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SINGULAR THEOREMS FOR LIGHTLIKE SUBMANIFOLDS IN A SEMI-RIEMANNIAN SPACE FORM

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Abstract. We study the geometry of lightlike submanifolds of a semi-Riemannian manifold. The purpose of this paper is to prove two singular theorems for irrotational lightlike submanifolds M of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection such that the structure vector field of $\bar{M}(c)$ is tangent to M.

1. Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics, especially in the general relativity. The study of such notion was initiated by Duggal and Bejancu [3] and later studied by many authors (see up-to date results in two books [4, 7]). Recently many authors have studied lightlike submanifolds M of indefinite almost contact metric manifolds $\bar{M}$ (see [5, 6, 7, 8, 14, 16]). The authors in above papers principally assumed that the structure vector field ζ of $\bar{M}$ is tangent to M. Călin proved the following result in his thesis:

• *Călin's result* [2]*: If the structure vector field ζ of $\bar{M}$ is tangent to M, then it belongs to the screen distribution $S(TM)$ of M.*

After Călin's work, many earlier works [5, 6, 7, 14, 16], which have been written on lightlike submanifolds of indefinite almost contact metric manifolds, obtained their results by using the Călin's result described in above.

The notion of a semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe and Chafle [1]. Although now we have lightlike version of a large variety of Riemannian submanifolds, the geometry of lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections has been few known. Several works ([9]∼[13]), which have been written on lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections, also obtained their results by

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using the Călin's result. In this paper, first of all, we prove that the afore cited Călin's result is not true for any irrotational lightlike submanifolds of a semi-Riemannian space form admitting a semi-symmetric non-metric connection. Next, some authors [8, 16] guessed that two type screen conformalities of M, named by *screen conformal* and *screen quasi-conformal*, are dependent to each other. We prove that such two type screen conformalities are independent.

2. Semi-symmetric non-metric connections

Let $(\bar{M}, \bar{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold. A connection $\bar{\nabla}$ on $\bar{M}$ is called a *semi-symmetric non-metric connection* [1, 17] if, for any vector fields X, Y and Z on $\bar{M}$, $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_X \bar{g})(Y,Z) = -\pi(Y)\bar{g}(X,Z) - \pi(Z)\bar{g}(X,Y), \tag{2.1}$$

$$\bar{T}(X,Y) = \pi(Y)X - \pi(X)Y, \tag{2.2}$$

where π is a 1-form associated with a non-vanishing smooth vector field ζ, which is called the *structure vector field*, of $\bar{M}$ by $\pi(X) = \bar{g}(X, \zeta)$.

Let (M, g) be an m-dimensional lightlike submanifold of $\bar{M}$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \le r \le \min\{m,\, n\})$. Therefore, in general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen* and *co-screen distributions*, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{2.3}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M, by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E and by $(2.3)_i$ the i-th equation of (2.3). We use same notations for any others. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $ltr(TM)$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = \bar{g}(X, N_i) = \bar{g}(W, N_i) = 0,$$

for all $X \in \Gamma(S(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where the set $\{\xi_1, \cdots, \xi_r\}$ is a lightlike basis of $Rad(TM)$. Then $T\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{2.4}$$

A lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is called

(1) *r-lightlike* if $1 \le r < \min\{m,\, n\}$;
(2) *co-isotropic* if $1 \le r = n < m$;
(3) *isotropic* if $1 \le r = m < n$;
(4) *totally lightlike* if $1 \le r = m = n$.

The above three classes (2)∼(4) are particular cases of the class (1) as follows:

$S(TM^{\perp}) = \{0\}, S(TM) = \{0\}$ and $S(TM) = S(TM^{\perp}) = \{0\}$ respectively. The geometry of r-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, we consider only r-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$, with the following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1, \cdots, \xi_r, N_1, \cdots, N_r, F_{r+1}, \cdots, F_m, W_{r+1}, \cdots, W_n\}, \tag{2.5}$$

where $\{F_{r+1}, \cdots, F_m\}$ and $\{W_{r+1}, \cdots, W_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$ respectively. We use the following range of indices:

$$i, j, k, \cdots \in \{1, \cdots, r\}, \qquad \alpha, \beta, \gamma, \cdots \in \{r+1, \cdots, n\}$$

and ϵ_α denote the causal character of respective vector field W_α.

