

CHARACTERIZATIONS OF FUNCTIONS ON DUAL OCTONION VARIABLES IN CLIFFORD ANALYSIS

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ABSTRACT. The aim of this paper is to define hyperholomorphic functions with dual octonion variables in $\mathbb{C}^4 \times \mathbb{C}^4$. We characterize properties of dual octonion variables in Clifford analysis.

1. Introduction

The octonions are a normed division algebra over the real numbers in Clifford algebra. There are four such algebras, the other three being the real numbers \mathbb{R} , the complex numbers \mathbb{C} , and the quaternions \mathbb{H} . Octonions are an extension of quaternions which are double the number. In addition, octonions satisfy noncommutative and nonassociative, however, they satisfy a weaker form of associativity which is alternative. Even though octonions are not that famous compared to the quaternions and complex numbers which are much more studied and used, they have some interesting properties and are related to a number of exceptional structures in mathematics. Additionally, octonions have applications in fields such as string theory, special relativity theory, and quantum logic. The octonions were invented in 1843 by John T. Graves, inspired by William Hamilton's discovery of quaternions. Graves called his discovery octaves. In 1973, Deutoni and Sce [1] defined octonionic regular functions and some properties of octonionic regular functions. And K. Nôno [4–6] found some properties of hyperholomorphic functions. In 2013, Lim and Shon [3] researched the properties of hyperholomorphic functions and obtained the properties of corresponding o-Cauchy-Riemann equation with octonion variables in \mathbb{C}^4 . The function $g(z) = g_1(z) + g_2(z)e_2 + g_3(z)e_4 + g_4(z)e_6$ where $z = (z_1, z_2, z_3, z_4)$ and the functions $g_1(z), g_2(z), g_3(z)$ and $g_4(z)$ are harmonic in Ω satisfies the condition of harmonicity in an open set Ω in \mathbb{C}^4 . Besides, we [2] found the theorem about hyperholomorphic functions of dual quaternion in an open subset of $\mathbb{C}^2 \times \mathbb{C}^2$.

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In this paper, we investigate the properties of hyperholomorphic functions of dual octonion variables and find the condition of integrability of the corresponding o-Cauchy-Riemann equation in an open subset of $\mathbb{C}^4 \times \mathbb{C}^4$.

2. Preliminaries

The field $\mathcal{O} \cong \mathbb{C}^4$ of octonions

$$z = x_0 + \sum_{j=0}^7 e_j x_j, \quad (x_j \in \mathbb{R}, j = 0, \dots, 7) \tag{1}$$

is an eight dimensional non-commutative and non-associative \mathbb{R} -field generated by eight base elements $e_0, e_1, e_2, e_3, e_4, e_5, e_6$ and e_7 with the following non-commutative multiplication rules :

$$e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad e_i e_j e_k = e_i (e_j e_k) \quad (i \neq j \neq k, i \neq 0, j \neq 0, k \neq 0)$$

$$e_1 e_2 = e_3, e_3 e_5 = e_6, e_6 e_7 = e_1, e_1 e_4 = e_5, e_5 e_7 = e_2, e_2 e_6 = e_4, e_4 e_7 = e_3.$$

The element e_0 is the identities of \mathcal{O} and e_1 identify the imaginary unit $\sqrt{-1}$ in the \mathbb{C} -field of complex numbers. An octonion z given by (1) is regarded as $z = z_1 + z_2 e_2 + z_3 e_4 + z_4 e_6 \in \mathcal{O}$ where $z_1 := x_0 + e_1 x_1, z_2 := x_2 + e_1 x_3, z_3 := x_4 + e_1 x_5$ and $z_4 := x_6 + e_1 x_7$ are complex numbers in \mathbb{C} . Thus, we identify \mathcal{O} with \mathbb{C}^4 . the octonionic conjugation z^* , the absolute value $|z|$ of z and a inverse z^{-1} of z in \mathcal{O} are defined, respectively, by

$$z^* = x_0 - \sum_{j=1}^7 e_j x_j, \quad |z| = \sqrt{\sum_{j=1}^4 |z_j|^2}, \quad z^{-1} = \frac{z^*}{|z|^2} \quad (z \neq 0).$$

3. Some properties of hyperholomorphic functions on dual octonion variables

The dual numbers extended the real numbers by adjoining one new element ε with the property $\varepsilon^2 = 0$. Every dual number has the form $z = x + \varepsilon y$ with x and y uniquely determined real numbers. And the conjugate dual number z^* of z is defined by $z^* = x - \varepsilon y$ and we obtain $|z|^2 = x^2$. If we use matrices, dual numbers can be represented as

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad z = x + \varepsilon y = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}.$$

