - \beta_i,\ \forall (i, j, k) \in \Delta \\
& w_k + \alpha_k - \beta_k + w_i + \alpha_i - \beta_i \ge w_j + \alpha_j - \beta_j,\ \forall (i, j, k) \in \Delta \\
& \alpha_l \ge 0,\ \forall l \in \{1, \cdots, m\} \\
& \beta_l \ge 0,\ \forall l \in \{1, \cdots, m\}
\end{aligned} \tag{3}$$

Clearly, the formulation (3) is a linear programming problem. Hence, the inverse metric spanning tree problem under the L_1-norm can be solved in polynomial time by applying linear programming techniques. In particular, one can use the $O(q^3L)$ algorithm proposed in (Ye, 1991), where q is the number of variables and L is the number of bits in the input. Hence, we obtain the following result.

Proposition 1: *Under the L_1-norm, the inverse metric spanning tree problem, and hence, the inverse traveling salesman problem against MST in a metric space can be solved in polynomial time.*

Let us now consider the inverse metric traveling salesman problem against *MST* under the L_∞-norm. The corresponding mathematical formulation can be written as follows:

$$\begin{aligned}
\min\ & \max\ |w_l - w_l'| \\
\text{s.t. } & w'_i \le w'_j \ \text{ for any } i \in P_j,\ j \in \{n, \cdots, m\} \\
& w'_i + w'_j \ge w'_k \ \text{ for any } (i, j, k) \in \Delta \\
& w'_j + w'_k \ge w'_i \ \text{ for any } (i, j, k) \in \Delta \\
& w'_k + w'_i \ge w'_j \ \text{ for any } (i, j, k) \in \Delta \\
& w_l \in \mathbb{R},\ l \in \{1, \cdots, m\}
\end{aligned} \tag{4}$$

Obviously, this formulation is equivalent to the following linear programming problem.

$$\begin{aligned}
\min\ & \alpha \\
\text{s.t. } & w_l - w'_l \le \alpha \ \text{ for any } l \in \{1, \cdots, m\} \\
& -w_l + w'_l \le \alpha \ \text{ for any } l \in \{1, \cdots, m\} \\
& w'_i \le w'_j \ \text{ for any } i \in P_j,\ j \in \{n, \cdots, m\} \\
& w'_i + w'_j \ge w'_k \ \text{ for any } (i, j, k) \in \Delta \\
& w'_j + w'_k \ge w'_i \ \text{ for any } (i, j, k) \in \Delta \\
& w'_k + w'_i \ge w'_j \ \text{ for any } (i, j, k) \in \Delta \\
& w_l \in \mathbb{R},\ l \in \{1, \cdots, m\}
\end{aligned} \tag{5}$$

So, we obtain the following result.

Proposition 2: *Under the L_∞-norm, the inverse traveling salesman problem against MST in a metric space can be solved in polynomial time.*

4. CONCLUSIONS

In this paper, we have considered the inverse metric traveling salesman problem against the minimum spanning tree algorithm. We have shown its tractability by means of a polynomial time transformation to the inverse spanning tree problem with triangle inequality conditions. As future research, it would be interesting to devise for this problem some polynomial time algorithms with better computing time.

ACKNOWLEDGEMENT

This work was supported by the National Research foundation of Korea Grant, funded by the Korean government (NRF-2011-330-B00076).

REFERENCES

Ahuja, R. K., T. L. Magnanti, and J. B. Orlin, *Network ows*, *Theory*, *algorithms and applications*, Prentice-Hall, New York, 1993.

Ahuja, R. K. and J. B. Orlin, "A faster algorithm for the inverse spanning tree problem," *Journal of Algorithms* 34, 1 (2000), 177-193.

Chung, Y., *Inverse combinatorial optimization problems and applications*, Paris 1 University, Ph.D. thesis, Department of Computer Science, 2010.

Chung, Y. and M. Demange, "Some inverse traveling salesman problems," *4OR* 10, 2 (2012), 193-209.

Garey, M. R. and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, 1979.

Sokkalingam, P. T., P. K. Ahuja, and J. B. Orlin, "Solving inverse spanning tree problems through network flow techniques," *Operations Research* 47 (1999), 291-298.

Ye, Y., "An $o(n^3l)$ potential reduction algorithm for linear programming," *Mathematical Programming* 50 (1991), 239-258.

Zhang, J., Z. Liu, and Z. Ma, "On the inverse problem of minimum spanning tree with partition constraints," *Mathematical Methods of Operations Research* 44 (1996), 171-187.

Zhang, J., S. Xu, and Z. Ma, "An algorithm for inverse minimum spanning tree problem," *Optimization Methods and Software* 8, 1 (1997), 69-84.