In the entire discussion of this article, we shall assume that ζ *to be spacelike unit vector field to* M. We take $X, Y, Z \in \Gamma(TM)$ unless otherwise specified.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingartan formulas M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y)W_\alpha, \tag{2.6}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X)W_\alpha, \tag{2.7}$$

$$\bar{\nabla}_X W_\alpha = -A_{W_\alpha}X + \sum_{i=1}^{r} \phi_{\alpha i}(X)N_i + \sum_{\beta=r+1}^{n} \theta_{\alpha\beta}(X)W_\beta\,; \tag{2.8}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY)\xi_i, \tag{2.9}$$

$$\nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r} \tau_{ji}(X)\xi_j, \tag{2.10}$$

where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, h_i^{ℓ} and h_α^s are called the *local second fundamental forms* on TM respectively, h_i^* is called the *local second fundamental forms* on $S(TM)$. A_{N_i}, $A^*_{\xi_i}$ and A_{W_α} are linear operators on TM, which are called *shape operators*, and τ_{ij}, $\rho_{i\alpha}$, $\phi_{\alpha i}$ and $\theta_{\alpha\beta}$ are 1-forms on TM. We say that

$$h(X,Y) = \sum_{i=1}^{r} h_i^{\ell}(X,Y)N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y)W_\alpha$$

is the *second fundamental tensor* of M. Using (2.1), (2.2) and (2.6), we get

$$\begin{aligned}(\nabla_X g)(Y,Z) = &- \pi(Y)g(X,Z) - \pi(Z)g(X,Y) \\ &+ \sum_{i=1}^{r}\{h_i^{\ell}(X,Y)\eta_i(Z) + h_i^{\ell}(X,Z)\eta_i(Y)\},\end{aligned} \tag{2.11}$$

$$T(X,Y) \;=\; \pi(Y)X - \pi(X)Y, \tag{2.12}$$

and h_i^ℓ and h_α^s are symmetric on TM for each i and α, where T is the torsion tensor with respect to ∇ and η_i is a 1-form on TM such that

$$\eta_i(X) = \bar{g}(X, N_i), \quad \forall\, i \in \{1, \cdots, r\}.$$

From the facts $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $h_\alpha^s(X,Y) = \epsilon_\alpha \bar{g}(\bar{\nabla}_X Y, W_\alpha)$, we know that h_i^ℓ and h_α^s are independent of the choice of $S(TM)$. The above local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X, Y) - \sum_{j=1}^{r} h_j^\ell(X, \xi_i)\eta_j(Y), \tag{2.13}$$

$$\bar{g}(A^*_{\xi_i}X, N_j) = 0, \qquad h_i^\ell(X, \xi_j) + h_j^\ell(X, \xi_i) = 0,$$

$$\epsilon_\alpha h_\alpha^s(X,Y) = g(A_{W_\alpha}X, Y) - \sum_{i=1}^{r} \phi_{\alpha i}(X)\eta_i(Y), \tag{2.14}$$

$$\bar{g}(A_{W_\alpha}X, N_i) = \epsilon_\alpha \rho_{i\alpha}(X), \quad h_\alpha^s(X, \xi_i) \;=\; -\epsilon_\alpha \phi_{\alpha i}(X),$$

$$h_i^*(X, PY) = g(A_{N_i}X, PY) + f_i g(X, PY) + \eta_i(X)\pi(PY), \tag{2.15}$$

$$\mu_{ij} + \mu_{ji} = 0, \qquad \epsilon_\beta \theta_{\alpha\beta} + \epsilon_\alpha \theta_{\beta\alpha} = 0,$$

where f_i is a smooth function given by $f_i = \pi(N_i)$ and μ_{ij} is a skew symmetric 1-forms defined by

$$\mu_{ij}(X) = \eta_j(A_{N_i}X + f_i X) = \bar{g}(A_{N_i}X + f_i X, N_j). \tag{2.16}$$

Now we recall the following results due to Jin:

Theorem 2.1 [12]. *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ admitting a semi-symmetric non-metric connection. Then the following assertions are equivalent:*