The dual numbers are elements of the 2-dimensional real algebra

$$B = \{z = x + \varepsilon y \mid x, y \in \mathbb{R}, \varepsilon^2 = 0\}$$

generated by 1 and ε . The dual octonion $z = \sum_{j=0}^7 e_j x_j + \varepsilon \sum_{j=0}^7 e_j y_j$ of B is written as $z = a + \varepsilon b$. The conjugation number z^* , the absolute value $|z|$ and

the inverse z^{-1} of $z = a + \varepsilon b$ are given by the followings:

$$z^* = x_0 - \sum_{j=1}^7 e_j x_j + \varepsilon(y_0 - \sum_{j=1}^7 e_j y_j) = a^* + \varepsilon b^*,$$

$$|z|^2 = z z^* = \sum_{j=0}^7 x_j^2 + 2\varepsilon \sum_{j=0}^7 x_j y_j = \sum_{j=0}^7 \zeta_j^2, \quad z^{-1} = \frac{z^*}{|z|^2},$$

where $\zeta_j = x_j + \varepsilon y_j$ ($j = 0, \dots, 7$), a^* and b^* are conjugate numbers of a and b , respectively. We consider the following differential operators:

$$D := \frac{1}{2} \left(\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} - e_4 \frac{\partial}{\partial x_4} + e_5 \frac{\partial}{\partial x_5} - e_6 \frac{\partial}{\partial x_6} + e_7 \frac{\partial}{\partial x_7} \right. \\ \left. + \varepsilon \left(\frac{\partial}{\partial y_0} - e_1 \frac{\partial}{\partial y_1} - e_2 \frac{\partial}{\partial y_2} + e_3 \frac{\partial}{\partial y_3} - e_4 \frac{\partial}{\partial y_4} + e_5 \frac{\partial}{\partial y_5} - e_6 \frac{\partial}{\partial y_6} + e_7 \frac{\partial}{\partial y_7} \right) \right),$$

$$D^* = \frac{1}{2} \left(\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} - e_3 \frac{\partial}{\partial x_3} + e_4 \frac{\partial}{\partial x_4} - e_5 \frac{\partial}{\partial x_5} + e_6 \frac{\partial}{\partial x_6} - e_7 \frac{\partial}{\partial x_7} \right. \\ \left. + \varepsilon \left(\frac{\partial}{\partial y_0} + e_1 \frac{\partial}{\partial y_1} + e_2 \frac{\partial}{\partial y_2} - e_3 \frac{\partial}{\partial y_3} + e_4 \frac{\partial}{\partial y_4} - e_5 \frac{\partial}{\partial y_5} + e_6 \frac{\partial}{\partial y_6} - e_7 \frac{\partial}{\partial y_7} \right) \right).$$

Then, we have the following for dual quaternion operators:

$$DD^* = D^*D = \frac{1}{4} \left(\sum_{j=0}^7 \frac{\partial^2}{\partial x_j^2} + 2\varepsilon \left(\sum_{j=0}^7 \frac{\partial^2}{\partial x_j \partial y_j} \right) \right) = \frac{1}{4} \left(\sum_{j=0}^7 \frac{\partial^2}{\partial \tau_j^2} \right) = \frac{1}{4} \Delta_z,$$

where $\frac{\partial}{\partial \tau_j} = \frac{\partial}{\partial x_j} + \varepsilon \frac{\partial}{\partial y_j}$ ($j = 0, 1, \dots, 7$).

Definition 2. Let Ω be an open subset of $\mathbb{C}^4 \times \mathbb{C}^4$. A function $F(z) = f(a) + \varepsilon g(b) = \sum_{j=0}^7 e_j u_j(a) + \varepsilon (\sum_{j=0}^7 e_j v_j(b))$ is said to be hyperholomorphic on Ω if

- (1) $u_j(a)$ and $v_j(b)$ ($j = 0, 1, \dots, 7$) are continuously differentiable on Ω ,
- (2) $D^*F(z) = 0$ on Ω .

Equation (2) of Definition 2 operates to $F(z)$ as follows:

$$2D^*F = \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} - \frac{\partial u_4}{\partial x_4} + \frac{\partial u_5}{\partial x_5} - \frac{\partial u_6}{\partial x_6} + \frac{\partial u_7}{\partial x_7} \\ + e_1 \left(\frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} + \frac{\partial u_5}{\partial x_4} + \frac{\partial u_4}{\partial x_5} + \frac{\partial u_7}{\partial x_6} + \frac{\partial u_6}{\partial x_7} \right) \\ + e_2 \left(\frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} + \frac{\partial u_0}{\partial x_2} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_6}{\partial x_4} - \frac{\partial u_7}{\partial x_5} + \frac{\partial u_4}{\partial x_6} + \frac{\partial u_5}{\partial x_7} \right) \\ + e_3 \left(\frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_0}{\partial x_3} + \frac{\partial u_7}{\partial x_4} - \frac{\partial u_6}{\partial x_5} - \frac{\partial u_5}{\partial x_6} + \frac{\partial u_4}{\partial x_7} \right)$$