(1) *$A^*_{\xi_i}$ are self-adjoint on $\Gamma(TM)$ with respect to g, for all i.*
(2) *h_i^ℓ satisfy $h_i^\ell(X, \xi_j) = 0$ for all $X \in \Gamma(TM)$, i and j.*
(3) *$A^*_{\xi_i}\xi_j = 0$ for all i and j.*

Theorem 2.2 [12]. *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ admitting a semi-symmetric non-metric connection. Then the following assertions are equivalent:*

(1) *A_{W_α} are self-adjoint on $\Gamma(TM)$ with respect to g, for all α.*
(2) *h_α^s satisfy $h_\alpha^s(X, \xi_i) = 0$ for all $X \in \Gamma(S(TM))$, α and i.*
(3) *$\phi_{\alpha i}(X) = 0$ for all $X \in \Gamma(S(TM))$, α and i.*

3. Structure equations

Denote by $\bar{R}$, R and R^* the curvature tensors of the connections $\bar{\nabla}$, ∇ and ∇^* respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z \\ &\quad + \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\ &\quad + \sum_{\alpha=r+1}^{n}\{h_\alpha^s(X,Z)A_{W_\alpha}Y - h_\alpha^s(Y,Z)A_{W_\alpha}X\} \\ &+ \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ &\quad - \pi(X)h_i^\ell(Y,Z) + \pi(Y)h_i^\ell(X,Z) \\ &\quad + \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\ &\quad + \sum_{\alpha=r+1}^{n}[\phi_{\alpha i}(X)h_\alpha^s(Y,Z) - \phi_{\alpha i}(Y)h_\alpha^s(X,Z)]\}N_i \\ &+ \sum_{\alpha=r+1}^{n}\{(\nabla_X h_\alpha^s)(Y,Z) - (\nabla_Y h_\alpha^s)(X,Z) \\ &\quad - \pi(X)h_\alpha^s(Y,Z) + \pi(Y)h_\alpha^s(X,Z) \\ &\quad + \sum_{i=1}^{r}[\rho_{i\alpha}(X)h_i^\ell(Y,Z) - \rho_{i\alpha}(Y)h_i^\ell(X,Z)\} \\ &\quad + \sum_{\beta=r+1}^{n}[\theta_{\beta\alpha}(X)h_\beta^s(Y,Z) - \theta_{\beta\alpha}(Y)h_\beta^s(X,Z)]\}W_\alpha,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\bar{R}(X,Y)N_i &= -\nabla_X(A_{N_i}Y) + \nabla_Y(A_{N_i}X) + A_{N_i}[X,\,Y] \\ &\quad + \sum_{j=1}^{r}\{\tau_{ij}(X)A_{N_j}Y - \tau_{ij}(Y)A_{N_j}X\} \\ &\quad + \sum_{\alpha=r+1}^{n}\{\rho_{i\alpha}(X)A_{W_\alpha}Y - \rho_{i\alpha}(Y)A_{W_\alpha}X\} \\ &+ \sum_{j=1}^{r}\{h_j^\ell(Y,A_{N_i}X) - h_j^\ell(X,A_{N_i}Y) + 2d\tau_{ij}(X,Y) \\ &\quad + \sum_{k=1}^{r}[\tau_{ik}(Y)\tau_{kj}(X) - \tau_{ik}(X)\tau_{kj}(Y)] \\ &\quad + \sum_{\alpha=r+1}^{n}[\rho_{i\alpha}(Y)\phi_{\alpha j}(X) - \rho_{i\alpha}(X)\phi_{\alpha j}(Y)]\}N_j\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&+\sum_{\alpha=r+1}^{n}\{h^s_\alpha(Y,A_{N_i}X)-h^s_\alpha(X,A_{N_i}Y)+2d\rho_{i\alpha}(X,Y)\\
&\qquad+\sum_{j=1}^{r}[\tau_{ij}(Y)\rho_{j\alpha}(X)-\tau_{ij}(X)\rho_{j\alpha}(Y)]\\
&\qquad+\sum_{\beta=r+1}^{n}[\rho_{i\beta}(Y)\theta_{\beta\alpha}(X)-\rho_{i\beta}(X)\theta_{\beta\alpha}(Y)]\}W_\alpha,
\end{aligned}
$$