Therefore, the equation (2) of Definition 2 for $F(z)$ is equivalent to the following system of equations:

$$\begin{aligned}
 & \frac{\partial u_0}{\partial x_0} + \frac{\partial u_3}{\partial x_3} + \frac{\partial u_5}{\partial x_5} + \frac{\partial u_7}{\partial x_7} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_4}{\partial x_4} + \frac{\partial u_6}{\partial x_6}, \\
 & \frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} + \frac{\partial u_5}{\partial x_4} + \frac{\partial u_4}{\partial x_5} + \frac{\partial u_7}{\partial x_6} + \frac{\partial u_6}{\partial x_7} = 0, \\
 & \frac{\partial u_2}{\partial x_0} + \frac{\partial u_0}{\partial x_2} + \frac{\partial u_4}{\partial x_6} + \frac{\partial u_5}{\partial x_7} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_6}{\partial x_4} + \frac{\partial u_7}{\partial x_5}, \\
 & \frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_7}{\partial x_4} + \frac{\partial u_4}{\partial x_7} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_0}{\partial x_3} + \frac{\partial u_6}{\partial x_5} + \frac{\partial u_5}{\partial x_6}, \\
 & \frac{\partial u_4}{\partial x_0} + \frac{\partial u_6}{\partial x_2} + \frac{\partial u_7}{\partial x_3} + \frac{\partial u_0}{\partial x_4} = \frac{\partial u_5}{\partial x_1} + \frac{\partial u_1}{\partial x_5} + \frac{\partial u_2}{\partial x_6} + \frac{\partial u_3}{\partial x_7}, \\
 & \frac{\partial u_5}{\partial x_0} + \frac{\partial u_4}{\partial x_1} + \frac{\partial u_6}{\partial x_3} + \frac{\partial u_3}{\partial x_6} = \frac{\partial u_7}{\partial x_2} + \frac{\partial u_1}{\partial x_4} + \frac{\partial u_0}{\partial x_5} + \frac{\partial u_2}{\partial x_7}, \\
 & \frac{\partial u_6}{\partial x_0} + \frac{\partial u_2}{\partial x_4} + \frac{\partial u_3}{\partial x_5} + \frac{\partial u_0}{\partial x_6} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_4}{\partial x_2} + \frac{\partial u_5}{\partial x_3} + \frac{\partial u_1}{\partial x_7}, \\
 & \frac{\partial u_7}{\partial x_0} + \frac{\partial u_6}{\partial x_1} + \frac{\partial u_5}{\partial x_2} + \frac{\partial u_2}{\partial x_5} = \frac{\partial u_4}{\partial x_3} + \frac{\partial u_3}{\partial x_4} + \frac{\partial u_1}{\partial x_6} + \frac{\partial u_0}{\partial x_7}, \\
 & \frac{\partial v_0}{\partial x_0} + \frac{\partial v_3}{\partial x_3} + \frac{\partial v_5}{\partial x_5} + \frac{\partial v_7}{\partial x_7} + \frac{\partial u_0}{\partial y_0} + \frac{\partial u_3}{\partial y_3} + \frac{\partial u_5}{\partial y_5} + \frac{\partial u_7}{\partial y_7} \\
 & = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_4}{\partial x_4} + \frac{\partial v_6}{\partial x_6} + \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} + \frac{\partial u_4}{\partial y_4} + \frac{\partial u_6}{\partial y_6}, \\
 & \frac{\partial v_1}{\partial x_0} + \frac{\partial v_0}{\partial x_1} + \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} + \frac{\partial v_5}{\partial x_4} + \frac{\partial v_4}{\partial x_5} + \frac{\partial v_7}{\partial x_6} + \frac{\partial v_6}{\partial x_7} \\
 & + \frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} + \frac{\partial u_3}{\partial y_2} + \frac{\partial u_2}{\partial y_3} + \frac{\partial u_5}{\partial y_4} + \frac{\partial u_4}{\partial y_5} + \frac{\partial u_7}{\partial y_6} + \frac{\partial u_6}{\partial y_7} = 0, \\
 & \frac{\partial v_2}{\partial x_0} + \frac{\partial v_0}{\partial x_2} + \frac{\partial v_4}{\partial x_6} + \frac{\partial v_5}{\partial x_7} + \frac{\partial u_2}{\partial y_0} + \frac{\partial u_0}{\partial y_2} + \frac{\partial u_4}{\partial y_6} + \frac{\partial u_5}{\partial y_7} \\
 & = \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} + \frac{\partial v_6}{\partial x_4} + \frac{\partial v_7}{\partial x_5} + \frac{\partial u_3}{\partial y_1} + \frac{\partial u_1}{\partial y_3} + \frac{\partial u_6}{\partial y_4} + \frac{\partial u_7}{\partial y_5}, \\
 & \frac{\partial v_3}{\partial x_0} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_7}{\partial x_4} + \frac{\partial v_4}{\partial x_7} + \frac{\partial u_3}{\partial y_0} + \frac{\partial u_2}{\partial y_1} + \frac{\partial u_7}{\partial y_4} + \frac{\partial u_4}{\partial y_7} \\
 & = \frac{\partial v_1}{\partial x_2} + \frac{\partial v_0}{\partial x_3} + \frac{\partial v_6}{\partial x_5} + \frac{\partial v_5}{\partial x_6} + \frac{\partial u_1}{\partial y_2} + \frac{\partial u_0}{\partial y_3} + \frac{\partial u_6}{\partial y_5} + \frac{\partial u_5}{\partial y_6}, \\
 & \frac{\partial v_4}{\partial x_0} + \frac{\partial v_6}{\partial x_2} + \frac{\partial v_7}{\partial x_3} + \frac{\partial v_0}{\partial x_4} + \frac{\partial u_4}{\partial y_0} + \frac{\partial u_6}{\partial y_2} + \frac{\partial u_7}{\partial y_3} + \frac{\partial u_0}{\partial y_4} \\
 & = \frac{\partial v_5}{\partial x_1} + \frac{\partial v_1}{\partial x_5} + \frac{\partial v_2}{\partial x_6} + \frac{\partial v_3}{\partial x_7} + \frac{\partial u_5}{\partial y_1} + \frac{\partial u_1}{\partial y_5} + \frac{\partial u_2}{\partial y_6} + \frac{\partial u_3}{\partial y_7},
 \end{aligned}$$

$$\begin{aligned}
& \frac{\partial v_5}{\partial x_0} + \frac{\partial v_4}{\partial x_1} + \frac{\partial v_6}{\partial x_3} + \frac{\partial v_3}{\partial x_6} + \frac{\partial u_5}{\partial y_0} + \frac{\partial u_4}{\partial y_1} + \frac{\partial u_6}{\partial y_3} + \frac{\partial u_3}{\partial y_6} \\
&= \frac{\partial v_7}{\partial x_2} + \frac{\partial v_1}{\partial x_4} + \frac{\partial v_0}{\partial x_5} + \frac{\partial v_2}{\partial x_7} + \frac{\partial u_7}{\partial y_2} + \frac{\partial u_1}{\partial y_4} + \frac{\partial u_0}{\partial y_5} + \frac{\partial u_2}{\partial y_7}, \\
& \frac{\partial v_6}{\partial x_0} + \frac{\partial v_2}{\partial x_4} + \frac{\partial v_3}{\partial x_5} + \frac{\partial v_0}{\partial x_6} + \frac{\partial u_6}{\partial y_0} + \frac{\partial u_2}{\partial y_4} + \frac{\partial u_3}{\partial y_5} + \frac{\partial u_0}{\partial y_6} \\
&= \frac{\partial v_3}{\partial x_1} + \frac{\partial v_4}{\partial x_2} + \frac{\partial v_5}{\partial x_3} + \frac{\partial v_1}{\partial x_7} + \frac{\partial u_3}{\partial y_1} + \frac{\partial u_4}{\partial y_2} + \frac{\partial u_5}{\partial y_3} + \frac{\partial u_1}{\partial y_7}, \\
& \frac{\partial v_7}{\partial x_0} + \frac{\partial v_6}{\partial x_1} + \frac{\partial v_5}{\partial x_2} + \frac{\partial v_2}{\partial x_5} + \frac{\partial u_7}{\partial y_0} + \frac{\partial u_6}{\partial y_1} + \frac{\partial u_5}{\partial y_2} + \frac{\partial u_2}{\partial y_5} \\
&= \frac{\partial v_4}{\partial x_3} + \frac{\partial v_3}{\partial x_4} + \frac{\partial v_1}{\partial x_6} + \frac{\partial v_0}{\partial x_7} + \frac{\partial u_7}{\partial y_0} + \frac{\partial u_6}{\partial y_1} + \frac{\partial u_5}{\partial y_2} + \frac{\partial u_2}{\partial y_5}. \tag{2}
\end{aligned}$$

Now, we add the following condition of integrability:

$$\begin{aligned}
& \frac{\partial v_0}{\partial x_0} + \frac{\partial v_3}{\partial x_3} + \frac{\partial v_5}{\partial x_5} + \frac{\partial v_7}{\partial x_7} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_4}{\partial x_4} + \frac{\partial v_6}{\partial x_6}, \\
& \frac{\partial v_1}{\partial x_0} + \frac{\partial v_0}{\partial x_1} + \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} + \frac{\partial v_5}{\partial x_4} + \frac{\partial v_4}{\partial x_5} + \frac{\partial v_7}{\partial x_6} + \frac{\partial v_6}{\partial x_7} = 0, \\
& \frac{\partial v_2}{\partial x_0} + \frac{\partial v_0}{\partial x_2} + \frac{\partial v_4}{\partial x_6} + \frac{\partial v_5}{\partial x_7} = \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} + \frac{\partial v_6}{\partial x_4} + \frac{\partial v_7}{\partial x_5}, \\
& \frac{\partial v_3}{\partial x_0} + \frac{\partial v_2}{\partial x_1} + \frac{\partial v_7}{\partial x_4} + \frac{\partial v_4}{\partial x_7} = \frac{\partial v_1}{\partial x_2} + \frac{\partial v_0}{\partial x_3} + \frac{\partial v_6}{\partial x_5} + \frac{\partial v_5}{\partial x_6}, \\
& \frac{\partial v_4}{\partial x_0} + \frac{\partial v_6}{\partial x_2} + \frac{\partial v_7}{\partial x_3} + \frac{\partial v_0}{\partial x_4} = \frac{\partial v_5}{\partial x_1} + \frac{\partial v_1}{\partial x_5} + \frac{\partial v_2}{\partial x_6} + \frac{\partial v_3}{\partial x_7}, \\
& \frac{\partial v_5}{\partial x_0} + \frac{\partial v_4}{\partial x_1} + \frac{\partial v_6}{\partial x_3} + \frac{\partial v_3}{\partial x_6} = \frac{\partial v_7}{\partial x_2} + \frac{\partial v_1}{\partial x_4} + \frac{\partial v_0}{\partial x_5} + \frac{\partial v_2}{\partial x_7}, \\
& \frac{\partial v_6}{\partial x_0} + \frac{\partial v_2}{\partial x_4} + \frac{\partial v_3}{\partial x_5} + \frac{\partial v_0}{\partial x_6} = \frac{\partial v_3}{\partial x_1} + \frac{\partial v_4}{\partial x_2} + \frac{\partial v_5}{\partial x_3} + \frac{\partial v_1}{\partial x_7}, \\
& \frac{\partial v_7}{\partial x_0} + \frac{\partial v_6}{\partial x_1} + \frac{\partial v_5}{\partial x_2} + \frac{\partial v_2}{\partial x_5} = \frac{\partial v_4}{\partial x_3} + \frac{\partial v_3}{\partial x_4} + \frac{\partial v_1}{\partial x_6} + \frac{\partial v_0}{\partial x_7}. \tag{3}
\end{aligned}$$

We let

$$\begin{aligned}
dxy &= dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7 \\
&\quad \wedge dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \wedge dy_5 \wedge dy_6 \wedge dy_7.
\end{aligned}$$

Theorem 3.1. *Under the condition of integrability (3), let $F(z)$ be a hyperholomorphic function in an open set Ω of $\mathbb{C}^4 \times \mathbb{C}^4$ and*

$$\begin{aligned}
\kappa &= d\hat{y}_0 + e_1 d\hat{y}_1 + e_2 d\hat{y}_2 - e_3 d\hat{y}_3 + e_4 d\hat{y}_4 - e_5 d\hat{y}_5 + e_6 d\hat{y}_6 - e_7 d\hat{y}_7 \\
&+ \varepsilon(d\hat{x}_0 + e_1 d\hat{x}_1 + e_2 d\hat{x}_2 - e_3 d\hat{x}_3 + e_4 d\hat{x}_4 - e_5 d\hat{x}_5 + e_6 d\hat{x}_6 - e_7 d\hat{x}_7),
\end{aligned}$$

where \hat{dx}_j is the dx_j -removed form on $dx y$, and \hat{dy}_j is the dy_j -removed form on $dx y$ ($j = 0, 1, \dots, 7$). Then for any domain $G \subset \Omega$ with smooth distinguished boundary bG ,

$$\int_{bG} \kappa F(z) = 0,$$

where $\kappa F(z)$ is the product of dual octonions of the form κ on the function $F(z)$.