$$
\begin{aligned}
\bar{R}(X,Y)W_\alpha &= -\nabla_X(A_{W_\alpha}Y)+\nabla_Y(A_{W_\alpha}X)+A_{W_\alpha}[X,\,Y] \qquad (3.3)\\
&\qquad+\sum_{i=1}^{r}\{\phi_{\alpha i}(X)A_{N_i}Y-\phi_{\alpha i}(Y)A_{N_i}X\}\\
&\qquad+\sum_{\beta=r+1}^{n}\{\theta_{\alpha\beta}(X)A_{W_\beta}Y-\theta_{\alpha\beta}(Y)A_{W_\beta}X\}\\
&+\sum_{i=1}^{r}\{h^\ell_i(Y,A_{W_\alpha}X)-h^\ell_i(X,A_{W_\alpha}Y)+2d\phi_{\alpha i}(X,Y)\\
&\qquad+\sum_{j=1}^{r}[\phi_{\alpha j}(Y)\tau_{ji}(X)-\phi_{\alpha j}(X)\tau_{ji}(Y)]\\
&\qquad+\sum_{\beta=r+1}^{n}[\theta_{\alpha\beta}(Y)\phi_{\beta i}(X)-\theta_{\alpha\beta}(X)\phi_{\beta i}(Y)]\}N_i\\
&+\sum_{\beta=r+1}^{n}\{h^s_\beta(Y,A_{W_\alpha}X)-h^s_\beta(X,A_{W_\alpha}Y)+2d\theta_{\alpha\beta}(X,Y)\\
&\qquad+\sum_{j=1}^{r}[\phi_{\alpha j}(Y)\rho_{j\beta}(X)-\phi_{\alpha j}(X)\rho_{j\beta}(Y)]\\
&\qquad+\sum_{\gamma=r+1}^{n}[\theta_{\alpha\gamma}(Y)\theta_{\gamma\beta}(X)-\theta_{\alpha\gamma}(X)\theta_{\gamma\beta}(Y)]\}W_\beta,
\end{aligned}
$$

$$
\begin{aligned}
R(X,Y)PZ &= R^*(X,Y)PZ \qquad (3.4)\\
&\qquad+\sum_{i=1}^{r}\{h^*_i(X,PZ)A^*_{\xi_i}Y-h^*_i(Y,PZ)A_{\xi_i}X\}\\
&+\sum_{i=1}^{r}\{(\nabla_X h^*_i)(Y,PZ)-(\nabla_Y h^*_i)(X,PZ)\\
&\qquad+\pi(Y)h^*_i(X,PZ)-\pi(X)h^*_i(Y,PZ)\\
&\qquad+\sum_{k=1}^{r}[\tau_{ik}(Y)h^*_k(X,PZ)-\tau_{ik}(X)h^*_k(Y,PZ)]\}\xi_i,
\end{aligned}
$$

$$\begin{aligned} R(X,Y)\xi_i = & -\nabla^*_X(A^*_{\xi_i}Y) + \nabla^*_Y(A^*_{\xi_i}X) + A^*_{\xi_i}[X,\, Y] \\ & + \sum_{j=1}^{r}\{\tau_{ji}(Y)A^*_{\xi_j}X - \tau_{ji}(X)A^*_{\xi_j}Y\} \\ & + \sum_{j=1}^{r}\{h^*_j(Y, A^*_{\xi_i}X) - h^*_j(X, A^*_{\xi_i}Y) - 2d\tau_{ji}(X,Y) \\ & + \sum_{k=1}^{r}[\tau_{jk}(X)\tau_{ki}(Y) - \tau_{jk}(Y)\tau_{ki}(X)]\}\xi_j. \end{aligned} \tag{3.5}$$

A complete simply connected semi-Riemannian manifold $\bar{M}$ of constant curvature c is called a *semi-Riemannian space form* and denote it by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\bar{R}(X,Y)Z = c\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(\bar{M}(c)). \tag{3.6}$$

In case the ambient manifold $\bar{M}$ is a semi-Riemannian space form $\bar{M}(c)$. Taking the scalar product with ξ_i and W_α to (3.6) by turns, we show that

$$\bar{g}(\bar{R}(X,Y)Z,\, \xi_i) = \bar{g}(\bar{R}(X,Y)Z,\, W_\alpha) = 0, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM).$$

From this results and (3.1), for any $X,\, Y,\, Z \in \Gamma(TM)$, we obtain

$$\begin{aligned} \bar{R}(X,Y)Z = & R(X,Y)Z + \sum_{i=1}^{r}\{h^\ell_i(X,Z)A_{N_i}Y - h^\ell_i(Y,Z)A_{N_i}X\} \\ & + \sum_{\alpha=r+1}^{n}\{h^s_\alpha(X,Z)A_{W_\alpha}Y - h^s_\alpha(Y,Z)A_{W_\alpha}X\}. \end{aligned} \tag{3.7}$$

4. Characterization theorems

Definition 1. An r-lightlike submanifold M of $\bar{M}$ is said to be *irrotational* [15] if $\bar{\nabla}_X\xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(Rad(TM))$.

Due to (2.6) and $(2.14)_3$, we show that M is irrotational if and only if

$$h^\ell_j(X,\xi_i) = 0, \qquad h^s_\alpha(X,\xi_i) = \phi_{\alpha i} = 0, \qquad \forall\, i,\ j,\ \alpha. \tag{4.1}$$

In this case, from $(2.13)_1$, $(4.1)_1$ and the fact $S(TM)$ is non-degenerate, we get

$$A^*_{\xi_i}\xi_j = 0, \quad \forall\, i,\ j. \tag{4.2}$$

It follow from Theorem 2.1 and Theorem 2.2 that the shape operators $A^*_{\xi_i}$ and A_{W_α} of an irrotational lightlike submanifold M are self-adjoint.

Lemma 4.1 [12] *Let M be an irrotational r-lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ admitting a semi-symmetric non-metric connection. If the structure vector field ζ is tangent to M, then ζ satisfies $h(X,\zeta) = 0$.*

Note that $h(X,\zeta) = 0$ is equivalent to the following two equations:

$$h^\ell_i(X,\zeta) = \pi(A^*_{\xi_i}X) = 0, \quad h^s_\alpha(X,\zeta) = \pi(A_{W_\alpha}X) = 0, \quad \forall\, i,\ \alpha. \tag{4.3}$$

In case M is an irrotational r-lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection, we have the following equations: Taking the scalar product with ξ_i to (3.1) and using (3.6) and the fact $\phi_{\alpha i} = 0$, we have

$$(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) = \pi(X)h_i^\ell(Y,Z) - \pi(Y)h_i^\ell(X,Z) + \sum_{j=1}^r \{\tau_{ji}(Y)h_j^\ell(X,Z) - \tau_{ji}(X)h_j^\ell(Y,Z)\}. \tag{4.4}$$

Taking the scalar product with N_i to (3.7) and then, substituting (3.4) and (3.6) into the resulting equation and using $(2.14)_2$ and (2.16), we obtain

$$\begin{aligned} & c\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y)\} \\ & + \sum_{j=1}^r \{\mu_{ji}(X)h_j^\ell(Y,PZ) - \mu_{ji}(Y)h_j^\ell(X,PZ)\} \\ & - \sum_{j=1}^r f_j\{\eta_i(X)h_j^\ell(Y,PZ) - \eta_i(Y)h_j^\ell(X,PZ)\} \\ & + \sum_{\alpha=r+1}^n \epsilon_\alpha\{\rho_{i\alpha}(X)h_\alpha^s(Y,PZ) - \rho_{i\alpha}(Y)h_\alpha^s(X,PZ)\} \\ = \ & (\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ) \\ & + \pi(Y)h_i^*(X,PZ) - \pi(X)h_i^*(Y,PZ) \\ & + \sum_{j=1}^r \{\tau_{ij}(Y)h_j^*(X,PZ) - \tau_{ij}(X)h_j^*(Y,PZ)\}. \end{aligned} \tag{4.5}$$

Definition 2. An r-lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ admitting a semi-symmetric non-metric connection is called *screen quasi-conformal* [8, 16] if the second fundamental forms h_i^* and h_i^ℓ are related by

$$h_i^*(X,PY) = \varphi_i\, h_i^\ell(X,PY) + \eta_i(X)\pi(PY), \quad \forall\, i, \tag{4.6}$$

where φ_is are non-vanishing functions on a coordinate neighborhood $\mathcal{U}$ in M.