Proof. By the rule of the multiplication of dual octonions, we have

$$\begin{aligned} \kappa F(z) &= (\hat{dy}_0 + e_1 \hat{dy}_1 + e_2 \hat{dy}_2 - e_3 \hat{dy}_3 + e_4 \hat{dy}_4 - e_5 \hat{dy}_5 + e_6 \hat{dy}_6 - e_7 \hat{dy}_7 \\ &\quad + \varepsilon(\hat{dx}_0 + e_1 \hat{dx}_1 + e_2 \hat{dx}_2 - e_3 \hat{dx}_3 + e_4 \hat{dx}_4 - e_5 \hat{dx}_5 + e_6 \hat{dx}_6 \\ &\quad - e_7 \hat{dx}_7)) \cdot \left(\sum_{j=0}^7 e_j u_j + \varepsilon \sum_{j=0}^7 e_j v_j \right) \\ &= u_0 \hat{dy}_0 + e_1 u_1 \hat{dy}_0 + e_2 u_2 \hat{dy}_0 + e_3 u_3 \hat{dy}_0 + e_4 u_4 \hat{dy}_0 + e_5 u_5 \hat{dy}_0 \\ &\quad + e_6 u_6 \hat{dy}_0 + e_7 u_7 \hat{dy}_0 + e_1 u_0 \hat{dy}_1 - u_1 \hat{dy}_1 + e_3 u_2 \hat{dy}_1 - e_2 u_3 \hat{dy}_1 \\ &\quad + e_5 u_4 \hat{dy}_1 - e_4 u_5 \hat{dy}_1 + e_7 u_6 \hat{dy}_1 - e_6 u_7 \hat{dy}_1 + e_2 u_0 \hat{dy}_2 - e_3 u_1 \hat{dy}_2 \\ &\quad - u_2 \hat{dy}_2 + e_1 u_3 \hat{dy}_2 - e_6 u_4 \hat{dy}_2 + e_7 u_5 \hat{dy}_2 + e_4 u_6 \hat{dy}_2 - e_5 u_7 \hat{dy}_2 \\ &\quad - e_3 u_0 \hat{dy}_3 - e_2 u_1 \hat{dy}_3 + e_1 u_2 \hat{dy}_3 + u_3 \hat{dy}_3 - e_7 u_4 \hat{dy}_3 - e_6 u_5 \hat{dy}_3 \\ &\quad + e_5 u_6 \hat{dy}_3 + e_4 u_7 \hat{dy}_3 + e_4 u_0 \hat{dy}_4 - e_5 u_1 \hat{dy}_4 + e_6 u_2 \hat{dy}_4 - e_7 u_3 \hat{dy}_4 \\ &\quad - u_4 \hat{dy}_4 + e_1 u_5 \hat{dy}_4 - e_2 u_6 \hat{dy}_4 + e_3 u_7 \hat{dy}_4 - e_5 u_0 \hat{dy}_5 - e_4 u_1 \hat{dy}_5 \\ &\quad + e_7 u_2 \hat{dy}_5 + e_6 u_3 \hat{dy}_5 + e_1 u_4 \hat{dy}_5 + u_5 \hat{dy}_5 - e_3 u_6 \hat{dy}_5 - e_2 u_7 \hat{dy}_5 \\ &\quad + e_6 u_0 \hat{dy}_6 - e_7 u_1 \hat{dy}_6 - e_4 u_2 \hat{dy}_6 + e_5 u_3 \hat{dy}_6 + e_2 u_4 \hat{dy}_6 - e_3 u_5 \hat{dy}_6 \\ &\quad - u_6 \hat{dy}_6 + e_1 u_7 \hat{dy}_6 - e_7 u_0 \hat{dy}_7 - e_6 u_1 \hat{dy}_7 - e_5 u_2 \hat{dy}_7 - e_4 u_3 \hat{dy}_7 \\ &\quad + e_3 u_4 \hat{dy}_7 + e_2 u_5 \hat{dy}_7 + e_1 u_6 \hat{dy}_7 + u_7 \hat{dy}_7 + \varepsilon(v_0 \hat{dy}_0 + e_1 v_1 \hat{dy}_0 \\ &\quad + e_2 v_2 \hat{dy}_0 + e_3 v_3 \hat{dy}_0 + e_4 v_4 \hat{dy}_0 + e_5 v_5 \hat{dy}_0 + e_6 v_6 \hat{dy}_0 + e_7 v_7 \hat{dy}_0 \\ &\quad + e_1 v_0 \hat{dy}_1 - v_1 \hat{dy}_1 + e_3 v_2 \hat{dy}_1 - e_2 v_3 \hat{dy}_1 + e_5 v_4 \hat{dy}_1 - e_4 v_5 \hat{dy}_1 \\ &\quad + e_7 v_6 \hat{dy}_1 - e_6 v_7 \hat{dy}_1 + e_2 v_0 \hat{dy}_2 - e_3 v_1 \hat{dy}_2 - v_2 \hat{dy}_2 + e_1 v_3 \hat{dy}_2 \\ &\quad - e_6 v_4 \hat{dy}_2 + e_7 v_5 \hat{dy}_2 + e_4 v_6 \hat{dy}_2 - e_5 v_7 \hat{dy}_2 - e_3 v_0 \hat{dy}_3 - e_2 v_1 \hat{dy}_3 \\ &\quad + e_1 v_2 \hat{dy}_3 + v_3 \hat{dy}_3 - e_7 v_4 \hat{dy}_3 - e_6 v_5 \hat{dy}_3 + e_5 v_6 \hat{dy}_3 + e_4 v_7 \hat{dy}_3 \\ &\quad + e_4 v_0 \hat{dy}_4 - e_5 v_1 \hat{dy}_4 + e_6 v_2 \hat{dy}_4 - e_7 v_3 \hat{dy}_4 - v_4 \hat{dy}_4 + e_1 v_5 \hat{dy}_4 \\ &\quad - e_2 v_6 \hat{dy}_4 + e_3 v_7 \hat{dy}_4 - e_5 v_0 \hat{dy}_5 - e_4 v_1 \hat{dy}_5 + e_7 v_2 \hat{dy}_5 + e_6 v_3 \hat{dy}_5 \\ &\quad + e_1 v_4 \hat{dy}_5 + v_5 \hat{dy}_5 - e_3 v_6 \hat{dy}_5 - e_2 v_7 \hat{dy}_5 + e_6 v_0 \hat{dy}_6 - e_7 v_1 \hat{dy}_6 \\ &\quad - e_4 v_2 \hat{dy}_6 + e_5 v_3 \hat{dy}_6 + e_2 v_4 \hat{dy}_6 - e_3 v_5 \hat{dy}_6 - v_6 \hat{dy}_6 + e_1 v_7 \hat{dy}_6 \\ &\quad - e_7 v_0 \hat{dy}_7 - e_6 v_1 \hat{dy}_7 - e_5 v_2 \hat{dy}_7 - e_4 v_3 \hat{dy}_7 + e_3 v_4 \hat{dy}_7 + e_2 v_5 \hat{dy}_7 \end{aligned}$$

$$\begin{aligned}
& +e_1v_6\hat{d}y_7 + v_7\hat{d}y_7 + u_o\hat{d}x_0 + e_1u_1\hat{d}x_0 + e_2u_2\hat{d}x_0 + e_3u_3\hat{d}x_0 \\
& +e_4u_4\hat{d}x_0 + e_5u_5\hat{d}x_0 + e_6u_6\hat{d}x_0 + e_7u_7\hat{d}x_0 + e_1u_0\hat{d}x_1 - u_1\hat{d}x_1 \\
& +e_3u_2\hat{d}x_1 - e_2u_3\hat{d}x_1 + e_5u_4\hat{d}x_1 - e_4u_5\hat{d}x_1 + e_7u_6\hat{d}x_1 - e_6u_7\hat{d}x_1 \\
& +e_2u_0\hat{d}x_2 - e_3u_1\hat{d}x_2 - u_2\hat{d}x_2 + e_1u_3\hat{d}x_2 - e_6u_4\hat{d}x_2 + e_7u_5\hat{d}x_2 \\
& +e_4u_6\hat{d}x_2 - e_5u_7\hat{d}x_2 - e_3u_0\hat{d}x_3 - e_2u_1\hat{d}x_3 + e_1u_2\hat{d}x_3 + u_3\hat{d}x_3 \\
& -e_7u_4\hat{d}x_3 - e_6u_5\hat{d}x_3 + e_5u_6\hat{d}x_3 + e_4u_7\hat{d}x_3 + e_4u_0\hat{d}x_4 - e_5u_1\hat{d}x_4 \\
& +e_6u_2\hat{d}x_4 - e_7u_3\hat{d}x_4 - u_4\hat{d}x_4 + e_1u_5\hat{d}x_4 - e_2u_6\hat{d}x_4 + e_3u_7\hat{d}x_4 \\
& -e_5u_0\hat{d}x_5 - e_4u_1\hat{d}x_5 + e_7u_2\hat{d}x_5 + e_6u_3\hat{d}x_5 + e_1u_4\hat{d}x_5 + u_5\hat{d}x_5 \\
& -e_3u_6\hat{d}x_5 - e_2u_7\hat{d}x_5 + e_6u_0\hat{d}x_6 - e_7u_1\hat{d}x_6 - e_4u_2\hat{d}x_6 + e_5u_3\hat{d}x_6 \\
& +e_2u_4\hat{d}x_6 - e_3u_5\hat{d}x_6 - u_6\hat{d}x_6 + e_1u_7\hat{d}x_6 - e_7u_0\hat{d}x_7 - e_6u_1\hat{d}x_7 \\
& -e_5u_2\hat{d}x_7 - e_4u_3\hat{d}x_7 + e_3u_4\hat{d}x_7 + e_2u_5\hat{d}x_7 + e_1u_6\hat{d}x_7 + u_7\hat{d}x_7).
\end{aligned}$$