Due to (2.13) and (2.15), we know that an r-lightlike submanifold M of $\bar{M}$ is screen quasi-conformal if and only if A_{N_i} and $A^*_{\xi_i}$ are related by

$$A_{N_i}X = \varphi_i\, A^*_{\xi_i}X - f_iX + \sum_{j=1}^r \mu_{ij}(X)\xi_j, \quad \forall\, i, \tag{4.7}$$

for some non-vanishing functions φ_i on a coordinate neighborhood $\mathcal{U}$ in M.

Theorem 4.2. *Let M be an irrotational screen quasi-conformal r-lightlike submanifolds M of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection. If the structure vector field ζ of $\bar{M}$ is tangent to M but it does not belong to $S(TM)$, then $c = 1$.*

Proof. Taking the scalar product with PZ to (3.2) and (3.7) with $Z = \xi_i$ by turns and using (2.13), (2.14), (2.15), (3.5), (4.1) and (4.6), we get

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)N_i,\, PZ) &= g(-\nabla_X(A_{N_i}Y) + \nabla_Y(A_{N_i}X) + A_{N_i}[X,Y],\, PZ) \\ &+ \sum_{j=1}^{r} \varphi_j\{\tau_{ij}(X)h_j^{\ell}(Y,PZ) - \tau_{ij}(Y)h_j^{\ell}(X,PZ)\} \\ &+ \sum_{\alpha=r+1}^{n} \epsilon_\alpha\{\rho_{i\alpha}(X)h_\alpha^s(Y,PZ) - \rho_{i\alpha}(Y)h_\alpha^s(X,PZ)\} \\ &- \sum_{j=1}^{r} f_j\{\tau_{ij}(X)g(Y,PZ) - \tau_{ij}(Y)g(X,PZ)\},\end{aligned} \tag{4.8}$$

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)\xi_i,\, PZ) &= g(-\nabla^*_X(A^*_{\xi_i}Y) + \nabla^*_Y(A^*_{\xi_i}X) + A^*_{\xi_i}[X,Y],\, PZ) \\ &+ \sum_{j=1}^{r}\{\tau_{ji}(Y)h_j^{\ell}(X,PZ) - \tau_{ji}(X)h_j^{\ell}(Y,PZ)\}.\end{aligned} \tag{4.9}$$

Applying ∇_Y to (4.7) and then, taking the scalar product with PZ, we have

$$\begin{aligned}g(\nabla_X(A_{N_i}Y), PZ) &= X[\varphi_i]\, h_i^{\ell}(Y,PZ) + \varphi_i\, g(\nabla_X(A^*_{\xi_i}Y), PZ) \\ &\quad -X[f_i]g(Y,PZ) - f_i g(\nabla_X Y, PZ) - \sum_{j=1}^{r}\mu_{ij}(Y)h_j^{\ell}(X,PZ).\end{aligned}$$

Substituting this into (4.8) and using (3.6), (3.7), (4.1) and (4.9), we get

$$\begin{aligned}&X[\varphi_i]\, h_i^{\ell}(Y,Z) - Y[\varphi_i]\, h_i^{\ell}(X,Z) \\ &= \sum_{j=1}^{r}\{\varphi_i\tau_{ji}(X) + \varphi_j\tau_{ij}(X) - \mu_{ij}(Y)\}h_j^{\ell}(Y,Z) \\ &- \sum_{j=1}^{r}\{\varphi_i\tau_{ji}(Y) + \varphi_j\tau_{ij}(Y) - \mu_{ij}(X)\}h_j^{\ell}(X,Z) \\ &+ \sum_{\alpha=r+1}^{n}\epsilon_\alpha\{\rho_{i\alpha}(X)h_\alpha^s(Y,Z) - \rho_{i\alpha}(Y)h_\alpha^s(X,Z)\} \\ &+ \{X[f_i] - \sum_{j=1}^{r} f_j\tau_{ij}(X) - f_i\pi(X) + c\eta_i(X)\}g(Y,Z) \\ &- \{Y[f_i] - \sum_{j=1}^{r} f_j\tau_{ij}(Y) - f_i\pi(X) + c\eta_i(Y)\}g(X,Z).\end{aligned} \tag{4.10}$$

Taking $X = Z = \zeta$ and $Y = \xi_i$ to this and using (4.3), we have

$$\xi_i[f_i] - \sum_{j=1}^{r} f_j\tau_{ij}(\xi_i) + c = 0. \tag{4.11}$$

Applying $\bar{\nabla}_X$ to $\eta_i(Y) = \bar{g}(Y, N_i)$ and using (2.1), (2.5) and (2.6), we have

$$\begin{aligned} X(\eta_i(Y)) &= -\pi(Y)\eta_i(X) - f_i g(X,Y) + \bar{g}(\nabla_X Y, N_i) \\ &\quad - g(A_{N_i}X, Y) + \sum_{j=1}^{r} \tau_{ij}(X)\eta_j(Y). \end{aligned}$$

Substituting this into $2d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])$ and using (2.12), (4.7) and the fact that each $A^*_{\xi_i}$ is self-adjoint, we get

$$2d\eta(X,Y) = \sum_{j=1}^{r}\{\tau_{ij}(X)\eta_j(Y) - \tau_{ij}(Y)\eta_j(X)\}. \tag{4.12}$$

Applying ∇_X to $h^*_i(Y, PZ) = \varphi_i h^\ell_i(Y, PZ) + \eta_i(Y)\pi(PZ)$, we have

$$\begin{aligned} (\nabla_X h^*_i)(Y, PZ) &= X[\varphi_i]\, h^\ell_i(Y, PZ) + \varphi_i(\nabla_X h^\ell_i)(Y, PZ) \\ &+ \{X(\eta_i(Y)) - \eta_i(\nabla_X Y)\}\pi(PZ) + \eta_i(Y)\{X(\pi(PZ)) - \pi(\nabla^*_X PZ)\}. \end{aligned}$$

Substituting this into (4.5) and using (2.12), $(2.15)_2$, (4.4), (4.10) and (4.12), we obtain

$$\begin{aligned} &\sum_{j=1}^{r} f_j\{\eta_i(Y)h^\ell_j(X, PZ) - \eta_i(X)h^\ell_j(Y, PZ)\} \\ &= \{X[f_i] - \sum_{j=1}^{r} f_j\tau_{ij}(X) - f_i\pi(X)\}g(Y, PZ) \\ &- \{Y[f_i] - \sum_{j=1}^{r} f_j\tau_{ij}(Y) - f_i\pi(X)\}g(X, PZ) \\ &+ \eta_i(Y)\{X(\pi(PZ)) - \pi(\nabla^*_X PZ)\} \\ &- \eta_i(X)\{Y(\pi(PZ)) - \pi(\nabla^*_Y PZ)\}. \end{aligned} \tag{4.13}$$

Applying ∇_X to $\pi(PZ) = g(\zeta, PZ)$ and using (2.11), we have

$$\begin{aligned} &X(\pi(PZ)) - \pi(\nabla^*_X PZ) \\ &= -\, g(X, PZ) - \pi(X)\pi(PZ) + \sum_{j=1}^{r} f_j h^\ell_j(X, PZ) + g(\nabla_X\zeta, PZ). \end{aligned}$$

Substituting this equation into (4.13), we obtain

$$\begin{aligned} &\{X[f_i] - \sum_{j=1}^{r} f_j\tau_{ij}(X) - f_i\pi(X)\}g(Y, PZ) \\ &- \{Y[f_i] - \sum_{j=1}^{r} f_j\tau_{ij}(Y) - f_i\pi(X)\}g(X, PZ) \\ &= \eta_i(Y)\{g(X, PZ) + \pi(X)\pi(PZ) - g(\nabla_X\zeta, PZ)\} \\ &- \eta_i(X)\{g(Y, PZ) + \pi(Y)\pi(PZ) - g(\nabla_Y\zeta, PZ)\}. \end{aligned} \tag{4.14}$$

Applying $\bar{\nabla}_X$ to $g(\zeta,\zeta)=1$ and using (2.1) and (2.5), we have

$$g(\nabla_X\zeta,\zeta)=\pi(X). \tag{4.15}$$

Taking $X=Z=\zeta$ and $Y=\xi_i$ to (4.14) and using (4.15), we get

$$\xi_i[f_i]-\sum_{j=1}^{r} f_j\tau_{ij}(\xi_i)+1=0. \tag{4.16}$$

From (4.11) and (4.16), we have $c=1$.