Therefore,

$$\begin{aligned}
d(\kappa F) &= \left(\sum_{j=0}^7 \frac{\partial}{\partial x_j} dx_j + \varepsilon \sum_{j=0}^7 \frac{\partial}{\partial y_j} dy_j \right) (\kappa F) \\
&= \varepsilon \left(\frac{\partial u_0}{\partial y_0} - \frac{\partial u_1}{\partial y_1} - \frac{\partial u_2}{\partial y_2} + \frac{\partial u_3}{\partial y_3} - \frac{\partial u_4}{\partial y_4} + \frac{\partial u_5}{\partial y_5} - \frac{\partial u_6}{\partial y_6} + \frac{\partial u_7}{\partial y_7} \right. \\
&\quad + \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} - \frac{\partial u_4}{\partial x_4} + \frac{\partial u_5}{\partial x_5} - \frac{\partial u_6}{\partial x_6} + \frac{\partial u_7}{\partial x_7} \Big) dx y \\
&\quad + e_1 \left(\frac{\partial u_1}{\partial y_0} + \frac{\partial u_0}{\partial y_1} + \frac{\partial u_3}{\partial y_2} + \frac{\partial u_2}{\partial y_3} + \frac{\partial u_5}{\partial y_4} + \frac{\partial u_4}{\partial y_5} + \frac{\partial u_7}{\partial y_6} + \frac{\partial u_6}{\partial y_7} \right. \\
&\quad + \frac{\partial u_1}{\partial x_0} + \frac{\partial u_0}{\partial x_1} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} + \frac{\partial u_5}{\partial x_4} + \frac{\partial u_4}{\partial x_5} + \frac{\partial u_7}{\partial x_6} + \frac{\partial u_6}{\partial x_7} \Big) dx y \\
&\quad + e_2 \left(\frac{\partial u_2}{\partial y_0} - \frac{\partial u_3}{\partial y_1} + \frac{\partial u_0}{\partial y_2} - \frac{\partial u_1}{\partial y_3} - \frac{\partial u_6}{\partial y_4} - \frac{\partial u_7}{\partial y_5} + \frac{\partial u_4}{\partial y_6} + \frac{\partial u_5}{\partial y_7} \right. \\
&\quad + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} + \frac{\partial u_0}{\partial x_2} - \frac{\partial u_1}{\partial x_3} - \frac{\partial u_6}{\partial x_4} - \frac{\partial u_7}{\partial x_5} + \frac{\partial u_4}{\partial x_6} + \frac{\partial u_5}{\partial x_7} \Big) dx y \\
&\quad + e_3 \left(\frac{\partial u_3}{\partial y_0} + \frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2} - \frac{\partial u_0}{\partial y_3} + \frac{\partial u_7}{\partial y_4} - \frac{\partial u_6}{\partial y_5} - \frac{\partial u_5}{\partial y_6} + \frac{\partial u_4}{\partial y_7} \right. \\
&\quad + \frac{\partial u_3}{\partial x_0} + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} - \frac{\partial u_0}{\partial x_3} + \frac{\partial u_7}{\partial x_4} - \frac{\partial u_6}{\partial x_5} - \frac{\partial u_5}{\partial x_6} + \frac{\partial u_4}{\partial x_7} \Big) dx y \\
&\quad + e_4 \left(\frac{\partial u_4}{\partial y_0} - \frac{\partial u_5}{\partial y_1} + \frac{\partial u_6}{\partial y_2} + \frac{\partial u_7}{\partial y_3} + \frac{\partial u_0}{\partial y_4} - \frac{\partial u_1}{\partial y_5} - \frac{\partial u_2}{\partial y_6} - \frac{\partial u_3}{\partial y_7} \right. \\
&\quad + \frac{\partial u_4}{\partial x_0} - \frac{\partial u_5}{\partial x_1} + \frac{\partial u_6}{\partial x_2} + \frac{\partial u_7}{\partial x_3} + \frac{\partial u_0}{\partial x_4} - \frac{\partial u_1}{\partial x_5} - \frac{\partial u_2}{\partial x_6} - \frac{\partial u_3}{\partial x_7} \Big) dx y
\end{aligned}$$

$$\begin{aligned}
 &+e_5\left(\frac{\partial u_5}{\partial y_0} + \frac{\partial u_4}{\partial y_1} - \frac{\partial u_7}{\partial y_2} + \frac{\partial u_6}{\partial y_3} - \frac{\partial u_1}{\partial y_4} - \frac{\partial u_0}{\partial y_5} + \frac{\partial u_3}{\partial y_6} - \frac{\partial u_2}{\partial y_7}\right. \\
 &+ \left.\frac{\partial u_5}{\partial x_0} + \frac{\partial u_4}{\partial x_1} - \frac{\partial u_7}{\partial x_2} + \frac{\partial u_6}{\partial x_3} - \frac{\partial u_1}{\partial x_4} - \frac{\partial u_0}{\partial x_5} + \frac{\partial u_3}{\partial x_6} - \frac{\partial u_2}{\partial x_7}\right)dx y \\
 &+e_6\left(\frac{\partial u_6}{\partial y_0} - \frac{\partial u_7}{\partial y_1} - \frac{\partial u_4}{\partial y_2} - \frac{\partial u_5}{\partial y_3} + \frac{\partial u_2}{\partial y_4} + \frac{\partial u_3}{\partial y_5} + \frac{\partial u_0}{\partial y_6} - \frac{\partial u_1}{\partial y_7}\right. \\
 &+ \left.\frac{\partial u_6}{\partial x_0} - \frac{\partial u_7}{\partial x_1} - \frac{\partial u_4}{\partial x_2} - \frac{\partial u_5}{\partial x_3} + \frac{\partial u_2}{\partial x_4} + \frac{\partial u_3}{\partial x_5} + \frac{\partial u_0}{\partial x_6} - \frac{\partial u_1}{\partial x_7}\right)dx y \\
 &+e_7\left(\frac{\partial u_7}{\partial y_0} + \frac{\partial u_6}{\partial y_1} + \frac{\partial u_5}{\partial y_2} - \frac{\partial u_4}{\partial y_3} - \frac{\partial u_3}{\partial y_4} + \frac{\partial u_2}{\partial y_5} - \frac{\partial u_1}{\partial y_6} - \frac{\partial u_0}{\partial y_7}\right. \\
 &+ \left.\frac{\partial u_7}{\partial x_0} + \frac{\partial u_6}{\partial x_1} + \frac{\partial u_5}{\partial x_2} - \frac{\partial u_4}{\partial x_3} - \frac{\partial u_3}{\partial x_4} + \frac{\partial u_2}{\partial x_5} - \frac{\partial u_1}{\partial x_6} - \frac{\partial u_0}{\partial x_7}\right)dx y).
 \end{aligned}$$

By Equations (2) and (3), we have $d(\kappa F) = 0$. By Stokes theorem, we have

$$\int_{bG} \kappa F = \int_G d(\kappa F) = 0.$$

□

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