Corollary 1. *There exist no irrotational screen quasi-conformal r-lightlike submanifolds M of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ belongs to $S(TM)$.*

Proof. If ζ belongs to $S(TM)$, then we get $f_i=\bar{g}(\zeta,N_i)=0$ for all i. It follows from (4.16) that $1=0$. It is a contradiction. Thus there exist no irrotational screen quasi-conformal r-lightlike submanifolds M of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection such that ζ belongs to $S(TM)$.

Remark 1. For any lightlike submanifolds M of indefinite almost contact metric manifolds $\bar{M}$ such that the structure vector field ζ of $\bar{M}$ is tangent to M, if ζ belongs to $Rad(TM)$, then ζ is decompose as $\zeta=\sum_{i=1}^{r}a_i\xi_i$ and $a\neq 0$. It follow that $1=\bar{g}(\zeta,\zeta)=\sum_{i,\,j=1}^{r}a_ia_j\bar{g}(\xi_i,\xi_j)=0$. It is a contradiction. Thus ζ does not belong to $Rad(TM)$. This enables one to choose a screen distribution $S(TM)$ which contains ζ. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^{\sharp}=TM/Rad(TM)$ [15]. Thus all screen distributions are mutually isomorphic. This implies that *if ζ is tangent to M, then it belongs to $S(TM)$*. Călin [2] proved this result. Duggal and Sahin also proved this result in their book (see p.318 - 319 of [7]). After Călin's work, many earlier works [5, 6, 7, 14, 16], which have been written on lightlike submanifolds of indefinite almost contact manifolds or lightlike submanifolds of semi-Riemannian manifolds admitting semi-symmetric non-metric connections, obtained their results by using the Călin's result. However, we regret to indicate that the above Călin's result is not true for any lightlike submanifolds M of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection by Theorem 4.2 and its corollary.

Definition 3. An r-lightlike submanifold M is *screen conformal* [4, 7, 10] if the second fundamental forms B and C satisfy

$$h_i^*(X,PY)=\varphi_i\,h_i^{\ell}(X,PY), \quad \forall\, i, \tag{4.17}$$

where φ_is are non-vanishing functions on a coordinate neighborhood $\mathcal{U}$ in M.

Theorem 4.3. *Let M be an irrotational r-lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$ admitting a semi-symmetric non-metric connection such that the structure vector field ζ of $\bar{M}$ is tangent to M. If M is screen conformal, then we have $c = 0$.*

Proof. Applying ∇_X to $h_i^*(Y, PZ) = \varphi_i h_i^\ell(Y, PZ)$, we have

$$(\nabla_X h_i^*)(Y, PZ) = X[\varphi_i] h_i^\ell(Y, PZ) + \varphi_i(\nabla_X h_i^\ell)(Y, PZ).$$

Substituting this equation into (4.5) and using (4.4) and (4.17), we have

$$\begin{aligned}
& c\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\
&= X[\varphi_i]h_i^\ell(Y, PZ) - Y[\varphi_i]h_i^\ell(X, PZ) \\
&\quad + \sum_{j=1}^{r}\{\varphi_i\tau_{ji}(Y) + \varphi_j\tau_{ij}(Y) + \mu_{ij}(Y) + f_j\eta_i(Y)\}g(X, PZ) \\
&\quad - \sum_{j=1}^{r}\{\varphi_i\tau_{ji}(X) + \varphi_j\tau_{ij}(X) + \mu_{ij}(X) + f_j\eta_i(X)\}g(Y, PZ) \\
&\quad + \sum_{\alpha=r+1}^{n}\epsilon_\alpha\{\rho_{i\alpha}(Y)h_\alpha^s(X, PZ) - \rho_{i\alpha}(X)h_\alpha^s(Y, PZ).
\end{aligned}$$

Taking $X = \xi_i$ and $Y = Z = \zeta$ to this and using (4.3), we have $c = 0$.

Remark 2. From Theorem 4.2 and Theorem 4.3, we show that two type screen conformalities of M, named by *screen conformal* and *screen quasi-conformal*, are not mutually dependent to each other but mutually independent.

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